Seminar 25.03.2024. Holomorphic convexity

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Definition 1. A *Reinhardt domain* in $\mathbb{C}^n$ (centered at the origin) is a domain invariant under the ${}^{\mathrm{n}}$-action by coordinatewise rotations around the origin. We consider the map

$$\lambda : \mathbb{C}^n \to \mathbb{R}^n, \ z \mapsto (\ln |z_1|, \dots, \ln |z_n|).$$

Definition 2. A Reinhardt domain is *logarithmically convex,* if the image $\lambda(D) \subset \mathbb{R}^n$ is convex.
Example. We already know that the convergence domain of a power series based at the origin is a Reinhardt domain containing the origin.

Problem 1. A Reinhardt domain is a convergence domain for a power series, if and only if it contains the origin and is logarithmically convex.

Sketch of proof. Step 1 (Task 1, Problem 3). A convergence domain is a logarithmically convex Reinhardt domain.

Step 2. A logarithmically convex Reinhardt domain containing the origin is a union of polydisks centered at the origin. Or equivalently, its logarithmic image (which is a convex subset in $\mathbb{R}^n$ containing at least one negative quadrant) is a union of negative quadrants.

Step 3. A logarithmically convex Reinhardt domain D containing the origin is holomorphically convex. Hence, it is a domain of holomorphy of a function f, by the proved part of Oka's Theorem. This together with Step 2 and Abel's Lemma implies that this is the convergence domain of the function f.

Proof of Step 3. Consider a closed convex subset in $\mathbb{R}^n_{x_1,\dots,x_n}$ whose interior is a union of negative quadrants. Then it is the intersection of half-spaces defined by inequalities of the type $\sum_{j=1}^n a_j x_j \leq c$ with $a_j \geq 0$, $a_j \in \mathbb{Q}$. Multiplying the latter inequality by a natural number (product of denominators of the rational numbers a_j) and substituting $x_j = \ln |z_j|$, we get an equivalent inequality of the type $|z^m| = |z_1^{m_1} \dots z_n^{m_n}| \leq C$.

First consider the case, when D is bounded. Set $D_\varepsilon := (1 - \varepsilon)D$, where $\varepsilon \in (0, 1)$ is small enough. One has $\overline{D}_\varepsilon \Subset D$, and $\overline{D}_\varepsilon$ is F-convex, where F is the class of all the monomials z^m, by the above discussion. Any compact subset $K \Subset D$ is contained in $\overline{D}_\varepsilon$ for every ε small enough. Hence, its F-convex hull is also contained there, and thus, is compact. This implies that D is F-convex, and hence, holomorphically convex.

Consider now the case, when D is unbounded. Fix an arbitrary compact subset $K \Subset D$ and a polydisk Δ_r containing K. The intersection $D \cap \Delta_r$ is a logarithmically convex bounded Reinhardt domain. Hence, it is F-convex, by the above discussion. Therefore, the F-convex hull of the set K is a compact subset in $D \cap \Delta_r$. This proves F-convexity of the domain D, and hence, its holomorphic convexity.