Several Complex Variables

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Contents

1 Holomorphic functions of several complex variables. Cauchy–Riemann equations, Cauchy formula, Taylor series

Definition 1.1 Let $\Omega \subset \mathbb{C}^n$ be an open subset. Recall that a function $f : \Omega \to \mathbb{C}$ is said to be ($\mathbb{R}$-)differentiable at a point $p \in \Omega$, if it is differentiable there as a function of real variables: there exists an $\mathbb{R}$-linear mapping $df(p) : T_p\mathbb{C}^n \simeq \mathbb{R}^{2n} \to T_p\mathbb{C} \simeq \mathbb{R}^2$ such that

$$f(z) - f(p) = df(p)(z-p) + o(z-p), \text{ as } z \to p.$$

A function f is said to be $\mathbb{C}$-*differentiable at a point* p, if it is differentiable there and its differential $df(p)$ is $\mathbb{C}$-linear. A function f is said to be *holomorphic* on Ω, if it is $\mathbb{C}$-differentiable at each point $x_0 \in \Omega$. A function f is said to be *holomorphic at a point* $x_0 \in \mathbb{C}^n$, if it is $\mathbb{C}$-differentiable in some its neighborhood. A holomorphic mapping $F = (F_1, \dots, F_m) : U \to V$, $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ is defined in literally the same way: it is holomorphic, if and only if so are its components $F_1, \dots, F_m$.

Holomorphicity of a differentiable function is equivalent to Cauchy–Riemann Equations. To write them, let us first recall the following preparatory linear algebra.

Let $\mathbb{C}$ be equipped with a complex coordinate $z = x + iy$. Each $\mathbb{R}$-linear operator $L : \mathbb{C} \to \mathbb{C}$ can be written in the two following forms

$$L = \alpha x + \beta y = Az + B\bar{z}; \quad \alpha, \beta, A, B \in \mathbb{C}.$$

The expression of the coefficients A and B via α and β is obtained by the substitutions

$$x = \frac{1}{2}(z + \bar{z}),\ y = \frac{1}{2i}(z - \bar{z}):$$

$$L = \alpha x + \beta y = \frac{\alpha}{2}(z + \bar{z}) + \frac{\beta}{2i}(z - \bar{z}) = Az + B\bar{z},$$

$$A = \frac{1}{2}(\alpha - i\beta),\ B = \frac{1}{2}(\alpha + i\bar{\beta}). \tag{1.1}$$

Let $f : U \to \mathbb{C}$ be a differentiable mapping of a domain $U \subset \mathbb{C}$. For every $p \in U$ the differential $df(p) : T_p\mathbb{C} \simeq \mathbb{C} \to T_{f(p)}\mathbb{C} \simeq \mathbb{C}$ is an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$. One has

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}\overline{dz}; \quad \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}),\ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}), \tag{1.2}$$

which follows from formula (1.1) applied to the differential $L = df(p)$, taking into account that in our case

$$\alpha = \frac{\partial f}{\partial x}, \ \beta = \frac{\partial f}{\partial y}.$$

Proposition 1.2 *(Cauchy–Riemann Equations). A differentiable function $f(z_1, \dots, z_n)$ on a domain in $\mathbb{C}^n$ is holomorphic, if and only if*

$$\frac{\partial f}{\partial \bar{z}_j} \equiv 0 \text{ for every } j = 1, \dots, n. \tag{1.3}$$

The latter equation number j is equivalent to the system of equations

$$\begin{cases} \frac{\partial \operatorname{Re} f}{\partial x_j} = \frac{\partial \operatorname{Im} f}{\partial y_j} \\ \frac{\partial \operatorname{Re} f}{\partial y_j} = -\frac{\partial \operatorname{Im} f}{\partial x_j}. \end{cases} \tag{1.4}$$

Proof The tangent space $T_p\mathbb{C}^n$ is the direct sum of complex "coordinate lines" parallel to the coordinate axes. Thus, the $\mathbb{C}$-linearity of the differential $df(p)$ is equivalent to the $\mathbb{C}$-linearity of its restrictions to all the coordinate lines. The latter is equivalent to (1.3). Equivalence of equation (1.3) and system (1.4) follows from (1.2). This proves the proposition. □

Example 1.3 Holomorphicity is preserved under arithmetic combinations and compositions. In particular, polynomials and rational functions and in general, all the elementary functions (restricted to their appropriate definition domains) are holomorphic.

Remark 1.4 In the case, when $n = 1$ the above definition coincides with the classical definition of holomorphic function of one complex variable. If a function f is holomorphic in Ω, then for every complex line $L \subset \mathbb{C}^n$ the restriction $f|_{L\cap\Omega}$ is holomorphic as a function of one variable. The next Big Hartogs' Theorem implies that the converse is also true.

Theorem 1.5 *(Hartogs). A function $f(z_1, \dots, z_n)$ is holomorphic on a domain $\Omega = \Omega_1 \times \cdots \times \Omega_n \subset \mathbb{C}^n$, if and only if it is* **separately holomorphic:** *for every $j = 1, \dots, n$ and every given collection of points $z_s \in \Omega_s$, $s \neq j$, the function $g(z) = f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$ is holomorphic on Ω_j.*

Remark 1.6 The nontrivial part of the theorem says that if a function is separately holomorphic, then it is holomorphic as a function of several variables. Under the additional assumption that f is differentiable, this statement follows immediately from Proposition 1.2. We will not prove Theorem

1.5 in full generality. We will prove its weaker version under continuity assumption (Osgood Lemma).

Holomorphic functions in several variables share the basic properties of holomorphic functions in one variable: existence of converging Taylor series, uniqueness of analytic extension, openness, Maximum Principle, Liouville Theorem. At the same time we will see that the following new phenomena hold for holomorphic functions in several complex variables, which are in contrast with the case of one variable:

- *no isolated singularities;*

- *erasing compact singularities:* holomorphic functions on a complement of a domain $V \subset \mathbb{C}^n$ to a compact subset $K \Subset V$ extend holomorphically to all of V.

Everywhere below for every $\delta > 0$ and $z \in \mathbb{C}$ we denote

$$D_\delta(z) = \{|w - z| < \delta\} \subset \mathbb{C};\ D_\delta = D_\delta(0).$$

The corresponding balls in $\mathbb{C}^n$ of radius δ will be denoted by $B_\delta(z)$ and B_δ respectively. For every $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the *polydisk* of multiradius r centered at z is the product of disks of radii r_j, which we will denote by

$$\Delta_r(z) = \prod_j D_{r_j}(z_j) = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n \mid |w_j - z_j| < r_j\};\ \Delta_r = \Delta_r(0).$$

For $\delta > 0$ we denote $\Delta_\delta(z) = \Delta_{(\delta,\dots,\delta)}(z)$, $\Delta_\delta = \Delta_\delta(0)$. In the case, when we would like to specify the dimension of the ambient space of the polydisk, we will write Δ^n_r, $\Delta^n_\delta(z)$ etc.

The next theorem generalizes Cauchy formula for holomorphic functions in one variable.

Theorem 1.7 *(Multidimensional Cauchy formula). Let $f : \overline{\Delta}_r \to \mathbb{C}$ be a continuous function that is* **separately holomorphic** *on Δ_r: holomorphic in each variable $z_j \in D_{r_j}$, $j = 1, \dots, n$. (In particular, this holds for every function holomorphic on Δ_r and continuous on its closure). Then for every $z = (z_1, \dots, z_n) \in \Delta_r$ one has*

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1| = r_1} \dots \oint_{|\zeta_n| = r_n} \frac{f(\zeta)}{\prod_{j=1}^n (\zeta_j - z_j)} d\zeta_1 \dots d\zeta_n. \qquad (1.5)$$

Remark 1.8 Let $g(\zeta)$ denote the sub-integral function in the latter right-hand side. The multiple integral in (1.5) is independent of integration order

(Fubini's theorem and continuity of the function $g(\zeta)$). It is equal to the integral of the complex-valued differential n-form $g(\zeta)d\zeta_1 \wedge \cdots \wedge d\zeta_n$ on the n-torus $\mathbb{T}^n = \prod_{j=1}^n S_j^1$, $S_j^1 = \{|\zeta_j| = r_j\}$, oriented as a product of positively (i.e., counterclockwise) oriented circles. That is, an orienting basis $v_1, \dots, v_n \in T_\zeta \mathbb{T}^n$ is formed by vectors $v_j \in T_{\zeta_j} S_j^1$ oriented counterclockwise.

Proof It suffices to prove the statement of the theorem in the case, when f is holomorphic in each variable on a domain containing the closed polydisk $\overline{\Delta}_r$: the general case is reduced to it via scaling the function f to $f_\varepsilon(z) = f(\varepsilon z)$, $0 < \varepsilon < 1$ (which is holomorphic in each variable on $\overline{\Delta}_r$) and passing to the limit under the integral, as $\varepsilon \to 1$. We prove formula (1.5) by induction in n.

Induction base: for $n = 1$ this is the classical Cauchy formula for one variable.

Induction step. Let formula (1.5) be proved for the given $n = k$. Let us prove it for $n = k + 1$. For every $w = (w_1, \dots, w_k) \in \mathbb{C}^k$ set

$$f_w(t) = f(w_1, \dots, w_k, t).$$

For every fixed $z_{k+1} \in D_{r_{k+1}}$ the function $g(w_1, \dots, w_k) = f_w(z_{k+1})$ is holomorphic on $\overline{\Delta}_{(r_1,\dots,r_k)}$. Hence,

$$f(z_1, \dots, z_{k+1}) = \frac{1}{(2\pi i)^k} \oint_{|\zeta_1|=r_1} \cdots \oint_{|\zeta_k|=r_k} \frac{f_\zeta(z_{k+1})}{\prod_{j=1}^k (\zeta_j - z_j)} d\zeta_1 \dots d\zeta_k, \quad (1.6)$$

by the induction hypothesis. The function $f_\zeta(t)$ being holomorphic in $t \in \overline{D}_{r_{k+1}}$ for every $\zeta = (\zeta_1, \dots, \zeta_k)$, it is expressed by Cauchy Formula

$$f_\zeta(t) = \frac{1}{2\pi i} \oint_{|\zeta_{k+1}|=r_{k+1}} \frac{f_\zeta(\zeta_{k+1})}{\zeta_{k+1} - t} d\zeta_{k+1} \text{ for every } t \in D_{r_{k+1}}.$$

Substituting the latter formula with $t = z_{k+1}$ to (1.6) yields (1.5), by continuity and Fubini Theorem. □

Lemma 1.9 *(Osgood). Every continuous function on a domain in $\mathbb{C}^n$ that is holomorphic in each individual variable is holomorphic.*

Proof It sufficed to prove the statement of the lemma for a function continuous on a closed polydisk $\overline{\Delta}_r$. Then Multidimensional Cauchy Formula (1.5) holds, and its subintegral expression is a continuous family of rational functions in $z \in \Delta_r$. Therefore, the subintegral expressions are holomorphic on Δ_r. They are uniformly bounded and continuous together with derivatives on compact subsets in Δ_r. Therefore, the integral is C^1-smooth and its partial derivatives are equal to the integrals of partial derivatives in z of the subintegral expression (here one can differentiate the integral by the above boundedness and continuity statements). It satisfies Cauchy–Riemann equations, as do the subintegral functions, and hence, is holomorphic. The lemma is proved. □

Proposition 1.10 *Each holomorphic function is C^∞-smooth. If f is holomorphic on a polydisk Δ_r and continuous on its closure, then its derivatives are given by the formulas*

$$\frac{\partial^k f(z)}{\partial z_1^{k_1}\dots\partial z_n^{k_n}} = \frac{k_1!\dots k_n!}{(2\pi i)^n}\oint_{|\zeta_1|=r_1}\dots\oint_{|\zeta_n|=r_n}\frac{f(\zeta_1,\dots,\zeta_n)}{\prod_{j=1}^n(\zeta_j - z_j)^{k_j+1}}d\zeta_n\dots d\zeta_1. \tag{1.7}$$

Proof In the multidimensional Cauchy formula the subintegral expression a non-vanishing rational function. It is holomorphic, thus its $\frac{\partial}{\partial \bar{z}_j}$-derivatives vanish. It is differentiable infinitely many times, and its k-th derivatives, $k = (k_1,\dots,k_n)$, are equal to $\frac{k_1!\dots k_n! f(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)^{k_1+1}\dots(\zeta_n-z_n)^{k_n+1}}$. This together with Cauchy formula and uniform boundedness of every latter derivative, as $|\zeta_j| = r_j$ and z varies on a compact subset in Δ_r, implies that the corresponding derivative of the Cauchy integral is the integral of the derivative. This proves (1.7). The C^∞-smoothness statement then follows immediately. □

Theorem 1.11 *Let a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ converge uniformly on compact subsets. Then its limit is holomorphic on Ω.*

Proof Let f_m be our converging functions. Let us restrict them to a closed polydisk $\overline{\Delta} \subset \Omega$ and write multidimensional Cauchy formula for them on the polydisk Δ. For each $z \in \Delta$ its left-hand side $f_m(z)$ is a converging sequence, and so is the Cauchy integral in the right-hand side, by uniform convergence of $f_m(\zeta_1,\dots,\zeta_n)$. Therefore, the limit function satisfies the Cauchy formula as well. For every continuous function $f(\zeta_1,\dots,\zeta_n)$ the corresponding Cauchy integral is holomorphic in $z \in \Delta$. Indeed, so is the subintegral function (which is rational in z). We can permute integration in ζ and differentiation in z, since the module of the derivative of the latter rational function is uniformly bounded on each compact subset in Δ_r. In particular, the bar-derivative in z of the integral vanishes, as does that of the subintegral expression. This together with the Cauchy formula for the limit function implies holomorphicity of the limit. The theorem is proved. $\square$

Set

$$\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}.$$

Theorem 1.12 *Every function f holomorphic at $0 \in \mathbb{C}^n$ is a sum of power series converging to f uniformly on a neighborhood of 0:*

$$f(z) = \sum_{k \in \mathbb{Z}^n_{\geq 0}} c_k z^k; \ c_k \in \mathbb{C}, \ z^k = z_1^{k_1} \dots z_n^{k_n}, \ c_0 = f(0). \tag{1.8}$$

Proof Fix a $\delta > 0$ such that f is holomorphic on the closed polydisk $\overline{\Delta}_\delta = \overline{\Delta}_{(\delta,\dots,\delta)}$. Let us show that the right-hand side of the Cauchy formula written in the same polydisk is a sum of power series converging on Δ_δ. For every ζ_j and z_j with $|z_j| < \delta = |\zeta_j|$ one has

$$\frac{1}{\zeta_j - z_j} = \zeta_j^{-1} \frac{1}{1 - \frac{z_j}{\zeta_j}} = \sum_{l=0}^{+\infty} \zeta_j^{-l-1} z_j^l. \tag{1.9}$$

This series converges absolutely uniformly on every disk $|z_j| \leq \delta'$ with $\delta' < \delta$. Hence, the product of the latter series for all $j = 1, \dots, n$ also absolutely uniformly converges to $\frac{1}{\prod_j (\zeta_j - z_j)}$ on $\Delta_{\delta'}$. Substituting formulas (1.9) for all j to (1.5) together with permutability of integration and series summation (ensured by absolute uniform convergence of subintegral series and uniform boundedness of the function on $\partial\Delta$) yields (1.8) with

$$c_k = \frac{1}{(2\pi i)^n} \oint_{|\zeta_n|=\delta} \dots \oint_{|\zeta_1|=\delta} \frac{f(\zeta)}{\zeta_1^{-k_1-1} \dots \zeta_n^{-k_n-1}} d\zeta_1 \dots d\zeta_n. \tag{1.10}$$

Substituting $k = 0$ yields $c_0 = f(0)$, by (1.5). $\square$

2 Convergence of power series. Equivalent definition of holomorphic function

Here we study convergence of power series $\sum_k c_k z^k$ and present a higher-dimensional analogue of convergence radius theorem from the theory of functions of one complex variable.

Lemma 2.1 (Abel). *Consider a power series $\sum_{k\in\mathbb{Z}^n_{\geq 0}} c_k z^k$. Let its terms $c_k z^k$ at a given point $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ be uniformly bounded, set $r_j = |z_j|$, $r=(r_1,\dots,r_n)$. Let $r_j>0$ for all j. Then the series converges uniformly on compact subsets in the polydisk Δ_r.*

In the proof of the lemma and in what follows we will use the following convention.

Convention 2.2 For every $\delta, r\in\mathbb{R}^n_{\geq 0}$ we say that

$$\delta < r \ (\delta \leq r), \ \text{ if } \ \delta_j < r_j \ \text{ (respectively, } \delta_j \leq r_j) \ \text{ for every } \ j=1,\dots,n.$$

Proof of Lemma 2.1. Fix some $\delta=(\delta_1,\dots,\delta_n)$ with $\delta_j>0$, $\delta<r$. It suffices to show that $\sum |c_k|\delta^k<\infty$. Indeed, set

$$\nu_j = \frac{\delta_j}{r_j} < 1, \ C = \sup_k |c_k r^k| < +\infty.$$

Then $|c_k|\delta^k \leq C\nu^k$. But

$$\sum_k \nu^k = \prod_{j=1}^n \Big(\sum_{s=0}^{+\infty} \nu_j^s\Big) = \frac{1}{\prod_j (1-\nu_j)} < +\infty.$$

Therefore, the series $\sum_k |c_k|\delta^k$ is majorated by a converging series $C\sum_k \nu^k$, and hence, converges. The lemma is proved. □

Definition 2.3 The *convergence domain* of a power series $\sum_{k\in\mathbb{Z}^n_{\geq 0}} c_k z^k$ is the interior of the set of those points $z\in\mathbb{C}^n$ where it converges.

Consider the torus $\mathbb{T}^n = S^1\times\dots\times S^1$ identified with the product of unit circles in $\mathbb{C}$. Its points will be identified with collections $t=(t_1,\dots,t_n)$, $|t_1|=\dots=|t_n|=1$, thus, $t_j = e^{i\psi_j}$. It acts on $\mathbb{C}^n$ by coordinatewise rotations:

$$\mathbb{T}^n : \mathbb{C}^n \to \mathbb{C}^n, \ t(z_1,\dots,z_n) = (t_1 z_1,\dots,t_n z_n).$$

Corollary 2.4 *The convergence domain of a series $\sum_k c_k z^k$ is a union of polydisks centered at the origin. It is invariant under the above torus action.*

Proof Given a power series, let Ω denote its convergence domain. Given a point $z=(z_1,\dots,z_n)\in\Omega$, let us construct a polydisk $\Delta_r\subset\mathcal{D}$ containing z. For every $\lambda>1$ close enough to 1 (dependently on z) one has $w:=\lambda z\in\Omega$, by definition. Fix the above λ and w. Set $r_j = |w_j| = \lambda|z_j| > |z_j|$, $r = (r_1, \dots, r_n)$. The sequence $c_k r^k$ is uniformly bounded, by the convergence of the series $\sum_k c_k w^k$. Therefore, $\Delta_r \subset \Omega$ (Abel's Lemma 2.1) and $z \in \Delta_r$, by construction. The first statement of the corollary is proved. Its second statement follows from the first one and the invariance of each polydisk centered at 0 under the torus action. The corollary is proved. $\Box$

Proposition 2.5 *Each power series converges uniformly on compact subsets in its convergence domain* Ω.

Proof Each point $z \in \Omega$ is contained in two homothetic polydisks: a polydisk $\Delta_r \subset \Omega$ and in smaller homothetic polydisk $\Delta_{r'}$, $\overline{\Delta}_{r'} \subset \overline{\Delta}_r \subset \Delta_r$, $r' = \lambda r$, $0 < \lambda < 1$. The series converges uniformly on $\overline{\Delta}_{r'}$, since it converges at the point r and by Abel's Lemma. Every compact subset $K \Subset \Omega$ can be covered by a finite number of the above uniform convergence polydisks $\Delta_{r'}$. This implies uniform convergence on K. The proposition is proved. $\Box$

Example 2.6 The convergence domain of the series $\sum_{k\geq 0} z_1^k$ in two variables (z_1, z_2) is the cylinder $|z_1| < 1$. The convergence domain of the series $\sum z_1^{k_1} z_2^{k_2}$ is the unit bidisk $\Delta_{1,1}$. The convergence domain of the series $\sum (z_1 z_2)^k$ is the set $\{|z_1 z_2| < 1\}$.

Let us recall that the convergence radius r of a power series $\sum_k c_k z^k$ in one variable is given by the classical Cauchy-Hadamard formula $r = (\overline{\lim}_{k\to\infty} c_k^{\frac{1}{k}})^{-1}$, or equivalently,

$$\overline{\lim}_{k\to\infty} (c_k r^k)^{\frac{1}{k}} = 1.$$

The next proposition generalizes this formula to several variables. To state it, let us introduce the following notation. Consider the mapping

$$R : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}, \quad R(z) := (|z_1|, \dots, |z_n|).$$

It can be viewed as the map of the space $\mathbb{C}^n$ to its quotient $\mathbb{R}^n_{\geq 0}$ by the torus action.

Proposition 2.7 *Consider a given series $\sum_k c_k z^k$ in variable $z = (z_1, \dots, z_n)$. Let Ω denote its convergence domain. For every $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{\geq 0}$ set*

$$\phi(r) := \overline{\lim}_{k\to\infty} (|c_k| r^k)^{\frac{1}{|k|}}, \quad r^k = r_1^{k_1} \dots r_n^{k_n}. \tag{2.1}$$

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ be a point with $z_j \neq 0$ for all j.

1) One has $z \in \Omega$, if and only if $\phi(R(z)) < 1$.

2) One has $z \in \partial\Omega$, if and only if $\phi(R(z)) = 1$.

In the proof of the proposition we use following homogeneity and continuity properties of the upper limit function $\phi(r)$.

Proposition 2.8 *The function $\phi(r)$ has the two following properties:*

$$1)\ \textit{homogeneity}: \ \ \phi(\lambda r) = \lambda \phi(r) \ \textit{for every} \ \ \lambda > 0. \tag{2.2}$$

$$2)\ \textit{monotonicity}: \ \ \phi(r) \geq \phi(r') \ \textit{whenever} \ \ r \geq r'.$$

Proof The proposition follows from definition. □

Proposition 2.9 *Let a non-negative-valued function $\phi(r)$ be defined on a non-empty cone (i.e., homothety-invariant subset) $K \subset \mathbb{R}_+^n$ and have the above properties 1) and 2). Then it is continuous on K. In particular, for every series $\sum_k c_k z^k$ for which the corresponding function $\phi(r)$ is well-defined for at least one $r \in \mathbb{R}_+^n$, $\phi(r)$ is continuous on its definition domain.*

Proof Fix an $r \in K$ and a small $\varepsilon > 0$. Then the subset of those r' that satisfy the inequality

$$(1-\varepsilon)r \leq r' \leq (1+\varepsilon)r$$

contains the intersection with K of a neighborhood of the point r: the rectangular parallelepiped $\prod_{j=1}^n [r_j(1-\varepsilon), r_j(1+\varepsilon)]$. The function ϕ takes values $\varepsilon\phi(r)$-close to $\phi(r)$ there, since $\phi((1 \pm \varepsilon)r) = (1 \pm \varepsilon)\phi(r)$ (homogeneity) and by monotonicity. Therefore, it is continuous at r. □

Proof of Proposition 2.7. Let $z \in \mathbb{C}^n$ be a point with $z_j \neq 0$ for all j. Let us prove the first statement of Proposition 2.7: $z \in \Omega$ if and only if $\phi(R(z)) < 1$. Indeed, set $r = R(z)$. Let $z \in \Omega$. Then $\lambda z \in \Omega$ for some $\lambda > 1$ (openness; let us fix this λ). Hence, for every $\lambda' \in (1, \lambda)$ the series $\sum_k c_k (\lambda' z)^k$ converges, and thus, its terms are uniformly bounded in k. This implies that the upper limit of the $|k|$-th roots of its terms is no greater than 1. Thus, $\phi(r) \leq (\lambda')^{-1} < 1$. Conversely, let $\phi(r) < 1$. Fix a $\lambda > 1$ such that $\phi(r) < \lambda^{-1}$. Then $\phi(\lambda r) = \lambda\phi(r) < 1$. In other words, $\overline{\lim}_{k\to\infty}(|c_k|(\lambda r)^k)^{\frac{1}{|k|}} < 1$. This implies that the expression under limit is less than one, thus $|c_k|(\lambda r)^k < 1$, whenever $|k|$ is large enough. Therefore, $\Delta_{\lambda r} \subset \Omega$, by Abel's Lemma, and thus, $z \in \Delta_{\lambda r} \subset \Omega$. This proves Statement 1) of Proposition 2.7.

Let us prove Statement 2). Let $r = R(z)$ and let $\phi(r) = 1$. Then for every $\lambda \in (0,1)$ one has $\phi(\lambda r) < 1$, and hence, $\Delta_{\lambda r} \subset \Omega$ (Statement 1)). This implies that $z \in \overline{\Omega}$. But $z \notin \Omega$, since $\phi(R(z)) = 1$ and by Statement 1). Hence, $z \in \partial\Omega$. Conversely, let $z \in \partial\Omega$ and $R(z) \in \mathbb{R}^n_+$. Then z is the limit of a convergent sequence $w_m \to z$, $w_m \in \Omega$, $R(w_m) \in \mathbb{R}^n_+$, and $\phi(R(w_m)) < 1$, by Statement 1). Therefore, $\phi(R(z)) \leq 1$, by continuity. We know that $\phi(R(z))$ cannot be less than 1, since $z \notin \Omega$ and by Statement 1). Therefore, $\phi(R(z)) = 1$. Proposition 2.7 is proved. $\square$

Remark 2.10 All the above statements on power series remain valid for power series $\sum_k c_k(z-p)^k$ with arbitrary $p \in \mathbb{C}^n$: the convergence domain is a union of polydisks centered at p, etc.

The higher derivatives $\frac{\partial^l f}{\partial z^l}$, $\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l}$ of function of one variable and the higher derivatives

$$\frac{\partial^{k+l} f}{\partial z^k \partial \bar{z}^l} = \frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}}, \ k, l \in \mathbb{Z}_{\geq 0}$$

of a function of n complex variables are defined by subsequent differentiations. They are independent on the choice of order of differentiations (if the order of smoothness of the function is no less than the number of differentiations). This follows from the general fact that every two differential operators with constant coefficients commute.

Example 2.11 Let $f(z) = z_1^{s_1} \dots z_n^{s_n}$. Then

$$\frac{\partial^{k+l} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n} \partial \bar{z}_1^{l_1} \dots \partial \bar{z}_n^{l_n}} = 0 \text{ whenever } l \neq 0;$$

$$\frac{\partial^k f}{\partial z^k} = 0 \text{ whenever } k_j > s_j \text{ for a certain } j;$$

$$\frac{\partial^k f}{\partial z^k} = \prod_{j=1}^n \frac{s_j!}{(s_j - k_j)!} z^{s-k}, \text{ whenever } k_j \leq s_j \text{ for all } j.$$

Proposition 2.12 *Let a power series $f(z) = \sum_k c_k z^k$ have a non-empty convergence domain. Then its sum $f(z)$ is holomorphic there and*

$$c_0 = f(0), \ c_k = \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|} f}{\partial z^k}(p), \tag{2.3}$$

Proof Without loss of generality we consider that $p = 0$. The convergence domain is a union of convergence polydisks. Fix a convergence polydisk Δ_r and let us prove the above regularity statements in Δ_r. We claim that each derivative (of any order) of the series $\sum_k c_k z^k$ converges uniformly on compact subsets in Δ_r. Let $\phi(r)$, $\phi_1(r)$ denote respectively the upper limits (2.1) corresponding to the initial series and its derivative

$$\frac{\partial}{\partial z_1}(\sum_k c_k z^k) = \sum_k k_1 z_1^{-1} c_k z^k.$$

One has

$$\phi_1(r) = \overline{\lim}_{k\to\infty}((|k_1 r_1^{-1} c_k|r^k)^{\frac{1}{|k|-1}} \leq \overline{\lim}_{k\to\infty}(|c_k|r^k)^{\frac{1}{|k|-1}} = \phi(r) \leq 1.$$

Thus, the above derivative series converges uniformly on compact subsets in Δ_r, by Proposition 2.7. For higher derivatives the proof is analogous: the l-th derivation yields a new multiplier polynomial in k of fixed degree $|l|$, and its contribution to the above upper limit cancels out after taking a root of order $|k|$, as in the above inequality. This implies infinite differentiability of the function f, and each its partial derivative is equal to the sum of the corresponding derivative series. In particular, $\frac{\partial f}{\partial \bar{z}_j} = 0$, since this holds for each term of the power series. Hence, f is holomorphic. The value $\frac{\partial^{|k|} f}{\partial z^k}(0)$ is equal to the free term of the corresponding derivative series, i.e., $k_1! \dots k_n! c_k$. This proves (2.3) and the proposition. $\square$

Corollary 2.13 *A function f on a domain $V \subset \mathbb{C}^n$ is holomorphic, if and only if each point $p \in V$ has a neighborhood where f is a sum of a converging power series $\sum_k c_k(z-p)^k$. The coefficients c_k are given by formula (2.3).*

The corollary follows from the above proposition and Theorem 1.12.

3 Analytic extension. Erasing singularities. Hartogs Theorem

Theorem 3.1 *(Uniqueness of analytic extension). Every two holomorphic functions on a connected domain $\Omega \subset \mathbb{C}^n$ that are equal on an open subset coincide on all of Ω.*

Proof It is sufficient to show that if a holomorphic function f on a connected domain Ω vanishes on some open subset $V \subset \Omega$, then $f \equiv 0$ on all of

Ω. To do this, let us consider the subset

$$K = \cap_{k \in (\mathbb{Z}_{\geq 0})^n} \{ \frac{\partial^{|k|} f}{\partial z^k} = 0 \} \subset \Omega : \ K \supset V.$$

One has $f|_K \equiv 0$, since the latter intersection includes $k = 0$. The subset $K \subset \Omega$ is closed, being an infinite intersection of closed subsets, since $f \in C^\infty(\Omega)$ (Corollary 2.13). The set K is open. Indeed, at each point $p \in K$ the function f has vanishing Taylor series coefficients, by definition and formula (2.3). Hence, $f \equiv 0$ on a neighborhood of the point p, and thus, the latter neighborhood is contained in K. Therefore, K is a nonempty closed and open subset of a connected domain Ω, hence $K = \Omega$ and $f \equiv 0$ on Ω. $\square$

Proposition 3.2 (Openness Principle.) *Each non-constant holomorphic function on a connected domain is an open map: the image of each open subset is open.*

Proof Let f be a non-constant holomorphic function on a connected domain Ω. It suffices to show that for every point $z \in \Omega$ the image of arbitrary ball centered at z contains a neighborhood of the image $f(z)$. Fix a $z \in \Omega$ and a complex line L through z where $f|_L \not\equiv const$ in a neighborhood of z. The line L exists since f is locally non-constant (uniqueness of analytic extension). The restriction of the function f to a disk in $L \cap \Omega$ centered at z is an open map, being a non-constant holomorphic function of one complex variable. This implies that the image of every disk as above contains a neighborhood of the point $f(z)$, and hence, so does the image of arbitrary ball in Ω centered at z. The proposition is proved. $\square$

Corollary 3.3 (Maximum Principle.) *The module of a non-constant holomorphic function on a connected domain Ω cannot achieve its maximum in Ω.*

Proof If a module of a holomorphic function $f \not\equiv const$ achieves its maximum at a point $z \in \Omega$, then the image $f(\Omega)$ contains the point $f(z)$ but avoids the exterior of the circle through $f(z)$ centered at 0. Hence, it contains no its neighborhood, – a contradiction to Openness Principle. The corollary is proved. $\square$

Theorem 3.4 (Liouville). *Every bounded holomorphic function on all of $\mathbb{C}^n$ is constant.*

Proof The restriction of a bounded holomorphic function f to each complex line through the origin is constant, being a bounded holomorphic function on $\mathbb{C}$ (Liouville Theorem in one variable). Therefore, $f \equiv f(0)$ on $\mathbb{C}^n$. $\square$

It is known that for every domain $V \subset \mathbb{C}$ there exists a holomorphic function on V that extends analytically to no point of its boundary. This statement is false in higher dimensions. A basic counterexample, the *Hartogs Figure* is provided by the next theorem.

Theorem 3.5 *(Hartogs) Let $R = (R_1, \dots, R_n)$, $R_j > 0$, $1 \le k < n$, $r = (r_1, \dots, r_k)$, $r_s < R_s$. Set $R^k = (R_1, \dots, R_k)$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. Let $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ be an open subset. Let $z = (z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$,*

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}}, \ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n, \ \Omega = A \cup B.$$

(In the case, when $n = 2$, $k = 1$, $V = D_{r_2}$, $r_2 < R_2$, the domain Ω is the so-called **Hartogs Figure,** *see Fig.1.) Then every function holomorphic on Ω extends holomorphically to the whole polydisk $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.*

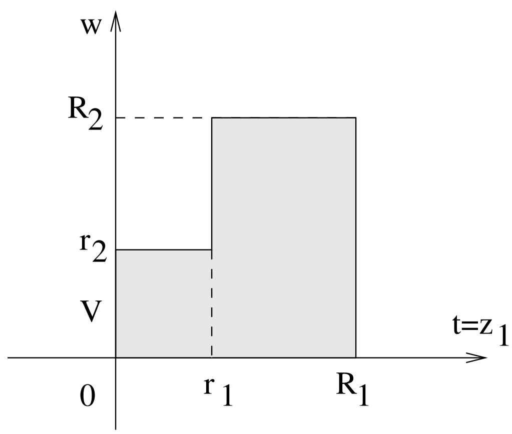

Figure 1: The Hartogs Figure for $n = 2$: picture in the positive quadrant

Proof For simplicity, let us prove the theorem in the case, when $n = 2$, $k = 1$: thus $R^k = R_1$, $R^{n-k} = R_2$, $z = (z_1, z_2)$, $t = z_1$, $w = z_2$. The proof in the general case is analogous. Let f be a function holomorphic on Ω. Fix an arbitrary $\delta \in (r_1, R_1)$. For every $w \in V$ the function $f(z_1, w)$ is holomorphic in $z_1 \in D_{R_1} \subset \mathbb{C}$, since $D_{R_1} \times \{w\} \subset B \subset \Omega$. Therefore, for every $z_1 \in D_\delta$ it is expressed as Cauchy integral

$$f(z_1, w) = \frac{1}{2\pi i} \oint_{|z_1|=\delta} \frac{f(\zeta, w)}{\zeta - z_1} d\zeta. \tag{3.1}$$

For every fixed $w \in D_{R_2}$ the subintegral function is holomorphic in $z_1 \in D_\delta$. Hence, the integral is also holomorphic in $z_1 \in D_\delta$, as in the proof of Osgood's Lemma. For every fixed $\zeta \in D_{R_1} \setminus D_\delta \supset \partial D_\delta$ the function $f(\zeta, w)$ is holomorphic in $w \in D_{R_2}$, since $\{\zeta\} \times D_{R_2} \subset A \subset \Omega$. Finally, the subintegral function is holomorphic in $(z_1, w) \in D_\delta \times D_{R_2}$, and hence, so is the integral. Thus, formula (3.1) extends the function $f(z_1, w)$ holomorphically to $D_\delta \times D_{R_2}$. This holomorphic extension is unique, by the Uniqueness Theorem for holomorphic extension. Thus, f is holomorphic there and hence, on all of $\Delta_R = D_{R_1} \times D_{R_2}$, since δ is an arbitrary number between r_1 and R_1. This proves the theorem for $n = 2$ and $k = 1$. Theorem 3.5 is proved. $\square$

Exercise 3.6 (Seminar.) Prove Theorem 3.5 in the general case using multidimensional Cauchy integral.

Theorem 3.7 *(Erasing compact singularities). Let $G \subset \mathbb{C}^n$ be an open subset, $K \Subset G$ be a compact subset. Let both G and the complement $G \setminus K$ be connected. Then every function holomorphic on $G \setminus K$ extends holomorphically to all of G.*

We prove this theorem only in the case, when the ambient domain is a polydisk. Its proof in general case is more complicated and can be done by using, e.g., Bochner–Martinelli integral formula.

Proof of Theorem 3.7 in the case, when G is a polydisk. Let us prove the theorem in the case when $n = 2$: in higher dimensions the proof is literally analogous. Let $G = \Delta_R$, $R = (R_1, R_2)$. Let K_1, K_2 denote respectively the images of the compact set K under the projections to the z_1- and z_2-axes: $K_1 \Subset D_{R_1}$, $K_2 \Subset D_{R_2}$. Fix an open subset $V \subset D_{R_2} \setminus K_2$ and a $0 < r_1 < R_1$ such that $K_1 \Subset D_{r_1}$. Let Ω be the Hartogs figure from Theorem 3.5 constructed by the chosen r_1, V and R. One has $\Omega \subset \Delta_R \setminus K$. Therefore, every function holomorphic on $\Delta_R \setminus K$ is holomorphic on Ω, and hence, extends to a function holomorphic on all of Δ_R, by Theorem 3.5. $\square$

Exercise 3.8 (Seminar). Prove that every function holomorphic on the complement of a polydisk centered at the origin to a coordinate subspace of codimension at least two extends holomorphically to the whole polydisk.

Hint. Construct an appropriate Hartogs figure in the complement to the coordinate subspace in question.

4 Implicit Function and Constant Rank Theorems. Complex manifolds. Extension theorems for functions on manifolds

4.1 Implicit Function and Constant Rank Theorems

Theorem 4.1 (Holomorphic Implicit Function Theorem) *Let $U \subset \mathbb{C}^n \times \mathbb{C}^\ell$, $(0,0) \in U$. Let $F : U \to \mathbb{C}^\ell$, $(X,Y) \mapsto F(X,Y)$ be a holomorphic map with $F(0,0) = 0$. Let the partial differential $\frac{\partial F}{\partial Y}(0) : T_0\mathbb{C}^\ell \to T_0\mathbb{C}^\ell$ be a non-degenerate linear operator. Then there exists a neighborhood $\Delta = V \times W$ of the origin in $\mathbb{C}^n \times \mathbb{C}^\ell$ such that the intersection $\Delta \cap \{F = 0\}$ is the graph $\{Y = Y(X)\}$ of a holomorphic mapping $Y : V \to W$. Its differential $dY(X_0)$ at each point X_0, set $Y_0 = Y(X_0)$, is equal to $-(\frac{\partial F}{\partial Y})^{-1}(X_0, Y_0)\frac{\partial F}{\partial X}(X_0, Y_0)dX$. That is, the latter matrix product is equal to the Jacobian matrix of the mapping $Y(X)$ at X_0.*

Proof The mapping F being considered as a real mapping of the domain $U \subset \mathbb{C}^n \times \mathbb{C}^\ell = \mathbb{R}^{2n} \times \mathbb{R}^{2\ell}$ to $\mathbb{C}^\ell = \mathbb{R}^{2\ell}$ satisfies the statement of the real Implicit Function Theorem from analysis. The above function $Y(X)$ is well-defined and C^1-smooth on some domain $V \subset \mathbb{C}^n$ containing the origin, and $Y(0) = 0$. The above formula for its derivative holds in terms of real linear operators. The derivatives of the map F in X and in Y are both $\mathbb{C}$-linear at each point $(X_0, Y_0) \in U$, by holomorphicity. Therefore, the differential $dY(X_0)$ is also $\mathbb{C}$-linear at each point $X_0 \in V$. But each C^1-smooth (vector) function on V with $\mathbb{C}$-linear differential at each point is holomorphic. Hence, $Y(X)$ is holomorphic on V. This proves the Holomorphic Implicit Function Theorem. □

Recall the following definition.

Definition 4.2 A mapping $F : U \to V$ of complex domains (manifolds) is *biholomorphic*, if it is holomorphic and has a holomorphic inverse.

Theorem 4.3 *(Holomorphic Inverse Map Theorem) Let $U \subset \mathbb{C}^n$ be a neighborhood of the origin. A holomorphic map $G : U \to \mathbb{C}^n$ with non-degenerate differential $dG(0)$ is always a biholomorphic map of some neighborhood of the origin onto an open subset in $\mathbb{C}^n$.*

Proof It suffices to apply the Implicit Function Theorem to the function $F(X, Y) = G(Y) - X$. $\square$

Remark 4.4 Each biholomorphic mapping is always a C^∞ diffeomorphism. There exist no biholomorphic mappings of domains of different dimensions, since this is true for diffeomorphisms.

Theorem 4.5 (Constant Rank Theorem). *Let $U \subset \mathbb{C}^n$ be a neighborhood of the origin. Let $F : U \to \mathbb{C}^m$ be a holomorphic map, $F(0) = 0$. Let its differential have constant rank $\ell \leq m$ on U. Then there exist neighborhoods $V \subset U$, $W \subset \mathbb{C}^m$ of the origin and biholomorphisms (coordinate changes) $g : V \to V_1 \times V_2 \subset \mathbb{C}^{n-\ell}_z \times \mathbb{C}^\ell_w$, $h : W \to W_1 \times W_2 \subset \mathbb{C}^\ell_x \times \mathbb{C}^{m-\ell}_y$ such that $F(V) \subset W$ and $h \circ F \circ g(z, w) = (w, 0)$.*

Proof The proof of this theorem repeats the classical proof of the similar theorem from calculus. It is done in two steps.

Step 1: case, when $\ell = m$. Let us split coordinates in $\mathbb{C}^n$ into two groups (z, w), $z = (z_1, \dots, z_{n-\ell})$, $w = (w_1, \dots, w_\ell)$ so that the partial differential $\frac{\partial F(0,0)}{\partial w}$ is epimorphic, i.e., invertible. Consider the auxiliary mapping $H : (z, w) \mapsto (z, F(z, w))$. It is well-defined and holomorphic on a neighborhood $V_1 \times V_2 \subset \mathbb{C}^{n-\ell} \times \mathbb{C}^\ell$ of the origin. Its differential at the origin is non-degenerate, by construction. Therefore, shrinking the above domains V_1, V_2, we get that it is a biholomorphism of the product $V_1 \times V_2$ onto its image: a neighborhood V of the origin in $\mathbb{C}^\ell$. Hence, the mapping H has a holomorphic inverse of the form $g : (z, y) \mapsto (z, G(z, y))$. By construction, $F \circ g(z, y) = y$. The theorem is proved with $h = Id$.

Step 2: case, when $\ell < m$. Let (z, w) be the above splitting of the coordinates on $\mathbb{C}^n$. Let us split the coordinates in the image space $\mathbb{C}^m$ in two groups (x, y), $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_{m-\ell})$, so that the map $\widetilde{F} := x \circ F$ has rank ℓ at the origin (and hence, on some its neighborhood). Then applying Step 1 to the map $\widetilde{F}$ we get that there exists a biholomorphic (invertible with holomorphic) map $g : V_1 \times V_2 \to \mathcal{W} \subset \mathbb{C}^n$, $g(0) = 0$, such that $\widetilde{F} \circ g(z, w) = w$. Hence, $F \circ g(z, w) = (w, \psi(z, w))$, ψ is holomorphic on a neighborhood of the origin. Shrinking V_1 and V_2, we consider that the latter neighborhood coincides with $V_1 \times V_2$. . The rank of the latter map $F \circ g$ should coincide with the rank of the map F, that is, with the dimension ℓ of the w-variable. This implies that the function $\psi(z, w)$ has zero derivative in z and hence, depends only on w. Post-composing the map $F \circ g(z, w) = (w, \psi(w))$ with the map $h : (x, y) \mapsto (x, y - \psi(x))$ yields the map $(z, w) \mapsto (w, 0)$. The Constant Rank Theorem is proved. $\square$

4.2 Complex manifolds and extension of functions

Definition 4.6 A *complex manifold* of complex dimension d is a real $2d$-dimensional manifold M admitting an atlas where all the transition functions are biholomorphic. In more detail, it is a topological space M that admits a covering by open sets U_j such that there exist homeomorphisms $H_j : U_j \to V_j \subset \mathbb{C}^d$ with the following property:

- for every two intersected open subsets U_i and U_j the transition maps $H_j \circ H_i^{-1} : H_i(U_i \cap U_j) \to H_j(U_i \cap U_j) \subset V_j$ are holomorphic (they are biholomorphic, since their inverses $H_i \circ H_j^{-1}$ are also holomorphic by definition).

Here we suppose that M has a countable basis of neighborhoods.

Definition 4.7 A function $f : M \to \mathbb{C}$ on a complex manifold M is *holomorphic* if for every j the function $f \circ H_j^{-1} : V_j \to \mathbb{C}$ is holomorphic. A holomorphic map $M \to \mathbb{C}^n$ and a holomorphic map between complex manifolds are defined analogously.

Definition 4.8 Let M be a n-dimensional complex manifold, and let $k \in \mathbb{N}$, $k \leq n$. A subset $A \subset M$ is a k-dimensional *complex (holomorphic) submanifold*, if it is closed and each point $x \in A$ has a neighborhood $U = U(x) \subset M$ that admits a biholomorphism h on a neighborhood of the origin in $\mathbb{C}^n_{z_1,\dots,z_n}$, $h(x) = 0$, such that h sends the intersection $A \cap U$ onto the intersection of the image $h(U)$ with the coordinate k-plane $\{z_{k+1} = \dots = z_n = 0\}$. The tangent space of a submanifold at its point is defined in the same way, as the tangent space of a real submanifold; in the holomorphic case under consideration the tangent space has a natural structure of a complex vector space.

Example 4.9 Let $f : M \to \mathbb{C}^{n-k}$ be a holomorphic vector function, $n = \dim M$, and let $A = \{f = 0\}$. Let 0 be not its critical value: the differential $df(x)$ at each point $x \in A$ is non-degenerate, that is, has rank $n - k$. Then A is a k-dimensional submanifold, by the Implicit Function Theorem.

Theorem 4.10 *(Erasing* codim ≥ 2 *singularities). Let M be a complex manifold, and let $A \subset M$ be a complex submanifold of codimension at least two. Then every function holomorphic on $M \setminus A$ extends holomorphically to all of M.*

Proof It suffices to show that each point $x \in A$ has a neighborhood $U = U(x) \subset M$ such that each holomorphic function $f : U \setminus A \to \mathbb{C}$ extends holomorphically to all of U. This holds for a neighborhood U that admits a biholomorphism onto a polydisk so that $A \cap U$ is sent to a coordinate subspace of codimension at least two: see Exercise 3.8. $\square$

5 Analytic sets

First in Subsection 5.1 we introduce notion of analytic subsets, present their basic properties and prove density of regular part. In Subsection 5.2 we prove that each bounded holomorphic function on a complement to a non-trivial analytic subset extends holomorphically to all its points. Then in Subsections 5.3, 5.4 we study the case of germs of local hypersurfaces, i.e., zero loci of germs of holomorphic functions, where we prove Weierstrass Preparation Theorem and factoriality of the local ring of holomorphic functions. Afterwards we will pass to the general theory of analytic subsets.

5.1 Introduction and main properties

Definition 5.1 An *analytic subset* in a complex manifold M is a subset $A \subset M$ such that each point $p \in M$ has a neighborhood $U = U(p) \subset M$ where there exists a finite collection of holomorphic functions $f_j : U \to \mathbb{C}$, $j \in J$, such that

$$A \cap U = \{f_j = 0 \mid j \in J\}.$$

Remark 5.2 Each analytic subset is closed. Any holomorphic submanifold is an analytic subset, but the converse is not true. For example, the coordinate cross $A = \{xy = 0\} \subset \mathbb{C}^2$ and the cusp curve $B = \{y^2 = x^3\} \subset \mathbb{C}^2$ are analytic subsets. But they are not submanifolds. See a brief explanation (with an exercise) below.

Definition 5.3 The *regular part* of an analytic subset $A \subset M$ is the subset A_{reg} consisting of those points $x \in A$ such that there exists a neighborhood $U = U(x) \subset M$ for which the intersection $U \cap A$ is a submanifold in U. This is an open subset in A. The complement $A_{sing} := A \setminus A_{reg}$ is a closed subset in M called the *singular part* of the set A.

Exercise 5.4 (Seminar). Let $U \subset \mathbb{C}^n$ be a domain. Consider a holomorphic function $f : U \to \mathbb{C}$. Set

$$Z_f := \{f = 0\} \subset U, \quad Z_f^o := \{x \in Z_f \mid df(x) \neq 0\}.$$

Let Z_f^o be dense in Z_f, and the complement $Z_f^s := Z_f \setminus Z_f^o$ be non-empty. Show that $Z_{f,sing} = Z_f^s$. Deduce the statements of the above remark.

Hint. Suppose to the contrary that Z_f is a local submanifold at a point $x \in Z_f^s$. Then there exists a neighborhood $W = W(x)$ and a biholomorphism H that sends W to a domain $V \subset \mathbb{C}^n$ and sends $Z_f \cap W$ to a coordinate subspace of codimension 1, say, $z_n = 0$. Then the line L through $H(x)$ parallel to the z_n-axis intersects $H(W)$ once, and so does any close line. But one can find a parallel line L' arbitrary close to L that intersects $H(W)$ at least twice: the restriction to L of the function $f \circ H^{-1}$ has zero $H(x)$ of total multiplicity bigger than one, since its differential at $H(x)$ is zero.

Proposition 5.5 *Any analytic subset A in a connected manifold M either coincides with all of M, or is nowhere dense. In the latter case its complement is dense.*

Proof The interior $Int(A)$ is obviously open. It suffices to prove that it is closed: this will imply that it is either emply, or all of M, by connectivity. Let $p \in M$ be an accumulation point of $Int(A)$. Then $p \in A$. Hence, there exists a connected neighborhood $U = U(p) \subset M$ such that the functions f_j defining the set A in U are holomorphic in U and vanish on a non-empty open subset $Int(A) \cap U$. Hence, they vanish identically, by uniqueness of analytic extension. This implies that $A \cap U = U$ and hence, $p \in Int(A)$ and $Int(A)$ is closed. The proposition is proved. □

Exercise 5.6 (Seminar). Show that in the above second case the complement $M \setminus A$ is connected.

Proposition 5.7 *A finite union of analytic subsets $A_1 \cup \dots A_k$ is analytic. A finite intersection of analytic subsets $A_1 \cap \dots \cap A_k$ is analytic.*

Proof It suffices to prove these statements for union (intersection) of two analytic subsets A_1 and A_2 of a complex manifold M.

Let us show that $A_1 \cap A_2$ is analytic. Let $x \in A_1 \cap A_2$. Let U be its neighborhood in M where each $A_j \cap U$ is defined as zero locus of a finite collection F_j of holomorphic functions. Then $A_1 \cap A_2 \cap U$ is the zero locus of the functions from the finite collection $F_1 \cup F_2$. Therefore, $A_1 \cap A_2$ is analytic.

Let us now show that $A_1 \cup A_2$ is analytic. Recall that the sets A_1 and A_2 are closed, being analytic. In the case, when they are disjoint, there is nothing to prove: each point $x \in A_1 \cup A_2$ lies only in one subset A_j, its neighborhood $U \subset M$ small enough intersects only this A_j, and thus, $U \cap (A_1 \cup A_2) = U \cap A_j$ is defined by the same collection F_j of holomorphic functions, as A_j; hence it is analytic. Let now $x \in A_1 \cap A_2$. Let $U \subset M$

be its neighborhood where each $A_j \cap U$ is zero locus of a finite collections F_j of holomorphic functions on U. Then the zero locus of the products fg, $f \in F_1$, $g \in F_2$, coincides with $U \cap (A_1 \cup A_2)$. Hence, $A_1 \cup A_2$ is analytic. The statements of the proposition are proved. $\square$

Theorem 5.8 *The regular part of an analytic subset $A \subset M$ is dense in A.*

Proof We prove Theorem 5.8 by induction in the dimension n of the ambient manifold M.

Induction base: for $n = 1$ the statement of the theorem is obvious.

Induction step. Let the statement of the theorem be proved for $n \leq k$. Let us prove it for $n = k+1$. Let $x \in A$. Let us show that A_{reg} accumulates to x. Fix some small neighborhood $V = V(x) \subset M$ and consider that x is the origin in a holomorphic chart containing V. We show that $A_{reg} \cap V \neq \emptyset$ and then we apply this statement to arbitrarily small V. Thus, we deal with A as an analytic subset in $V \subset \mathbb{C}^n$, $0 \in A$, where A is defined as the zero locus of a finite collection of functions holomorphic on V. Fix a holomorphic function $f \not\equiv 0$ on V, $f|_A \equiv 0$. There exists a (higher) partial derivative g of the function f (which may be f itself) that vanishes identically on A and such that some of its partial derivatives $\frac{\partial g}{\partial z_j}$ does not vanish identically on A. Indeed in the opposite case all the partial derivatives of the function f would vanish at 0, and hence, $f \equiv 0$, - a contradiction. Thus, $g|_A \equiv 0$, and thus, the analytic subset $\Gamma = \{g = 0\} \subset V$ contains A. The regular part of the set Γ contains the open subset $\Gamma^0 := \Gamma \cap \{\frac{\partial g}{\partial z_j} \neq 0\} \subset \Gamma$, since $\frac{\partial g}{\partial z_j}|_A \not\equiv 0$ and by the Implicit Function Theorem. The intersection $A^0 := \Gamma^0 \cap A$ is non-empty, by assumption. On the other hand, it is an analytic subset in the complex manifold Γ^0 of dimension $n-1$. Hence, its regular part A^0_{reg} is dense in $\Gamma^0 \cap A$ (and thus, non-empty), by the induction hypothesis. But A^0_{reg} is contained in A_{reg}. Indeed, for every $y \in A^0_{reg}$ and every neighborhood $W = W(y) \subset V$ such that $W \cap \Gamma \subset \Gamma^0$ and $A \cap W = A^0_{reg} \cap W$ the former intersection, which coincides with $W \cap \Gamma^0$, is a submanifold in W, and the latter intersection is a submanifold in the former. Hence, $A \cap W$ is a submanifold in W: a submanifold of a submanifold in W is obviously a submanifold in W. Finally, A_{reg} contains a non-empty subset $A^0_{reg} \subset V$. Applying the above arguments to arbitrarily small neighhborhood V we get that A_{reg} accumulates to x. Hence, A_{reg} is dense. Theorem 5.8 is proved. $\square$

5.2 Extension of holomorphic functions on complements to analytic subsets

Theorem 5.9 *Each holomorphic bounded function on a complement of a complex manifold to an analytic subset extends holomorphically to the whole ambient manifold.*

Proof It suffices to prove the local version of the theorem, for a bounded holomorphic function f on the complement of a domain $U \subset \mathbb{C}^n_{z_1,\dots,z_n}$ to the zero locus $Z_g = \{g = 0\}$ of a holomorphic function $g : U \to \mathbb{C}$. Fix an $x \in Z_g$. It suffices to show that f extends holomorphically to a neighborhood of the point x in Z_g. Passing to appropriate local chart we can and will consider that $x = 0$, set $z = (z_1, \dots, z_{n-1})$, $w = z_n$, and that g is holomorphic on the polydisk $\Delta = \Delta_r \times D_\delta \subset \mathbb{C}^{n-1}_z \times C_w$ and $g(0, w) \not\equiv 0$ in $w \in D_\delta$. Then one can find a circle $S^1 = \{|w| = s\} \subset D_\delta$ and $\sigma = (\sigma_1, \dots, \sigma_{n-1})$, $\sigma_j < r_j$, such that $\Delta_\sigma \times S^1$ is disjoint from the zero locus Z_g: it suffices to fix a $s \in (0, \delta)$ such that $g(0, w) \neq 0$ whenever $|w| = s$ and then to take σ small enough. The intersection of each disk $\{z\} \times D_\delta$ with Z_g is a discrete set of points. The function $f(z, w)$ with fixed z extends there holomorphically, being bounded (Erasing Singularity Theorem for holomorphic functions in one variable). Thus, for every $z \in \Delta_\sigma$ and every w with $|w| < s$ one has

$$f(z, w) = \frac{1}{2\pi i} \oint_{S^1} \frac{f(z, \eta)}{\eta - w} d\eta,$$

by Cauchy Formula. The subintegral expression is holomorphic in $(z, w) \in \Delta_\sigma \times D_s$. Therefore, the above Cauchy integral extends $f(z, w)$ holomorphically to the latter product, and hence, to the neighborhood $(\Delta_\sigma \times D_s) \cap Z_g$ of the point $x = 0$ in Z_g. The theorem is proved. □

5.3 The dimension and proper maps of analytic sets. Remmert Proper Mapping Theorem

Definition 5.10 The *dimension* $\dim_x A$ of an analytic set A at its regular point $x \in A_{reg}$ is the dimension at x of the submanifold $A \cap U$ in a small neighborhood $U = U(x)$. Its dimension at a point $x \in A_{sing}$ is

$$\dim_x A := \overline{\lim}_{y \in A_{reg},\ y \to x} \dim_y A.$$

The *dimension of analytic set* A is

$$\dim A := \sup_{x \in A} \dim_x A = \max_{x \in A_{reg}} \dim_x A.$$

Theorem 5.11 *(given without proof). For every analytic subset $A \subset M$ its singular part A_{sing} is an analytic subset in M of dimension strictly less than* $\dim A$.

Proposition 5.12 *The preimage of an analytic subset in a manifold N under a holomorphic mapping $M \to N$ is an analytic subset in M.*

The proposition obviously follows from definition.

Definition 5.13 A map $F : M \to N$ between topological spaces is *proper,* if the preimage of every compact subset in N is a compact subset in M.

A fundamental result of the theory is the following theorem

Theorem 5.14 (Remmert Proper Mapping Theorem). *Let M, N be complex manifolds, and let $A \subset M$ be an analytic subset. Let $F : M \to N$ be a holomorphic map whose restriction to A is proper. Then the image $F(A) \subset N$ is an analytic subset.*

Corollary 5.15 *Let M, N be complex manifolds, and let N be compact. Let $A \subset M \times N$ be an analytic subset. Then the projection of the set A to M is an analytic subset in M.*

5.4 Decomposition into irreducible components

Definition 5.16 An analytic subset $A \subset M$ of a complex manifold M is *irreducible*, if it cannot be presented as a union of two analytic subsets $A = A_1 \cup A_2$ such that $A_1, A_2 \neq A$.

Theorem 5.17 *An analytic subset is irreducible, it and only if its regular part is connected.*

Remark 5.18 The main part of the theorem is the statement saying that *irreducibility implies connectivity of regular part.* Its proof is very technical. Here we prove the easy part: the converse.

Proof of the easy part: connectivity implies irreducibility. Suppose the contrary: there exists an analytic subset A in a complex manifold M such that A_{reg} is connected but A is not irreducible: $A = A_1 \cup A_2$, $A_{1,2} \neq A$. Then the intersections $A_j \cap A_{reg}$ are analytic subsets in the connected complex manifold A_{reg}, and their union is all of A_{reg}. This is possible only in the case, when some of them, say $A_1 \cap A_{reg}$ coincides with all of A_{reg}: otherwise they are both nowhere dense closed subsets in a connected manifold A_{reg} and cannot cover it together. But then A_1 contains A, since A_1 is closed and A_{reg} is dense in A (Theorem 5.8). Therefore, $A_1 = A$. The contradiction thus obtained proves irreducibility. $\square$

Theorem 5.19 *(given without proof). Let A be an analytic subset of a complex manifold M. The closure of each connected component $A_{reg,j}$ of its regular part is an analytic subset in M, which will be denoted A_j. It is irreducible, by Theorem 5.17; it is called an* **irreducible component.** *The set A is a locally finite union $\cup_j A_j$ of its irreducible components. This is its unique decomposition as a locally finite union of irreducible analytic sets.*

Remark 5.20 Recall that a finite union of analytic subsets is analytic. This statement remains valid for infinite but locally finite unions, by the statement for finite unions and since the notion of analytic subset is local: it is locally defined as zero locus of a finite collection of holomorphic functions.

5.5 Weierstrass polynomials and Preparatory Theorem

Definition 5.21 A polynomial $P_z(w) = w^d + a_1(z)w^{n-1} + \dots + a_0(z)$ with variable coefficients depending holomorphically on $z = (z_1, \dots, z_n)$ from a neighborhood of the origin in $\mathbb{C}^n$ with $a_j(0) = 0$ is a holomorphic function in $n+1$ variables (z, w) called a *Weierstrass polynomial* in w.

Remark 5.22 For every fixed z a Weierstrass polynomial does not vanish identically in w and has the same number d of roots with multiplicity.

Definition 5.23 Let $f(z, w)$ be a germ of holomorphic function at $(0, 0)$ in $\mathbb{C}^n_z \times \mathbb{C}_w$, $f(0, 0) = 0$, that does not vanish identically on the w-axis. Let $\delta > 0$, $r = (r_1, \dots, r_n)$, $r_j > 0$ be such that the function f is holomorphic on $\Delta_r \times \overline{D}_\delta$, $f(0, w) \neq 0$ for $w \in \overline{D}_\delta \setminus \{0\}$ and $f|_{\Delta_r \times \partial D_\delta} \neq 0$. Then $\Delta := \Delta_r \times D_\delta$ is called a *Weierstrass polydisc* for the function f.

Remark 5.24 If $f(0, w) \not\equiv 0$ on a neighborhood of zero in the w-axis, then a Weierstrass polydisk always exists. In general, if f is a holomorphic function on a neighborhood of the origin in $\mathbb{C}^{n+1}$, then one can choose coordinates $(z_1, \dots, z_n, w)$ in such a way that $f(0, w) \not\equiv 0$, and hence, in the latter coordinates a Weierstrass polydisk exists.

Theorem 5.25 (Weierstrass preparatory theorem). *Let $f(z, w)$ be a holomorphic function on a neighborhood of the origin in $\mathbb{C}^{n+1} = \mathbb{C}^n_z \times \mathbb{C}_w$,*

$z = (z_1, \dots, z_n)$, with $f(0,0) = 0$ and $f(0,w) \not\equiv 0$. Let $\Delta = \Delta_r \times D_\delta$ be a Weierstrass polydisk. Then there exists a unique Weierstrass polynomial $P_z(w)$ such that on some neighborhood U of the origin one has $f(z,w) = h(z,w)P_z(w)$, $h(z,w)$ is a holomorphic function on the latter neighborhood U, $h(0,0) \neq 0$. Moreover, $P_z(w)$ is holomorphic on $\Delta_r \times \mathbb{C}_w$ and $h(z,w)$ is holomorphic and nonvanishing on $\Delta_r \times \overline{D}_\delta$.

Proof Fix a Weierstrass polydisc $\Delta = \Delta_r \times D_\delta$. Set $g_z(w) = f(z,w)$. The function g_0 has geometrically unique zero in $\overline{D}_\delta$: the origin. Let d denote its multiplicity. Then for every $z \in \Delta_r$ the function g_z has d roots with multiplicities in D_δ and does not vanish on its boundary. Let $b_1(z), \dots, b_d(z)$ denote its roots. The coefficients of the Weierstrass polynomial we are looking for are uniquely determined as the basic symmetric polynomials $\sigma_s = \sigma_s(z)$ in $b_j(z)$ up to sign. (This already proves the uniqueness.) They vanish at $z = 0$ by assumption. Let us show that they are holomorphic functions in z. Indeed, they are expressed as polynomials in the power sums $\hat{\sigma}_s(z) = \sum_j b_j^s(z)$, $s \in \mathbb{N}$. One has

$$\hat{\sigma}_s(z) = \frac{1}{2\pi i} \oint_{\partial D_\delta} \frac{\zeta^s \frac{\partial f}{\partial w}(z,\zeta)}{f(z,\zeta)} d\zeta. \tag{5.1}$$

Indeed, the latter integral is equal to the sum of residues of the subintegral expression. The nonzero residues may exist only at those ζ, where $g_z(\zeta) = f(z,\zeta) = 0$. The residue value corresponding to a root ζ of the function $g_z(w)$ of multiplicity ν is equal to $\nu\zeta^s$. Indeed, one has

$$g_z(u) = f(z,u) = c(u-\zeta)^\nu(1 + O(u-\zeta)), \text{ as } u \to \zeta;\ c \neq 0,$$

$$\frac{\partial f}{\partial w}(z,u) = c\nu(u-\zeta)^{\nu-1}(1+o(1)) + O((u-\zeta)^\nu) = \frac{\nu}{u-\zeta} f(z,u)(1+o(1)).$$

This implies that the residue at ζ is equal to $\nu\zeta^s$. This proves (5.1). The right-hand side in (5.1) is holomorphic in $z \in \Delta_r$, since the subintegral expression is holomorphic and its restriction to the integration circle is a uniformly bounded function whenever z run over arbitrary compact subset in Δ_r. Therefore, the integral and hence, the power sums $\hat{\sigma}_s(z)$ are holomorphic on Δ_r. Hence, the elementary symmetric polynomials σ_s are also holomorphic. Therefore, the function

$$P_z(w) = \prod_{j=1}^{d}(w - b_j(z)) = w^d + \sum_{s=1}^{d}(-1)^s \sigma_s(z) w^{d-s}$$

is a Weierstrass polynomial vanishing exactly on the zero set $\Gamma = \{f = 0\}$ of the function f. The ratio $h = \frac{f}{P}$ and its inverse h^{-1} are holomorphic functions on the complement $(\Delta_r \times \overline{D}_\delta) \setminus \Gamma$. Let us show that each of them extends holomorphically to Γ (say h; for h^{-1} the proof is the same): then the theorem follows immediately. For every fixed z the function $h(z, w)$ has a nonzero limit, as w tends to a root of the polynomial $P_z(w)$, since the latter root has the same multiplicity for both functions $P_z(w)$ and $g_z(w)$. Therefore, the function $h(z, w)$ is holomorphic in $w \in \overline{D}_\delta$ for every fixed $z \in \Delta_r$. Hence, it can be written as Cauchy integral

$$h(z, w) = \frac{1}{2\pi i} \oint_{|\zeta|=\delta} \frac{h(z, \zeta)}{\zeta - w} d\zeta, \ \ w \in D_\delta.$$

The subintegral expression is holomorphic in $(z, w) \in \Delta$ and uniformly bounded with derivatives and continuous on compact subsets in Δ. Therefore, the latter integral, and hence h are holomorphic there. Similarly, h^{-1} is holomorphic. Hence, h is a unity.

The uniqueness of Weierstrass polynomial satysfying the statements of the theorem follows by construction: for every $z \in \Delta_r$ this is the unique monic polynomial in w having the same roots, as $f(z, w)|_{|w|<\delta}$, and with the same multiplicities. □

5.6 Local rings. Factorization of holomorphic functions as products of irreducible ones

Definition 5.26 Let X be a topological space, $x \in X$. Two functions f and g defined on neighborhoods U_f and U_g of the point x are called *x-equivalent,* if there exists a neighborhood $W = W(x) \subset U_f \cap U_g$ where $f \equiv g$. The *germ* of a function at a point x is its x-equivalence class.

Remark 5.27 In general, two functions (e.g., smooth functions on a manifold) defining the same germ at x can be distinct. But *if two holomorphic functions on a connected manifold M have the same germ at some point, then they are identically equal on M,* by uniqueness of analytic extension. For every point $x \in M$ there is a 1-to-1 correspondence between germs at x of functions f holomorphic on some its neighborhoods U_f depending on f (i.e., functions holomorphic just at x) and germs of holomorphic functions at $0 \in \mathbb{C}^n$, $n = \dim M$, or equivalently, converging power series.

Definition 5.28 The ring of germs of holomorphic functions f at $0 \in \mathbb{C}^n$ will be called the *local ring* and denoted $\mathcal{O}_n$. Recall that a *unity* of a ring is an invertible element, i.e., an element u for which there exists an inverse u^{-1}, $uu^{-1} = 1$. Thus, a unity in $\mathcal{O}_n$ is a germ of holomorphic function that does not vanish at 0.

Remark 5.29 The Weierstrass Preparation Theorem implies that *each germ of holomorphic function $f(z, w)$ at $(0,0) \in \mathbb{C}^n_z \times \mathbb{C}_w$ with $f(0,0) = 0$ and $f(0, w) \not\equiv 0$ is the product of a Weierstrass polynomial and a unity in $\mathcal{O}_{n+1}$.*

Definition 5.30 An element of a ring is *irreducible,* if it is not a unity and cannot be presented as a product ab, where a and b are not unities. A ring is *factorial,* if each its non-zero element that is not a unity can be represented in a unique way (up to permutation and multiplication by unities) as a product of irreducible elements.

Here we prove the following theorem.

Theorem 5.31 *The local ring $\mathcal{O}_n$ is factorial.*

In the proof of Theorem 5.31 we use the Weierstrass Preparatory Theorem and the following well-known Gauss Lemma and property of Weierstrass polynomials.

Lemma 5.32 *(Gauss). Let R be a factorial ring. Then the polynomial ring $R[w]$ is also factorial.*

Proposition 5.33 *A Weierstrass polynomial $P_z(w)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $w \in \mathbb{C}$ has irreducible germ at $0 \in \mathbb{C}^{n+1}$ as an element of the local ring $\mathcal{O}_{n+1}$, if and only if it is irreducible as a polynomial over the local ring $\mathcal{O}_n$.*

In the proof of the proposition and in what follows we will use the next trivial remark.

Remark 5.34 A monic polynomial $P_z(w)$ with coefficients being holomorphic functions in z is Weierstrass (i.e., the coefficients vanish at $z = 0$), if and only if $P_0(w) = w^d$.

Proof of Proposition 5.33. A Weierstrass polynomial $P_z(w)$ with irreducible germ in $\mathcal{O}_{n+1}$ is irreducible as a polynomial over $\mathcal{O}_n$. Indeed, let, to the contrary, $P_z(w) = Q_{1,z}(w)Q_{2,z}(w)$, with $Q_1, Q_2 \in \mathcal{O}_n[w]$ being not unities. Then at least one of them, say Q_1 is a unity in $\mathcal{O}_{n+1}$, since $P_z(w)$ is irreducible in $\mathcal{O}_{n+1}$. Therefore, its restriction $Q_{1,0}(w)$ to the w-axis does vanish at 0 and is a divisor of the monomial $w^d = P_0(w)$. But the only divisors of positive degree of a monomial w^d are monomials w^k, $k \leq d$, which vanish at 0. Therefore, the polynomial $Q_{1,z}(w)$ has zero degree in w. Hence, it is a function of z; thus, it lies in $\mathcal{O}_n$ and does not vanish at 0. Thus, it is a unity in $\mathcal{O}_n$, and hence, in $\mathcal{O}_n[w]$, – a contradiction. Let now $P_z(w)$ be irreducible as a polynomial over $\mathcal{O}_n$. Let us prove that it defines an irreducible germ at $0 \in \mathbb{C}^{n+1}$. Suppose the contrary: $P_z(w) = fg$, $f(0,0) = g(0,0) = 0$. Then $f(0,w), g(0,w) \not\equiv 0$, since the same holds for their product $P_0(w)$. Therefore,

$$f(z,w) = u(z,w)P_{f,z}(w), \ g(z,w) = v(z,w)P_{g,z}(w),$$

where u and v are unities in $\mathcal{O}_{n+1}$ and $P_{f,z}(w)$, $P_{g,z}(w)$ are Weierstrass polynomials (Weierstrass Preparatory Theorem). By construction, they are monic polynomials, and for every fixed z the union of their root collections (with multiplicities added) coincides with the root collection (with multiplicities) of the polynomial $P_z(w)$. This together with the uniqueness part of the Weierstrass Preparatory Theorem implies that $P_z(w) = P_{f,z}(w)P_{g,z}(w)$, - a contradiction to the irreducibility of the polynomial $P_z(w)$ as an element of the ring $\mathcal{O}_n[w]$. The proposition is proved. $\square$

Proof of Theorem 5.31. Induction on n.

Induction base: $n = 1$. The statement is obvious: each non-identically zero germ of holomorphic function of one variable that vanishes at 0 is $h(w)w^d$, $h(0) \neq 0$; w is irreducible and h is a unity.

Induction step. Let we have shown that the ring $\mathcal{O}_n$ is factorial. Let us show that $\mathcal{O}_{n+1}$ is factorial.

Let $f \in \mathcal{O}_{n+1}$, $f(0) = 0$, $f \not\equiv 0$. Let us fix coordinates (z,w) centered at 0, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $w \in \mathbb{C}$, such that $f(0,w) \not\equiv 0$. Then up to muptiplication by unity, f is a Weierstrass polynomial $P_z(w)$. Therefore, it admits a unique factorization as a product of some of irreducible polynomials $P_{j,z}(w)$ in the ring $\mathcal{O}_n[w]$, since the latter ring is factorial (Gauss Lemma and the induction hypothesis). The highest degree coefficients of the polynomials $P_{j,z}(w)$ do not vanish at 0, since their product is equal to 1. Therefore, multiplying $P_{j,z}$ by unities in the coefficient ring $\mathcal{O}_n$, one make them monic.

Claim. *Thus normalized polynomial factors $P_{j,z}(w)$ are Weierstrass.*

Proof For $z = 0$ one has $\prod_j P_{j,0}(w) = P_0(w) = w^d$. Hence, $P_{j,0}(w) = w^{d_j}$ for some d_j, $\sum_j d_j = d$, and $P_{j,z}(w)$ are Weierstrass (Remark 5.34). $\square$

The polynomials $P_{j,z}(w)$ are irreducible as elements of the local ring $\mathcal{O}_{n+1}$, since they are irreducible as elements of the ring $\mathcal{O}_n[w]$ and by Proposition 5.33. The existence of decomposition in $\mathcal{O}_{n+1}$ of the function $P_z(w)$

as a product of irreducible ones is proved. Let us now prove uniqueness. Let $P_z(w) = \prod_j f_j(z,w)$ be some decomposition into product of irreducible factors f_j. Multiplying f_j by unities in the ring $\mathcal{O}_{n+1}$, we get Weierstrass polynomials $\hat{f}_{j,z}(w)$: the functions f_j do not vanish on the w-axis, since the same is true for the product of their powers. But then $P_z(w) = \prod_j \hat{f}_{j,z}(w)$, as in the proof of Proposition 5.33. The Weierstrass polynomials $\hat{f}_{j,z}(w)$ are irreducible as elements of the ring $\mathcal{O}_n[w]$, by the same proposition. Hence, the latter product decomposition coincides with the above one: $\hat{f}_{j,z} = P_{j,z}$ up to permutation, by factoriality of the ring $\mathcal{O}_n[w]$. The uniqueness is proved. The induction step is over, and Theorem 5.31 is proved. □

5.7 Zero locus as a ramified covering. Geometric factorization and irreducibility criterion

Definition 5.35 Let M be a topological space, $x \in M$. Two subsets $Y_1, Y_2 \subset M$ are *x-equivalent,* if there exists a neighborhood $U = U(x) \subset M$ such that $U \cap Y_1 = U \cap Y_2$. The x-equivalence class of a subset is called its *germ* at x. A germ of subset $Y \subset M$ at x is *connected* (or dense in another germ of subset $W \supset Y$) if there exists arbitrarily small neighborhood $U = U(x) \subset M$ such that $Y \cap U$ is connected (dense in $W \cap U$).

Theorem 5.36 *Let $f \in \mathcal{O}_n$ be a germ of holomorphic function,*

$$Z_f = \{f = 0\}, \quad Z_f^o := \{z \in Z_f \mid df(z) \neq 0\}.$$

(Note that $Z_f^o \subset Z_{f,reg}$.) A germ f is irreducible, if and only if the germ at 0 *of the set Z_f^o is dense in Z_f and is connected.*

The main step in the proof of Theorems 5.36 is the next theorem stating that the germ of zero locus has a covering structure. To state it, let us recall the following definition.

Definition 5.37 A *(topological) covering* is an epimorphic mapping of topological spaces $\pi : Y \to X$ such that each point $x \in X$ has a neighborhood $U = U(x)$ whose preimage $\pi^{-1}(U)$ is a disjoint union of open subsets $U_j \subset Y$ such that the projections $\pi_j : U_j \to U$ are homeomorphisms. The space X is called the *base*, and the space Y is called the *covering space over* X. If for every $x \in X$ the number of preimages $\pi^{-1}(x)$ is finite and the same for all $x \in X$, then their number is called the *degree* of the covering. (Recall that if X is path-connected, then the number of preimages $\pi^{-1}(x)$ is independent of x.) In the case, when Y and X are complex manifolds and π_j are biholomorphic, we will say that the *covering is holomorphic.*

Remark 5.38 In fact, for a topological covering between complex manifolds holomorphicity is equivalent to just holomorphicity of the projection π. This follows from the fact that *if a holomorphic map between domains $U, V \subset \mathbb{C}^n$ is a homeomorphism, then it is biholomorphic.* The proof of this statement is omitted for simplicity.

Theorem 5.39 *Let $f(z,w)$ be a holomorphic function on a neighborhood of $(0,0) \in \mathbb{C}^n_z \times \mathbb{C}_w$, $f(0,0) = 0$, $f(0,w) \not\equiv 0$. Let $\Delta = \Delta_r \times D_\delta$ be its Weierstrass polydisk,*

$$Z_f = \{f = 0\} \cap \Delta.$$

1) There exists an analytic subset $A \subset \Delta_r$, $A \neq \Delta_r$, set

$$V := \Delta_r \setminus A, \; \widetilde{Z}_f := Z_f \cap (V \times \mathbb{C}), \tag{5.2}$$

such that $\widetilde{Z}_f \subset V \times \mathbb{C}$ is a submanifold and the projection $\pi : \widetilde{Z}_f \to V$ is a holomorphic covering. The degree of this covering is no greated than the order of zero at 0 of the function $f(0,w)$.

2) Let $\Gamma^o_1, \dots, \Gamma^o_m$ denote the connected components of the covering manifold $\widetilde{Z}_f$. For every component Γ^o_j all its points (z_j, w_j) correspond to zeros w_j of the function $f(z_j, w)$ of the same multiplicity μ_j.

3) Let d_j denote the degree of the component Γ^o_j as a covering $\pi : \Gamma^o_j \to \Delta_r \setminus A$. For every $z \in \Delta \setminus A$ let $w_{j1}(z), \dots, w_{jd_j}(z)$ denote the w-coordinates of the π-preimages of the point z lying in Γ^o_j. The function

$$P_{j,z}(w) = \prod_{s=1}^{d_j} (w - w_{js}(z)) \tag{5.3}$$

is a Weierstrass polynomial holomorphic on Δ. It is irreducible in the ring of functions holomorphic on Δ, and its zero locus is the closure $\Gamma_j := \overline{\Gamma}^o_j$.

4) There exists a function $h(z,w)$ holomorphic and nowhere vanishing on Δ such that

$$f(z,w) = h(z,w) \prod_{j=1}^{m} P^{\mu_j}_{j,z}(w).$$

Proof Without loss of generality we consider that the function f is a Weierstrass polynomial $P_z(w)$ of degree d. Set

$$k_z := \text{ the number of distinct roots of the polynomial } P_z(w); \quad k := \max_{z \in \Delta_r} k_z,$$

$$A := \{z \in \Delta_r \mid k_z < k\}. \tag{5.4}$$

Proposition 5.40 *The subset $A \subset \Delta_r$ is analytic.*

Proof A polynomial P of degree d has ℓ distinct roots, if and only if it has $d-\ell$ common roots with its derivative. (Exercise: prove this.) The set A is thus exactly the set of those points $z \in \Delta_r$ for which the polynomial $P_z(w)$ and its derivative $P_z'(w)$ in w have at least $d-k+1$ common roots with multiplicities. Or equivalently, if for this fixed individual z they have a common polynomial divisor $h(w)$ of degree at least $d-k+1$. This holds if and only if there exist polynomials $R(w)$ and $S(w)$,

$\deg R = \deg P_z' - \deg h \leq d-1-d+k-1 = k-2$, $\deg S = \deg P_z - \deg h \leq k-1$,

$$R(w)P_z(w) - S(w)P_z'(w) = 0. \tag{5.5}$$

Namely, equation in (5.5) always holds for $R = P_z'h^{-1}$, $S = P_zh^{-1}$, where h is a higher degree common divisor of the polynomials P_z and P_z'. Conversely, if polynomials R and S as in (5.5) exist and are normalized to be coprime, then R is a divisor of P_z', S is a divisor of P_z, and $h = R^{-1}P_z' = S^{-1}P_z$ is the higher common divisor of degree at least $d-k+1$ of the polynomials P_z and P_z'. The above R and S exist, if and only if the system of $2k-1$ polynomials

$$P_z,\ wP_z, \dots, w^{k-2}P_z,\quad P_z',\ wP_z', \dots w^{k-1}P_z'$$

is linearly dependent, or equivalently, their coefficient matrix has rank at most $2k-2$. This linear dependence condition (vanishing of all the highest size minors of the above matrix) is a system of polynomial equations on the coefficients of the polynomial P_z. This yields a system of holomorphic equations on $z \in \Delta_r$ and proves analyticity of the subset A. $\square$

Proposition 5.41 *Set $V = \Delta_r \setminus A$, $\widetilde{Z}_f = Z_f \cap (V \times \mathbb{C})$. The subset $\widetilde{Z}_f$ is a complex submanifold in $V \times \mathbb{C}$, and the projection $\pi : \widetilde{Z}_f \to V$ is a holomorphic covering.*

Proof The projection $\pi : \widetilde{Z}_f \to V$ is a topological covering, which follows from definition: the roots $\zeta_j(z)$ of the Weierstrass polynomial do not bifurcate at $z \in V$ and are local well-defined and continuous functions of $z \in V$, by the definition of the set V. It suffices to show that each root $\zeta(z) = \zeta_j(z)$ depends locally holomorphically on $z \in V$: each point $x \in V$ has a neighborhood $W = W(x) \subset V$ where $\zeta(z)$ is holomorphic. Indeed, fix a point $(x, \zeta(x)) \in \widetilde{Z}_f$ and consider a Weierstrass polydisk

$\Delta^x = \Omega_1 \times \Omega_2 \subset V \times \mathbb{C}$ centered at x for the function $P_z(w)$. The Weierstrass polynomial corresponding to the function $P_z(w)|_{\Delta^x}$ has the zero locus $\widetilde{Z}_f \cap \Delta^x$ and has one root $\zeta(z)$ for every $z \in \Omega_1$. Hence, it has the form $(w - \zeta(z))^k = w^k - k\zeta(z)w^{k-1} + \ldots$. Thus, $-k\zeta(z)$ is its coefficient, and hence, is holomorphic. The proposition is proved. $\square$

The degree of the covering $\widetilde{Z}_f$ is equal to the number of distinct roots $\zeta_j(z)$ for $z \in V$, which is no greater than the degree of the Weierstrass polynomial $P_z(w)$. This together with Propositions 5.40 and 5.41 imply Statement 1) of Theorem 5.39. Its Statement 2) on multiplicity follows from the last argument in the proof of Proposition 5.41. The function $P_{j,z}(w)$ from its Statement 3) is clearly holomorphic and bounded on the complement $\Delta \setminus \pi^{-1}(A)$ to an analytic subset $\pi^{-1}(A)$. Hence, it extends holomorphically to all of Δ. It is a Weierstrass polynomial, by construction and since all the roots $w_{js}(z)$ of the Weierstrass polynomial $P_z(w)$ converge to zero, as $z \to 0$. Its zero locus is Γ_j, by construction and continuous dependence of its roots on z. Let us prove its irreducibility in Δ. Suppose the contrary: it is a product of two functions f_1, f_2, each vanishes somewhere in Δ. Then the intersection of zero locus of each f_ℓ with Γ_j^o is an analytic subset A_ℓ in the connected manifold Γ_j^o. Hence, some of them, say A_1, coincides with Γ_j^o, and $f_1 \equiv 0$ on Γ_j. Replacing f_1 by its Weierstrass polynomial, we get that $f_1 = P_{j,z}(w)$. Hence, $f_2 \equiv 1$, - a contradiction. Thus, $P_{j,z}(w)$ is irreducible in Δ. Statement 3) is proved. Statement 4) follows from construction and Weierstrass Preparation Theorem. This proves Theorem 5.39. $\square$

Proof of Theorem 5.36. Let $f(z, w)$ be a germ at $(0, 0)$ of holomorphic function in $(z, w) \in \mathbb{C}_z^n \times \mathbb{C}_w$, $f(0, 0) = 0$. We choose coordinates (z, w) so that $f(0, w) \not\equiv 0$. Without loss of generality we can and will consider that f is a Weierstrass polynomial. Let $\Delta = \Delta_r \times D_\delta$ be its Weierstrass polydisk. The decomposition (5.3) from Theorem 5.39 implies that if f is irreducible, then $\widetilde{Z}_f$ consists of only one connected component, i.e., $\widetilde{Z}_f = \Gamma_1^o$, whose points $(z_0, w_s(z_0))$ correspond to roots w_s of multiplicity one of the Weierstrass polynomial f. This implies that $df(z_0, w_s(z_0)) \neq 0$. Hence, $\widetilde{Z}_f \subset Z_f^o$ is an open dense connected subset in a complex manifold Z_f^o: the complement to the analytic subset $Z_f^o \cap \pi^{-1}(A)$. Thus, Z_f^o is also connected. Conversely, connectivity of the germ of the set Z_f^o is equivalent to connectivity of its intersection with a Weierstrass polydisk $\Delta_r \times D_\delta$ with r small enough, which in its turn equivalent to connectivity of the covering $\widetilde{Z}_f$ in the same polydisk and simplicity of the roots of the Weierstrass polynomial f corresponding to its points. This together with Theorem 5.39 implies irreducibility of the function f in the same (small) polydisk. Theorem 5.36 is proved. $\square$

5.8 Zero loci of functions of two variables and Newton diagrams

5.8.1 Zero loci of irreducible germs as parametrized curves

Below we consider germs of functions in two complex variables and show that the zero locus of an irreducible germ is a parametrized curve, see the following definition.

Definition 5.42 A germ of planar *parametrized curve* is the germ at $0 \in \mathbb{C}^2_{z,w}$ of image of a bijective holomorphic map $t \mapsto (\phi(t), \psi(t))$, $t \in D_\varepsilon \subset \mathbb{C}$.

Remark 5.43 In the case, when the parametrized curve does not lie in the w-axis, a conformal change of parameter transforms its parametrization to a one of the form either $t \mapsto (t, 0)$, or

$$t \mapsto (t^q, ct^p(1 + O(t)), \quad c \in \mathbb{C} \setminus \{0\}. \tag{5.6}$$

Theorem 5.44 *A germ of function $f(z, w)$ in $\mathcal{O}_2$ is irreducible, if and only if its zero locus Z_f is a germ of parametrized curve, and $df(x) \not\equiv 0$ in $x \in Z_f$ (or equivalently, $df(x) \neq 0$ for $x \in Z_f \setminus \{0\}$).*

Using Theorem 5.44, we will later present an explicit necessary irreducibility condition for a function of two variables in terms of a finite part of its Taylor terms: the so-called Newton diagram.

In the proof of Theorem 5.44 we use the following corollary of Theorem 5.39 in the case of two variables.

Corollary 5.45 *In Theorem 5.39 in the case of two variables choosing a Weierstrass polydisk, which is a bidisk $\Delta = D_r \times D_\delta$, with r small enough, one can achieve that either $A = \emptyset$, or $A = \{0\}$. In the former case the set $Z_f = \widetilde{Z}_f$ is a one-dimensional submanifold in $\Delta_r \times \mathbb{C}$ bijectively projected onto Δ_r. In the latter case the set $\widetilde{Z}_f$ is a one-dimensional submanifold in the complement of the ambient Weierstrass polydisk to the w-axis; each its connected component is a finite degree covering over the punctured disk $\dot{D}_r = D_r \setminus \{0\}$.*

Before starting the proof of Theorem 5.44 let us recall the following background material on coverings.

Theorem 5.46 (Covering Homotopy Theorem). *Let $\pi : M \to B$ be a covering. Then for every path $\alpha : [0, 1] \to B$ and every $x_0 \in \pi^{-1}(\alpha(0))$*

there exists a unique path $\widetilde{\alpha} : [0,1] \to M$ starting at x_0 and projected to α: $\pi \circ \widetilde{\alpha}(t) = \alpha(t)$. It is called the **lifting** *of the path α starting at x_0.*

The next proposition deals with coverings over a punctured disk $\dot{D}_r \subset \mathbb{C}$. It is well-known that its fundamental group $\pi_1(\dot{D}_r, z_0)$, $z_0 \in \dot{D}_r$, is isomorphic to $\mathbb{Z}$ and generated by the homotopy class $[\alpha]$ of a path α in $\dot{D}_r$ starting at z_0 and going counterclockwise around the origin along a circle centered at 0. For every covering $\pi : M \to \dot{D}_r$ of a finite degree d the fiber $\pi^{-1}(z_0) \in M$ consists of d points $x_1, \dots, x_d$. The fundamental group of the base acts on the fiber by permutations (thus, elements of the group S_d) as follows. For every $x_j \in \pi^{-1}(z_0)$ let $\widetilde{\alpha}_j : [0,1] \to M$ denote the lifting of the path α that starts at x_j. Its endpoint $\widetilde{\alpha}_j(1)$ lies in $\pi^{-1}(z_0)$, since the path α is closed, and hence, coincides with some point $x_{\sigma(j)}$; here $\sigma \in S_d$ is a permutation of the indices $1, \dots, d$.

Proposition 5.47 *The covering M is connected, if and only if the above permutation $\sigma \in S_d$ given by the action of the element $[\alpha]$ on $\pi^{-1}(z_0)$ is a cyclic permutation. Or equivalently, if and only if its action is transitive: for every $x_j, x_k \in \pi^{-1}(z_0)$ there exists an $m \in \mathbb{Z}$ such that $\sigma^m(x_j) = x_k$.*

Proof Connectivity is clearly equivalent to the statement that every two points $x_j, x_k \in \pi^{-1}(z_0)$ can be connected by a path in M. The projection of the latter path to the base $\dot{D}_r$ is homotopic to a power α^m. Then $\sigma^m(x_j) = x_k$, by definition. Conversely, transitivity implies connectivity, by the above argument. A permutation is transitive, if and only if it is cyclic. This proves the proposition. □

Proposition 5.48 *Every connected holomorphic covering $\pi : M \to \dot{D}_r$ of degree d is isomorphic to the standard degree d power covering $\pi_{st,d} : \dot{D}_{r^{\frac{1}{d}}} \to \dot{D}_r$, $\pi_{st,d}(t) = t^d$. This means that there exists a biholomorphism $\psi : \dot{D}_{r^{\frac{1}{d}}} \to M$ that forms a commutative diagram with the projections: $\pi \circ \psi = \pi_{st,d}$.*

Proof Fix a point $x_0 \in \pi_{st,d}^{-1}(z_0) \subset \dot{D}_{r^{\frac{1}{d}}}$ and a point $y_0 \in \pi^{-1}(z_0) \subset M$. We set $\psi(x_0) = y_0$. Let us extend ψ along paths. Namely, for every $x \in \dot{D}_{r^{\frac{1}{d}}}$ we take an arbitrary path $\widetilde{\beta}_1 : [0,1] \to \dot{D}_{r^{\frac{1}{d}}}$ going from x_0 to x. Let β denote its projection to $\dot{D}_r$. Let $\widetilde{\beta}_2 : [0,1] \to M$ denote the lifting to M of the path β that starts at y_0. Set $\psi(x) = y := \widetilde{\beta}_2(1)$. The map thus constructed is locally biholomorphic. It remains to show that it is well-defined together with its inverse ψ^{-1}. We will check well-definedness (independence of the path $\widetilde{\beta}_1$) of the map ψ: the same statement for its inverse is proved analogously. It suffices to show that thus constructed map ψ sends a closed path $\widetilde{\beta}$ with base point at x_0 to a closed path. Indeed, consider an arbitrary path $\widetilde{\beta}$ in a connected covering over $\dot{D}_r$ with endpoints lying in the fiber $\pi^{-1}(z_0)$. The path $\widetilde{\beta}$ is closed, if and only if its projection $\beta := \pi \circ \widetilde{\beta}$ is homotopic to α^{dm} for some $m \in \mathbb{Z}$. This follows from Proposition 5.47. Let the above path $\widetilde{\beta}$ lie in the standard covering $\dot{D}_{r^{\frac{1}{d}}}$ and be closed. Then β is homotopic to α^{dm}. The path $\psi(\widetilde{\beta}(t))$ in M is the lifting of the same path β, as $\widetilde{\beta}$. Therefore, it is also closed, as is $\widetilde{\beta}$, by the above closeness criterion. The proposition is proved. □

Proof of Theorem 5.44. Let $f(z,w)$ be a germ of holomorphic function of two variables, $f(0,0) = 0$, $f \not\equiv 0$. We choose coordinates (z,w) so that $f(0,w) \not\equiv 0$. Let $\Delta = D_r \times D_\delta$ be its Weierstrass polydisk chosen as in Corollary 5.45. Then either $A = \emptyset$, or $A = \{0\}$. Without loss of generality we consider that f is a Weierstrass polynomial, dividing it by a nonvanishing holomorphic function.

The function f is irreducible in Δ, if and only if the corresponding covering $\widetilde{Z}_f$ over D_r (or over $\dot{D}_r$) is connected and its points correspond to simple zeros of the Weierstrass polynomial f as a function of w (Theorem 5.39). In both cases we will restrict our covering to $\dot{D}_r$. If the covering is connected, then it is bijectively parametrized by a punctured disk $\dot{D}_{r^{\frac{1}{d}}}$: the parametrization realizes its covering isomorphism with the standard covering $t \mapsto t^d$ (Proposition 5.48). The latter parametrization being a bounded holomorphic function, it extends to the puncture and hence, sends 0 to 0 and yields a bijective parametrization $D_{r^{\frac{1}{d}}} \to Z_f := \{f = 0\} \cap \Delta$. Simplicity of zeros implies that $df(x) \neq 0$ for every x lying in the open and dense subset $\widetilde{Z}_f \subset Z_f$. Thus, we have proved that if f is irreducible in a Weierstrass bidisk Δ chosen as in Corollary 5.45, then its zero locus is a parametrized curve, and $df(x) \neq 0$ for $x \in Z_f \setminus \{0\}$.

Let us prove the converse. Fix a polydisk Δ as in Corollary 5.45. Let Z_f be a parametrized curve, and let $df(x) \neq 0$ for $x \in Z_f \setminus \{0\}$. Then $\widetilde{Z}_f$ is clearly connected and its points correspond to simple zeros of the Weierstrass polynomial f. Hence, f is irreducible in Δ, by Theorem 5.39.

Passing to a smaller Weierstrass bidisk does not change connectivity of covering, by Proposition 5.47. Parametrization of smaller covering is obtained by restriction of parametrization of the initial, bigger covering, to a smaller disk. This proves Theorem 5.44. □

Theorem 5.49 *Every germ of parametrized curve in* $(\mathbb{C}^2, 0)$ *is zero locus of an irreducible germ* $f \in \mathcal{O}_2$.

Proof The case, when the curve is a line is obvious: it is zero locus of a linear function. Let now a non-linear parametrized curve be given by a germ at 0 of injective holomorphic map $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $\gamma(0) = 0$. Then $\gamma_1(t) = ct^d(1 + O(t))$, as $t \to 0$; $d \in \mathbb{N}$, $c \neq 0$. Therefore, there exists a holomorphic function $\tau(t)$, $\tau(0) = 0$, $\tau'(0) \neq 0$, such that $\gamma_1(t) = (\tau(t))^d$. Let us take τ as a new parameter. Then the curve takes the form $\gamma(\tau) = (\tau^d, \zeta(\tau))$, $\zeta(\tau)$ is a holomorphic function on a disk $D_r \subset \mathbb{C}$, $\zeta(0) = 0$. For every $(z, w) \in \dot{D}_r \times \mathbb{C}$ set

$$f(z, w) := \prod_{u^d = z} (w - \zeta(u)). \tag{5.7}$$

The function $f(z, w)$ is holomorphic on the complement of the product $D_r \times \mathbb{C}$ to the analytic subset $\{0\} \times \mathbb{C}$ and bounded on its intersection with each ball in $\mathbb{C}^2$. Therefore, $f(z, w)$ extends analytically to all of $D_r \times \mathbb{C}$ so that $f(0, 0) = 0$ (Theorem 5.9). It is a Weierstrass polynomial whose zero locus is the intersection of the curve γ with $D_r \times \mathbb{C}$. Its points different from the origin correspond to simple zeros of the function $f(z, w)$ as a function of w. Both latter statements follow from construction. Therefore, the function $f(z, w)$ is irreducible in $\mathcal{O}_2$, by Theorem 5.44. Theorem 5.49 is proved. $\square$

5.8.2 Newton diagrams and irreducible factors

Here we present a partial description of irreducible factors of a holomorphic germ in $\mathcal{O}_2$ in terms of its Newton diagram, see the next definition. As a corollary, we prove lower bound of the number of essentially distinct irreducible factors by the number of edges in the Newton diagram.

Definition 5.50 Let $f(z, w)$ be a germ of holomorphic function at the origin, $f(0, 0) = 0$. To each monomial $a_{mn} z^m w^n$ entering its Taylor series with non-zero coefficient a_{mn} we put into correspondence the positive quadrant with vertex (m, n): the quadrant $K_{m,n} = \mathbb{R}^2_{\geq 0} + (m, n) \subset \mathbb{R}^2_{\geq 0}$. Set

$$K = K_f = \text{ the convex hull of } \cup_{a_{mn} \neq 0} K_{m,n}.$$

Remark 5.51 It follows from definition that the above convex hull can be taken as a convex hull of a finite union of appropriate quadrants $K_{m,n}$, and the number of edges of its boundary is always finite. There are exactly two semi-infinite edges; each of them either lies in a coordinate axis, or is parallel to an axis; one edge per axis.

Definition 5.52 The *Newton diagram* $\mathcal{N}_f$ of a germ $f(z,w)$ is the complement of the boundary ∂K_f to the coordinate axes. More precisely, we consider that all the boundary vertices of the diagram $\mathcal{N}_f$ are contained in $\mathcal{N}_f$, even those of them that lie in some of the axes.

Example 5.53 The Newton diagram of the function z consists of one edge: the vertical semi-interval $\{1\}\times[0,+\infty)$. The Newton diagram of the function zw consists of two edges: the boundary edges of the quadrant $K_{1,1}$, which are parallel to the coordinate axes. The Newton diagram of a homogeneous polynomial $a_0z^d+\dots+a_dw^d$ with $a_0, a_d\neq 0$ consists of one edge: the segment $[(d,0),(0,d)]$. The Newton diagram of the cusp polynomial w^2-z^3 consists of one edge: the segment $[(0,2),(3,0)]$. The Newton diagram of the polynomial $z^3+w^3+zw+zw^3$ consists of two edges: $[(3,0),(1,1)]$ and $[(1,1),(0,3)]$.

Remark 5.54 Multiplication of the germ f by unity in the local ring of germs does not change the Newton diagram. But (even a linear) change of variables may change the Newton diagram and even the number of its edges: the polynomial zw has two-edge Newton diagram in the coordinates (z,w) and one-edge diagram in the coordinates (x,y), $x=z-w$, $y=z+w$.

Theorem 5.55 *1) The Newton diagram of an irreducible germ of holomorphic function f at $0\in\mathbb{C}^2$, $f(0)=0$, consists of one edge.*

2) Let f be irreducible, $f(z,0), f(0,w)\not\equiv 0$, and let

$$\gamma: t\mapsto (t^q,\zeta(t)),\quad \zeta(t)=ct^p(1+O(t))),\quad q,p\in\mathbb{N},\ c\neq 0, \tag{5.8}$$

be a germ of injective parametrization of its zero locus. Then its Newton diagram is the edge $[(p,0),(0,q)]$.

Proof If an irreducible germ f of holomorphic function vanishes on the w-axis, then the w-axis is exactly its zero locus, and up to multiplication by unity, $f=z$. Thus, its Newton diagram consists of one edge: the vertical edge $1\times\mathbb{R}_{\geq 0}$. The case, when $f(z,0)\equiv 0$, is treated analogously. Let now $f(0,w)\not\equiv 0$, $f(z,0)\not\equiv 0$. Without loss of generality we consider that f is a Weierstrass polynomial. Then

$$f(z,w)=\prod_{u^q=z}(w-\zeta(u))=\prod_{u^q=z}(w-cu^p(1+o(1))=w^q+\sum_{k=1}^{q-1}a_k(z)w^{q-k}+a_0(z),$$

$$a_k(z)=O(z^{\frac{kp}{q}}),\ a_0(z)=c'z^p(1+o(1)),\ c'\neq 0,$$

see (5.7). Thus, its monomials with non-zero Taylor coefficients have bidegrees (m,n) with $n \leq q$ and $m \geq \frac{(q-n)p}{q}$. Or equivalently, the latter bidegrees satisfy the inequality $mq + np \geq pq$. The latter inequality is equivalent to the statement that (m,n) lies either in the edge $[(p,0),(0,q)]$, or above it. Taking into account that the vertices of the latter edge correspond to monomials z^p, w^q entering f with non-zero coefficients, we get that the Newton diagram coincides with the edge $[(p,0),(0,q)]$. Theorem 5.55 is proved. $\Box$

Theorem 5.56 *Consider an arbitrary germ of holomorphic function $f(z,w)$ and its factorization as a product of powers of its irreducible factors. To each edge E of its Newton diagram $\mathcal{N}_f$ corresponds at least one irreducible factor, whose Newton diagram consists of an edge parallel to E. And conversely, each edge of the Newton diagram of any irreducible factor is parallel to an edge of the Newton diagram $\mathcal{N}_f$.*

Corollary 5.57 *The number of distinct irreducible factors of any germ $f \in \mathcal{O}_2$ is no less than the number of edges in its Newton diagram $\mathcal{N}_f$.*

The proof Theorem 5.56 is based on the following proposition.

Proposition 5.58 *Let f and g be two non identically zero germs of holomorphic functions at $0 \in \mathbb{C}^2$. Then*

$$K_{fg} = K_f + K_g = \{u + v \mid u \in K_f,\ v \in K_g\}.$$

For every edge E of the Newton diagram of either f, or g there exists a unique edge parallel to E of the Newton diagram of the product fg. Vice versa, each edge of the Newton diagram of the product fg is parallel to an edge of Newton diagram of either f, or g.

Proof We will be dealing with *(may be infinite) convex polygons* in $\mathbb{R}^2$: (may be unbounded) convex subsets whose boundary is a locally finite union of adjacent segments. Basic examples are the subsets K_f corresponding to germs $f \in \mathcal{O}_2$.

Recall that a *vertex* of a convex polygon $K \subset \mathbb{R}^2$ that is a union of positive quadrants is a point of its boundary that is contained in no straightline interval lying in K. For every vertex $A \in \partial K$ there exists an oriented line L through A such that $K \cap L = \{A\}$ and $K \setminus \{A\}$ lies on the positive side from the line L. The latter line L has always negative slope. A half-plane bounded by L is called *positive,* it it contains a positive quadrant $(a,b)+\mathbb{R}^2_{\geq 0}$.

The fact that the sum of convex polygons is a convex polygon obviously follows from definition. Let us describe the vertices of the sum. To do this, let us introduce the following definition.

Definition 5.59 Let $K_1, K_2 \subset \mathbb{R}^2$ be two (may be infinite) polygons. Two vertices $A_1 \in K_1$ and $A_2 \in K_2$ are called *compatible,* if there exists a pair of co-oriented parallel lines L_1 and L_2 with negative slope such that $L_j \cap K_j = \{A_j\}$ and K_j lies on the positive side from the line L_j.

Remark 5.60 For a given vertex $A_1 \in K_1$ there may exist several compatible vertices $A_2 \in K_2$. For example, this is the case, if K_1 is a triangle and K_2 is a quadrilateral.

Proposition 5.61 *The vertices of the sum $K_1 + K_2$ of two convex polygons in $\mathbb{R}^2$ are the sums $A_1 + A_2$ through all the pairs of compatible vertices (A_1, A_2) of the polygons K_1 and K_2 respectively. Each edge of the sum $K = K_1 + K_2$ is parallel to an edge of either K_1, or K_2. Conversely, each edge of either K_1, or K_2 is parallel to an edge of their sum K.*

Proof The fact that the sums of compatible vertices A_1, A_2 are vertices of the sum $K_1 + K_2$ follows from the fact that the sum $K_1 + K_2$ lies in the positive half-plane with respect to the line $L = L_1 + L_2$ parallel to L_1, L_2 and touches L exactly at the point $A_1 + A_2$; the latter statement obviously follows from definition. Vice versa, it is obvious that vertices of the sum $K_1 + K_2$ should be sums of some vertices $A_1 + A_2$ of the polygons K_1 and K_2. Let us show that if $A = A_1 + A_2$ is a vertex and L is an oriented line that touches K exactly at A (so that $K \setminus \{A\}$ lies on its positive side), then A_1 and A_2 are compatible with respect to the line L. Indeed, let L_j denote the lines through A_j that are parallel to L. In the contrary case, one of the vertices A_j, say A_1 is adjacent to an edge E_1 that does not lie in the positive half-plane with respect to the line L_1. It is clear that the sum $E_1 + A_2 \subset K$ does not lie in the positive half-plane with respect to the oriented line L, - a contradiction with the assumption that the complement $K \setminus \{A\}$ lies in the latter half-plane. The statement of the proposition on vertices is proved. If a vertex A_1 is adjacent to an edge E_1, and A_2 is a vertex of the body K_2 compatible with A_1, then the sum $E_1 + A_2$ is an edge of the sum K, and all the edges of the body K are obtained in this way (and also with interchanged K_1 and K_2). This follows from the above argument and definition and finishes the proof of the proposition. $\square$

Now for the proof of Proposition 5.58 it remains to prove the equality $K_{fg} = K_f + K_g$: its second and third statements will then follow from

Proposition 5.61. Indeed, it follows from definition that $K_{fg} \subset K_f + K_g$. To show that the two latter polygons coincide, it suffices to show that K_{fg} contains the vertices of the sum $K_f + K_g$: the sums $A = A_f + A_g$ of compatible vertices $A_f \in K_f$ and $A_g \in K_g$. Fix a pair of compatible vertices A_f and A_g. Let L be the corresponding line that touches K exactly at A. We have to show that the product fg contains the monomial x^A, $x = (z, w)$, in the multi-index notation. The functions $h = f, g$ contain the monomials x^{A_h}, and the product of the latter monomials is x^A. We have to show that it does not cancel out with other terms of the product fg: with other products of Taylor monomials in f and g different from the above pair (x^{A_f}, x^{A_g}). But the multidegrees of the other products, which are represented by points in $K = K_f + K_g$, do not coincide with A: moreover, they lie in the positive half-plane with respect to the line L, as does $K \setminus \{A\}$, see the proof of Proposition 5.61. Therefore, the monomial x^A does not cancel out and hence, $A \in K_{fg}$. This proves Proposition 5.58. □

Proof of Theorem 5.56. Let $f = \prod_{j=1}^{N} f_j^{r_j}$ be a germ of holomorphic function represented as a product of powers of its irreducible factors. Then

$$K_f = \sum_j r_j K_{f_j}.$$

Each edge of the boundary of the latter sum is parallel to an edge of some of ∂K_{f_j} and vice versa: for every j each edge in ∂K_{f_j} is parallel to an edge in ∂K_f. Both statements follow from Proposition 5.58 by induction in the number of irreducible factors. This proves Theorem 5.56. □

6 Generalized Maximum Principle. Automorphisms

6.1 Generalized Maximum Principle and Schwarz Lemma

We will be dealing with norms $|| \; ||$ on $\mathbb{C}^n$ positive on non-zero vectors and satisfying the following conditions:

$$||v|| \geq 0, \qquad ||v|| = 0 \text{ if and only if } v = 0;$$

$$||v|| + ||w|| \geq ||v + w|| \text{ for every } v, w \in \mathbb{C}^n; \tag{6.1}$$

$$||\lambda v|| = |\lambda| ||v|| \text{ for every } \lambda \in \mathbb{C}. \tag{6.2}$$

The unit ball centered at 0 in a given norm $|| \; ||$ will be denoted by

$$B_{|| \; ||} = \{||v|| < 1\}.$$

Condition (6.1) is equivalent to the convexity of the unit ball in the norm under consideration. For a norm homogeneous under multiplication by real positive numbers ($||sv|| = s||v||$ for $s > 0$) condition (6.2) is equivalent to its invariance under multiplication by complex numbers with unit module.

Example 6.1 The Euclidean norm and the maximum module norm

$$||z||_E = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \ ||z||_{\max} = \max_j |z_j|$$

satisfy conditions (6.1) and (6.2).

Theorem 6.2 (Generalized Maximum Principle). *Let $U \subset \mathbb{C}$ be a connected domain, $f : U \to \mathbb{C}^n$ be a holomorphic mapping. Let $||\ ||$ be a norm on $\mathbb{C}^n$, and let the function $||f(z)||$ achieve its maximal value at some point $P \in U$. Then $||f(z)|| \equiv const$.*

In the proof of Theorem 6.2 we use the following general properties of convex sets and real hyperplanes in $\mathbb{C}^n$.

Theorem 6.3 *Let $C \subset \mathbb{R}^n$ be a convex subset. For every point $x \in \partial C$ there exists a hyperplane through x that does not intersect the interior $Int(C)$. Or equivalently, the interior of every convex subset is an intersection of half-spaces.*

Proof (sketch). It suffices to prove Theorem 6.3 for a bounded convex set: the intersection C_N of the set C with a ball centered at 0 of radius N. Namely, let H_N be hyperplanes through x that do not intersect $Int(C_N)$. Then we take H to be the limit of a converging subsequence H_{N_k} in the Grassmanian space of hyperplanes (which is compact).

Without loss of generality we consider that C is compact and $Int(C) \neq \emptyset$.

First we prove Theorem 6.3 in the case, when C is a polytope: the convex hull of a finite set. Afterwards we approximate C by polytopes and pass to the limit. Namely, for every $\varepsilon > 0$ fix a finite ε-net $S_\varepsilon \subset C$. Let $\Sigma_\varepsilon \subset C$ denote its convex hull, which is a polytope. For every $x_\varepsilon \to \partial\Sigma_\varepsilon$ there exists a hyperplane H_ε through x_ε satisfying the statement of the theorem for the convex set Σ_ε. Passing to the limit, as $\varepsilon \to 0$ and $x_\varepsilon \to x$, we take H to be the limit of a convergence subsequence H_{ε_k}. The hyperplane H passes through x and does not intersect $Int(C)$. This proves Theorem 6.3. □

Proposition 6.4 *1) Every real hyperlane $H \subset \mathbb{C}^n$ passing through the origin (that is, a vector subspace over $\mathbb{R}$ in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ of real codimension one)* *contains a complex vector subspace $H_{\mathbb{C}}$ of real codimension one in H. This is the intersection of the hyperplane H with its image under the multiplication by the imaginary unity i.*

2) This is the unique maximal complex subspace in H: every other complex vector subspace in H is contained in $H_{\mathbb{C}}$.

3) There exists a complex linear functional $\eta : \mathbb{C}^n \to \mathbb{C}$ such that

$$H_{\mathbb{C}} = \{\eta = 0\}, \quad H = \{\operatorname{Re} \eta = 0\}. \tag{6.3}$$

Proof The intersection $H_{\mathbb{C}} := H \cap (iH)$ is invariant under the multiplication by i, since $i^2 = -1$. Hence, it is a complex vector subspace. Each complex vector subspace in H is clearly contained in $H_{\mathbb{C}}$, being also invariant. The subspace $H_{\mathbb{C}}$ has real codimension two, being even-dimensional intersection of real-codimension one subspaces. Hence, it has complex codimension one, and it is the kernel of a complex linear functional $\eta : \mathbb{C}^n \to \mathbb{C}$. The image $\eta(H)$ is a line (a vector subspace over $\mathbb{R}$) $\ell \subset \mathbb{C}$. Multiplying η by complex number one can achieve that ℓ being the imaginary axis. Then $H = \{\operatorname{Re} \eta = 0\}$, by construction. The proposition is proved. □

Proof of Theorem 6.2. In the case, when $f(P) = 0$, the statement of the theorem is obvious. Thus, we consider that $f(P) \neq 0$. Fix a hyperplane H through the image $f(P)$ that does not intersect the ball $B = \{||w|| < R\}$, $R := ||f(P)||$: it exists by Theorem 6.3. Let $\widetilde{H}$ denote its translation image passing through the origin. Let $\widetilde{H}_{\mathbb{C}} \subset \widetilde{H}$ denote the corresponding maximal complex vector subspace from the above proposition. Let $H_{\mathbb{C}} \subset H$ denote its translation image. Let $\eta : \mathbb{C}^n \to \mathbb{C}$ denote the complex linear functional such that $\widetilde{H}_{\mathbb{C}} = \{\eta = 0\}$, $\widetilde{H} = \{\operatorname{Re} \eta = 0\}$. Then $H = \{\operatorname{Re} \eta = c\}$, $c \in \mathbb{R}$. Without loss of generality we can and will consider that $\operatorname{Re} \eta|_B < c$. One can achieve this by multiplying η by ± 1, since H is disjoint from the ball B. The function $g(z) := \eta \circ f(z)$ is holomorphic on U,

$$\operatorname{Re} g(z) \leq c \text{ for all } z \in U, \quad \operatorname{Re} g(P) = c.$$

This together with Openness Principle for holomorphic functions and connectivity of U implies that $g(z) \equiv const$, $\operatorname{Re} g(z) \equiv c$. That is, $f(z) \in H$ for all $z \in U$. Thus, $f(z) \in \overline{B} = \{||w|| \leq ||f(P)||\}$ and at the same time $f(z)$ lies in the hyperplane H disjoint from B. Therefore, $f(z) \in \partial B$ for all z, hence $||f(z)|| \equiv ||f(P)||$. Theorem 6.2 is proved. □

Lemma 6.5 (Generalized Schwarz Lemma). *Let $||\ ||_1$, $||\ ||_2$ be norms on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Let $f : B_{||\ ||_1} \to B_{||\ ||_2}$ be a holomorphic mapping such that $f(0) = 0$. Then $||f(z)||_2 \leq ||z||_1$.*

Proof Fix a complex line $L \subset \mathbb{C}^n$ through the origin. Its intersection with the unit ball $B_{|| \, ||_1}$ is a disk $D \subset L$ centered at zero. Let t be a linear complex coordinate on L in which D be the unit disk. Thus, the restriction to L of the norm $|| \, ||_1$ coincides with the function $|t|$. Consider the restriction of the mapping f to D as a holomorphic vector function in t. The function $g(t) = \frac{f(t)}{t}$ is holomorphic on the unit disk D, since $f(0) = 0$. There are two possible cases.

Case 1): $||g(t)||_2 \leq 1$ for every $t \in D$. This is equivalent to the inequality of the lemma for the restriction of the function f to D.

Case 2): $||g(t)||_2 > 1$ at some point t. The upper limit of the function $||g(t)||_2$, as $|t| \to 1$, is no greater than one, since $||f||_2 \leq 1$ on D, by assumption. Therefore, it takes its maximum greater than one at some point $t_0 \in D$. Hence, $||g(t)|| \equiv r > 1$ on D, by Theorem 6.2. That is, $||f(t)||_2 \equiv r|t|$, $r > 1$, and $||f(r^{-\frac{1}{2}})||_2 = r^{\frac{1}{2}} > 1$. This contradicts the condition of the lemma, which implies that $||f||_2$ takes values less than one on the disk D. Hence, this case is impossible. Lemma 6.5 is proved. □

Corollary 6.6 *For every norm on $\mathbb{C}^n$ as at the beginning of the section each biholomorphic automorphism of the corresponding unit ball that fixes the origin preserves the norm: $||f(z)|| \equiv ||z||$.*

6.2 Automorphisms of polydisk

By $Aut(D_1)$ we denote the group of conformal automorphisms of the unit disk. Recall that each of them extends to a conformal automorphism of the Riemann sphere and is given by fractional-linear transformation

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}, \quad |a|^2 - |b|^2 > 0.$$

Remark 6.7 The group $Aut(D_1)$ acts transitively on D_1.

Theorem 6.8 *The group of automorphisms of the unit polydisk $\Delta = \Delta_{(1,\ldots,1)} \subset \mathbb{C}^n$ is generated by the product $(Aut(D_1))^n$ and the symmetric group S_n acting by permutations of coordinate components: each automorphism is a composition of an element of the above product and a permutation.*

Remark 6.9 In fact the group $Aut(\Delta)$ is the semidirect product of the group $(Aut(D_1))^n$ and the permutation group S_n, with respect to the action of the group S_n on $(Aut(D_1))^n$ by conjugations.

Proof It suffices to prove the statement of the theorem for every automorphism $g \in Aut(\Delta)$ fixing the origin: each automorphism of the polydisk can be corrected to fix the origin by replacing it by its post-composition with an element of the group $(Aut(D_1))^n$, see Remark 6.7.

Proposition 6.10 *Let $g \in Aut(\Delta)$ fix 0. Then g is the composition of a permutation of coordinates and their multiplications by complex numbers with unit modules.*

Proof For every $j = 1, \dots, n$ let $V_j \subset \Delta$ denote the subset of those points $z = (z_1, \dots, z_n)$ for which $|z_j| > |z_s|$ for every $s \neq j$. The union $\cup_{j=1}^n V_j$ is an open and dense subset in Δ. One has $||z||_{\max} \equiv |z_j|$ on V_j, by definition. Let $k \in \{1, \dots, n\}$ be an index such that $V_k \cap g(V_j) \neq \emptyset$, or equivalently, $U_{jk} = V_j \cap g^{-1}(V_k) \neq \emptyset$. One has $||g(z)||_{\max} \equiv ||z||_{\max}$, by Corollary 6.6. Therefore, $|z_k(g(z))| \equiv |z_j|$ on U_{jk}. Thus, the ratio of two holomorphic functions z_j and $z_k \circ g$ on the open set U_{jk} is holomorphic and has module identically equal to one. Therefore the latter ratio is locally constant, by Opennes Principle for holomorphic functions. Thus, there exists a $\theta \in \mathbb{R}$ such that $z_k \circ g \equiv e^{i\theta} z_j$ on an open subset in Δ, and hence, on all of Δ, by uniqueness of analytic extension. Finally, for every $j = 1, \dots, n$ there exist a $k = k(j)$ and a $\theta_j \in \mathbb{R}$ such that $z_{k(j)} \circ g \equiv e^{i\theta_j} z_j$. One has $k(j_1) \neq k(j_2)$ whenever $j_1 \neq j_2$. Indeed, otherwise, if $k = k(j_1) = k(j_2)$, then $z_k \circ g \equiv e^{i\theta_{j_1}} z_{j_1} \equiv e^{i\theta_{j_2}} z_{j_2}$, $z_{j_1} \equiv e^{i(\theta_{j_2} - \theta_{j_1})} z_{j_2}$, - a contradiction. Thus, the mapping $j \mapsto k(j)$ is a permutation. Proposition 6.10 is proved. □

The proposition immediately implies the statement of Theorem 6.8. □

6.3 Cauchy inequality. Henri Cartan's theorem on automorphisms tangent to identity

Definition 6.11 A complex manifold is called a *domain of bounded type,* if it is biholomorphic to a bounded domain in $\mathbb{C}^n$.

Here we prove the following theorem

Theorem 6.12 (Henri Cartan). *Let B be a domain of bounded type, $O \in B$, $f : B \to B$ be a biholomorphic automorphism such that $f(O) = O$ and $df(O) = Id$. Then $f = Id$.*

In the proof of Cartan's Theorem we use Cauchy inequality, which follows immediately from Cauchy Integral Formula.

Theorem 6.13 (Cauchy Inequality). *Let $f : \Delta_r \to \mathbb{C}$ be a holomorphic function on a polydisk of multiradius $r = (r_1, \dots, r_n)$, and let $|f| \leq R$ on Δ_r. Let $m \in (\mathbb{Z}_{\geq 0})^n$, and let c_m be the Taylor coefficient of the function f at 0 at the monomial z^m. Then*

$$|c_m| \leq \frac{R}{r^m}. \tag{6.4}$$

Proof Without loss of generality we consider that f is holomorphic on the closed polydisk $\overline{\Delta}_r$, replacing r by λr, $0 < \lambda < 1$, and passing to the limit, as $\lambda \to 1$. One has

$$c_m = \left(\frac{1}{2\pi i}\right)^n \oint_{|\zeta_1|=r_1} \cdots \oint_{|\zeta_n|=r_n} \frac{f(\zeta)}{\zeta^m \zeta_1 \dots \zeta_n} d\zeta_n \dots d\zeta_1. \tag{6.5}$$

Indeed, in the Laurent series of the function $\frac{f(z)}{z^m z_1 \dots z_n}$ each monomial different from $\frac{c_m}{z_1 \dots z_n}$ contains at least one coordinate z_j in a power different from -1. Hence, its integral over the boundary ∂D_{r_j} vanishes, since the residue in the coordinate z_j vanishes. This implies that in the integral in the right-hand side of the formula (6.5) the only nontrivial contribution is given by the monomial $\frac{c_m}{\zeta_1 \dots \zeta_n}$, and the integral of the latter equals $(2\pi i)^n c_m$. This proves (6.5). The restriction to the product of the boundaries ∂D_{r_j} of the subintegral expression in (6.5) has module no greater than $\frac{R}{r^m r_1 \dots r_n}$, while the product of lengths of boundaries is equal to $(2\pi)^n r_1 \dots r_n$. This together with (6.5) implies (6.4). □

Proof of Theorem 6.12. Without loss of generality we consider that $B \subset \mathbb{C}^n$ is a bounded domain, O is the origin and B contains the polydisk $\Delta = \Delta_{(1,1,\dots,1)}$. Let R denote the minimal radius of the ball centered at the origin that contains B. For every $m \in (\mathbb{Z}_{\geq 0})^n$ let c_m denote the coefficient at z^m in the Taylor series at 0 of the mapping f. Suppose the contrary: $f \neq Id$, that is, the Taylor series of the mapping f contains some nonlinear terms. Set

$$d = \min\{|m| = |m_1| + \dots + |m_n| \mid m \in (\mathbb{Z}_{\geq 0})^n, \ c_m \neq 0, \ |m| \geq 2\},$$

$$P_d(z) = \sum_{|m|=d} c_m z^m.$$

The polynomial $P_d(z)$ is homogeneous nonzero of degree d. Consider the iterations $f^k = f \circ f \circ \dots \circ f$. For every $k \in \mathbb{N}$ one has

$$f^k(z) = z + kP_d(z) + O(|z|^{d+1}) :$$

taking k-th iterate of a mapping tangent to the identity (i.e., fixing 0 and having identity derivative there) multiplies lower nonlinear terms by k. This follows immediately from the fact that the Taylor series of the composition of mappings is the formal composition of their Taylor series and straightforward calculation. Therefore, for every m with $|m| = d$ and $c_m \neq 0$ for every $k \in \mathbb{N}$ the coefficient at z^m of the k-th iterate f^k equals kc_m. Thus, it becomes arbitrarily large, as k is large enough. On the other hand, the latter coefficients kc_m should be no greater than R for all k, by Theorem 6.13 and since all the iterates f^k are holomorphic on Δ and take values in the ball of radius R centered at the origin. The contradiction thus obtained proves Theorem 6.12. □

6.4 Automorphisms of ball

The next theorem describes the automorphisms of the unit ball B. To state it, let us consider the subgroup $U(1,n) \subset GL_{n+1}(\mathbb{C})$ acting naturally on the space $\mathbb{C}^{n+1}$ with the coordinates $\widetilde{z} = (\widetilde{z}_0, \dots, \widetilde{z}_n)$ that preserves the indefinite Hermitian form

$$Q(\widetilde{z}) = |\widetilde{z}_0|^2 - \sum_{j=1}^{n} |\widetilde{z}_j|^2.$$

Let $PU(1,n)$ denote its projectivization: its image under the natural projection $GL_{n+1}(\mathbb{C}) \to PGL_{n+1}(\mathbb{C})$ of factorization by $\mathbb{C}^*$. Set

$$K = \{Q > 0\} \subset \mathbb{C}^{n+1}, \ \Sigma = \{v \in \mathbb{C}^{n+1} \mid Q(v) = 1\} \subset K.$$

The images of the sets K and Σ under the tautological projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ coincide with the Euclidean unit ball B in the affine chart $\mathbb{C}^n = \{(1 : z_1 : \dots : z_n)\}$. The group $U(1,n)$ preserves both K and Σ. Therefore, each element of the group $PU(1,n)$ yields an automorphism of the unit ball.

Theorem 6.14 *The group of automorphisms of the unit ball $B \subset \mathbb{C}^n$ coincides with the group $PU(1,n)$: each its biholomorphism is the restriction to B of an element of the group $PU(1,n)$.*

The starting point of the proof of Theorem 6.14 is the following immediate corollary of Schwarz Lemma and Cartan's Theorem.

Lemma 6.15 *Every automorphism of the unit ball in $\mathbb{C}^n$ that fixes the origin is a unitary transformation.*

Proof Each automorphism $f(z)$ of the unit ball fixing the origin preserves the standard Euclidean norm: $||f(z)|| \equiv ||z||$, by Schwarz Lemma applied to f and to its inverse. Therefore, its differential $df(0)$ is a unitary operator. Without loss of generality we can and will consider that $df(0) = Id$: one can achieve this by taking a post-composition with a unitary transformation. Then $f = Id$, by Cartan's Theorem. This proves the lemma. $\square$

Remark 6.16 The group $U(n)$ of unitary transformations of the affine chart $\mathbb{C}^n$ embeds naturally into $PU(1,n)$. This follows from the fact that it lifts to the subgroup in $U(1,n)$ fixing the $\widetilde{z}_0$-axis and the coordinate $\widetilde{z}_0$ and acting as the unitary group $U(n)$ on the coordinates $(\widetilde{z}_1, \dots, \widetilde{z}_n)$.

Lemma 6.17 *The group $PU(1,n)$ acts transitively on the unit ball.*

Proof It suffices to show that $U(1,n)$ acts transitively on the unit sphere Σ in the pseudo-hermitian metric Q. That is, given two vectors $u, v \in \mathbb{C}^{n+1}$ with $Q(u) = Q(v) = 1$, let us show that there exists a transformation $g \in U(1,n)$ such that $g(u) = v$. Consider the orthogonal complements $u^\perp$ and $v^\perp$ with respect to the indefinite Hermitian form Q. One has $u \notin u^\perp$, $v \notin v^\perp$, since $Q(u) = Q(v) = 1 \neq 0$. The restriction to $u^\perp$ of the form Q is negative definite. Indeed, each indefinite Hermitian form has a well-defined signature: the number of positive squares minus the number of negative squares in a basis where its matrix is diagonal. The signature is independent on the choice of diagonalizing basis. The signature of the form Q is equal to $1 - n$. Its restriction to $u^\perp$ can be diagonalized: reduced to a sum of squared moduli of coordinates with signs. Then the signature of the form Q is equal to the signature of its restriction to $u^\perp$ plus one (corresponding to the vector u, where $Q(u) = 1 > 0$). This implies that the latter signature of restriction to $u^\perp$ equals $-n$, and thus, the latter restriction is negative definite. Finally, the restrictions of the form Q to both $u^\perp$ and $v^\perp$ are negative definite, and hence, can be transformed one into the other by a complex linear transformation $h : u^\perp \to v^\perp$. The transformation g sending u to v and coinciding with h on $u^\perp$ is a linear automorphism preserving the form Q, by construction, and hence, $g \in U(1,n)$. The lemma is proved. $\square$

The two latter lemmas immediately imply Theorem 6.14.

6.5 Introduction to complex dynamics: linearization theorem in dimension one

Here and in the next subsection we give an introduction to local complex dynamics given by a germ of biholomorphic transformation at a fixed point. We prove linearization theorems in one and two dimensions for contracting germs. Then we show that the attractive basin of an attracting non-resonant fixed point of an injective mapping $\mathbb{C}^2 \to \mathbb{C}^2$ is naturally biholomorphically equivalent to $\mathbb{C}^2$. This yields a wide class of domains in $\mathbb{C}^2$ that are smaller than $\mathbb{C}^2$ but biholomorphically equivalent to $\mathbb{C}^2$. This phenomena does not occur in one dimension, by Riemann Mapping Theorem.

Theorem 6.18 *Every germ of conformal mapping*

$$f : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \ f(z) = \lambda z + O(z^2), \ |\lambda| \neq 0, 1,$$

is conformally conjugated to its linear part. More precisely, there exists a unique germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$*,* $h(0) = 0$*,* $h'(0) = 1$*, such that*

$$\lambda h = h \circ f. \tag{6.6}$$

Proof Without loss of generality we consider that $0 < |\lambda| < 1$ (replacing f by f^{-1}, if this is not the case). Equation (6.6) is equivalent to the statement that h is a fixed point of the transformation

$$\mathcal{L} : h \mapsto \lambda^{-1} h \circ f.$$

We will show that $\mathcal{L}$ is a contraction in appropriate complete metric space and hence, has a unique fixed point there.

Fix a $\mu > 0$ such that

$$0 < \mu^2 < |\lambda| < \mu < 1. \tag{6.7}$$

Fix an $r > 0$ such that f is holomorphic on $\overline{D}_r$ and

$$|f(z)| \leq \mu |z| \text{ whenever } z \in \overline{D}_r. \tag{6.8}$$

In particular, (6.8) implies that $f(\overline{D}_r) \subset D_r$.

For every function $q(z)$ holomorphic on D_r and continuous on $\overline{D}_r$ such that $q(0) = q'(0) = 0$ set

$$||q|| := \sup_{|z| \leq r} \frac{|q(z)|}{|z|^2}.$$

Let M denote the space of functions h holomorphic on D_r and continuous on $\overline{D}_r$ such that

$$h(0) = 0, \ h'(0) = 1,$$

equipped with the distance $dist(h_1, h_2) = ||h_1 - h_2||$. This is a complete metric space. Indeed, a sequence fundamental in the norm converges uniformly, by definition. Hence, its limit is holomorphic, by Theorem 1.11, and vanishes at 0. The derivatives also converge uniformly in compact set to the derivative of the limit, by Cauchy integral formula for the derivative and convergence of the function. Therefore, the limit has unit derivative at 0. Finally, the limit of a converging sequence is an element of the space M, and hence, M is complete.

Proposition 6.19 $\mathcal{L}(M) \subset M$.

Proof If $h(0) = 0$, then $(\mathcal{L}h)(0) = 0$ and $(\mathcal{L}h)'(0) = h'(0)$. If h is holomorphic on D_r and continuous on $\overline{D}_r$, then so is the composition $h \circ f$, since f is holomorphic on $\overline{D}_r$ and $f(\overline{D}_r) \subset D_r$. This implies that $\mathcal{L}$ preserves the space M and proves the proposition. $\square$

Proposition 6.20 $||\mathcal{L}h_1 - \mathcal{L}h_2|| \leq \nu ||h_1 - h_2||$, $\nu = |\lambda|^{-1}\mu^2 < 1$.

Proof The operator $\mathcal{L}$ being linear, it suffices to show that $||\mathcal{L}q|| \leq \nu||q||$ for every q as above. One has

$$\frac{|(\mathcal{L}q)(z)|}{|z^2|} = |\lambda|^{-1} \frac{|q(f(z))|}{|f(z)|^2} \frac{|f(z)|^2}{|z|^2} \leq |\lambda|^{-1}||q||\mu^2,$$

by definition, (6.8) and since $f(z) \in D_r$ whenever $z \in \overline{D}_r$. This implies that the norm of the image $\mathcal{L}q$ is no greater than $\nu||q||$. The proposition is proved. $\square$

The two latter propositions together imply that $\mathcal{L} : M \to M$ is a contraction. Hence, $\mathcal{L}$ has a unique fixed point $h \in M$, which obviously represents a conjugating germ we are looking for. Its uniqueness follows from the above uniqueness of fixed point and the fact that the above argument holds for every r small enough. This proves Theorem 6.18. $\square$

6.6 Linearization theorem in dimension two

Here we prove a linearization theorem for a germ $F = (f_1, f_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of biholomorphic mapping at 0 with linear part of the type

$$dF(0) = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ 0 < |\lambda_1|, |\lambda_2| < 1.$$

Definition 6.21 A matrix Λ as above (or a vector $\lambda = (\lambda_1, \lambda_2)$) is said to be *resonant*, if it satisfies a relation of type

$$\lambda_j = \lambda^m = \lambda_1^{m_1}\lambda_2^{m_2},\ m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 0},\ m_1 + m_2 \geq 2,$$

which is called a *resonance relation*. If there are no resonance relations, then Λ is called *non-resonant.*

Remark 6.22 If $0 < |\lambda_1|, |\lambda_2| < 1$, then each resonance relation (if any) takes the form $\lambda_1 = \lambda_2^k$, $k \in \mathbb{N}$ (up to permutation of indices), since in this case $|\lambda^m| < |\lambda_j|$, whenever $m_1 + m_2 \geq 2$ and $m_j > 0$.

Theorem 6.23 *Every germ F as above with non-resonant linear part is biholomorphically conjugated to its linear part. More precisely, there exists a unique biholomorphic germ $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $H(0) = 0$, $dH(0) = Id$ such that*

$$\Lambda H = H \circ F. \tag{6.9}$$

The proof of Theorem 6.23 is analogous to the above proof of Theorem 6.18. Equation (6.9) is equivalent to the statement that H is a fixed point of the linear operator

$$\mathcal{L} : H \mapsto \Lambda^{-1} H \circ F.$$

First we replace F by its conjugate whose lower nonlinear terms have high enough degree. Then we will show that $\mathcal{L}$ is a contraction in appropriate complete metric space, which will imply the existence and uniqueness of fixed point.

In this subsection for simplicity for every $z \in \mathbb{C}^2$ we set

$$|z| := \sqrt{|z_1|^2 + |z_2|^2}.$$

Proposition 6.24 *For every $N \in \mathbb{N}$ there exists a unique*[1] *vector polynomial $H_N : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with components of degree at most N with $H_N(0) = 0$, $dH_N(0) = Id$ such that*

$$H_N \circ F \circ H_N^{-1}(z) = \Lambda z + O(|z|^{N+1}). \tag{6.10}$$

Proof Let us prove existence by induction in N.

Induction base: $N = 1$, $H_N = Id$.

[1] Uniqueness was not proved during the lectures. Its proof is presented in these notes for completeness

Induction step. Let the statement of the proposition be proved for $N = k$. Let us prove it for $N = k+1$. Let $H_k = H_{N-1}$ be the germ given by the induction hypothesis. Then

$$F_{N-1}(z) := H_{N-1} \circ F \circ H_{N-1}^{-1}(z) = \Lambda z + P_N(z) + O(|z|^{N+1}), \tag{6.11}$$

$$P_N(z) = \begin{pmatrix} \sum_{s=0}^N a_s z_1^s z_2^{N-s} \\ \sum_{s=0}^N b_s z_1^s z_2^{N-s} \end{pmatrix}. \tag{6.12}$$

We show that there exists a vector polynomial

$$h_N(z) = z + Q_N(z), \quad Q_N(z) = \begin{pmatrix} \sum_{s=0}^N \alpha_s z_1^s z_2^{N-s} \\ \sum_{s=0}^N \beta_s z_1^s z_2^{N-s} \end{pmatrix}$$

such that

$$h_N \circ F_{N-1} \circ h_N^{-1}(z) = \Lambda z + O(|z|^{N+1}). \tag{6.13}$$

Then $H_N = h_N \circ H_{N-1}$ satisfies (6.10). This will prove the induction step and the existence statement.

Homological equation on the vector polynomial Q_N.

Claim 1. *One has*

$$F_N(z) := h_N \circ F_{N-1} \circ h_N^{-1}(z) = F_{N-1} + Q_N(\Lambda z) - \Lambda Q_N(z) + O(|z|^{N+1}).$$

Proof By $\simeq$ we denote equality modulo $O(|z|^{N+1})$. One has

$$h_N^{\pm 1}(z) \simeq z \pm Q_N(z), \quad F_{N-1} \circ h_N^{-1}(z) \simeq F_{N-1}(z) - \Lambda Q_N(z),$$

$$\begin{aligned} F_N(z) &\simeq h_N(F_{N-1}(z) - \Lambda Q_N(z)) \simeq F_{N-1}(z) - \Lambda Q_N(z) + Q_N \circ F_{N-1}(z) \\ &\simeq F_{N-1}(z) - \Lambda Q_N(z) + Q_N(\Lambda z), \end{aligned}$$

by construction. This proves the claim. □

Equation (6.13) is equivalent to the equation

$$P_N(z) + Q_N(\Lambda z) - \Lambda Q_N(z) = 0, \tag{6.14}$$

by (6.11) and the claim. Equation (6.14) is called the *homological equation.* The coefficient at $z_1^s z_2^{N-s}$ of the first (second) component in its left-hand side equals respectively

$$a_s + \alpha_s(\lambda_1^s \lambda_2^{N-s} - \lambda_1) = 0,$$

$$b_s + \beta_s(\lambda_1^s \lambda_2^{N-s} - \lambda_2) = 0.$$

Note that the above expressions in the brackets (the multipliers at α_s and β_s) are non-zero by non-resonance condition. Therefore, the latter equations in α_s and β_s can be solved (and in a unique way), and the vector polynomial Q_N constructed from their solutions α_s, β_s satisfies (6.14), by construction. This proves the existence statement of the proposition.

Remark 6.25 The above argument shows that if Λ is nonresonant, then the linear operator $Q_N \mapsto Q_N \circ \Lambda - \Lambda Q_N$ is non-degenerate: a linear automorphism of the space of homogeneous vector polynomials of degree N.

Let us prove uniqueness. Suppose the contrary: there are two vector polynomials H_N and $\widetilde{H}_N$ of degrees no greater than N conjugating F to $\Lambda z + O(|z|^{N+1})$ with $H_N(0) = \widetilde{H}_N(0) = 0$, $dH_N(0) = d\widetilde{H}_N(0) = Id$. Let $d \in \mathbb{N}$, $d \leq N$, denote the smallest degree of nonzero terms in the difference $H_N - \widetilde{H}_N$. The compositional ratio $\widetilde{H}_N \circ H_N^{-1}$ is equal to $z + Q_d(z) + O(|z|^{d+1})$, where $Q_d \not\equiv 0$ is a homogeneous vector polynomial of degree d, by construction. It conjugates $F_N(z) = \Lambda z + O(|z|^{N+1})$ to a map with the same asymptotics, by construction. Therefore, the vector polynomial $z + Q_d(z)$ conjugates it to a map with asymptotics $\Lambda z + O(|z|^{d+1})$, since $d \leq N$. This together with Claim 1 and Remark 6.25 implies that $Q_d \equiv 0$. The contradiction thus obtained proves uniqueness. The proposition is proved. $\square$

Proof of Theorem 6.23. Without loss of generality we consider that $|\lambda_1| \leq |\lambda_2|$. Fix a $\mu > 0$ such that

$$0 < |\lambda_1| \leq |\lambda_2| < \mu < 1. \tag{6.15}$$

Let us choose a $N \in \mathbb{N}$ large enough so that

$$|\lambda_1|^{-1}\mu^N < 1. \tag{6.16}$$

Without loss of generality we consider that

$$F(z) = \Lambda z + O(|z|^N).$$

One can achieve this by conjugation from the above proposition. We will be looking for a linearizing conjugation of the type $H(z) = z + O(|z|^N)$. Fix an $r > 0$, such that F is holomorphic on the closed Euclidean ball $\overline{B}_r$ of radius r and

$$|F(z)| \leq \mu|z| \text{ whenever } z \in \overline{B}_r. \tag{6.17}$$

Here the norm is Euclidean. Let M denote the space of holomorphic mappings $H : B_r \to \mathbb{C}^2$ continuous on $\overline{B}_r$ such that $H(0) = 0$, $H(z) =$

$z + O(|z|^N)$. For every holomorphic mapping $Q : B_r \to \mathbb{C}^2$ continuous on $\overline{B}_r$ with

$$Q(z) = O(|z|^N) \text{ as } z \to 0$$

set

$$||Q|| = \sup_{z \in \overline{B}_r} \frac{|Q(z)|}{|z|^N}.$$

The space M equipped with the distance $d(H_1, H_2) = ||H_1 - H_2||$ is a complete metric space. The operator

$$\mathcal{L} : H \mapsto \Lambda^{-1} H \circ F$$

is a well-defined transformation of the space M to itself, since $F(\overline{B}_r) \subset B_r$, as in the previous subsection. Set

$$\nu = |\lambda_1|^{-1} \mu^N < 1.$$

Claim 2. *One has* $||\mathcal{L}Q|| \leq \nu ||Q||$ *for every* Q *as above.*

Proof One has

$$\frac{|\Lambda^{-1} Q \circ F(z)|}{|z|^N} \leq |\lambda_1|^{-1} \frac{|Q \circ F(z)|}{|F(z)|^N} \left(\frac{|F(z)|}{|z|} \right)^N \leq \lambda_1^{-1} \mu^N ||Q|| = \nu ||Q||,$$

as in the previous subsection. This implies the claim. □

The claim implies that $\mathcal{L} : M \to M$ is a contraction, and hence, it has a unique fixed point. This finishes the proof of the existence in Theorem 6.23, as at the end of the previous subsection. For every $N \in \mathbb{N}$ the Taylor polynomial of degree N of the linearizing conjugating map H is the unique vector polynomial satisfying the statement of Proposition 6.24, by construction. Hence the Taylor series of the map H is uniquely defined, and thus, H is unique. Theorem 6.23 is proved. □

6.7 Polynomial automorphisms of $\mathbb{C}^2$. Fatou–Bieberbach domains

Here we study polynomial automorphisms of $\mathbb{C}^2$ having an attractive fixed point of non-resonant type. We show that its basin of attraction is biholomorphic to $\mathbb{C}^2$. In the case, when the basin is not all of $\mathbb{C}^2$ (e.g., if there is another fixed point), it yields an example of domain in $\mathbb{C}^2$ different from $\mathbb{C}^2$ but biholomorphic to $\mathbb{C}^2$: the so-called Fatou-Bieberbach domain. This phenomena is specific to higher dimensions and does not occur in dimension one: every simply connected domain in $\mathbb{C}$ different from all of $\mathbb{C}$ is conformally equivalent to the unit disk, not to $\mathbb{C}$ (Riemann Mapping Theorem).

Example 6.26 Here are some examples of biholomorphic automorphisms of $\mathbb{C}^2$:

1) The group of affine transformations generated by the group $GL_2(\mathbb{C})$ and the group $\mathbb{C}^2$ of translations.

2) Elementary polynomial automorphisms of higher degrees:

$$\Psi : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 + P(z_2) \\ z_2 \end{pmatrix}.$$

3) Transcendental transformations, e.g., $(z_1, z_2) \mapsto (z_1 + e^{z_2}, z_2)$.

Definition 6.27 A *polynomial automorphism* of $\mathbb{C}^2$ is a mapping $\mathbb{C}^2 \to \mathbb{C}^2$ given by a vector polynomial whose inverse map exists and is also given by a vector polynomial.

Theorem 6.28 *(***Jung, 1942).**[2] *All the polynomial automorphisms form a group generated by affine and elementary polynomial automorphisms, see the above classes 1) and 2).*

We will not present a proof of Jung Theorem, since it requires additional techniques not covered by the cours.

Theorem 6.29 *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be an injective holomorphic mapping that has a fixed point at the origin. Let its linear part $\Lambda = dF(0)$ be diagonal non-resonant with nonzero eigenvalues lying in the unit disk. Consider the attractive basin*

$$V = \{z \in \mathbb{C}^2 \mid F^k(z) \to 0, \text{ as } k \to +\infty\},$$

which is an open subset in $\mathbb{C}^2$. Then the local linearizing germ $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ from Theorem 6.23 conjugating F to Λ (i.e., satisfying (6.9)) extends up to a biholomorphic isomorphism $H : V \simeq \mathbb{C}^2$.

Proof There exists a ball B centered at the origin such that F is well-defined and holomorphic on $\overline{B}$ and $F(\overline{B}) \subset B$ (see the proof of Theorem 6.23). Set

$$V_0 = B, \ V_1 = F^{-1}(V_0), \ V_2 = F^{-1}(V_1), \dots$$

One has

$$V_0 \subset V_1 \subset \dots = V,$$

[2] A beautiful geometric and relatively simple proof of Jung Theorem was obtained by a French mathematician Stéphane Lamy: Lamy, S. *Une prevue géométrique du théorème de Jung.* – Enseignement Mathématique, **48** (2002), 291–315.

since by definition, each point of the basin V is eventually sent to B by some iteration of the mapping F. We show that H extends holomorphically to every V_k by induction in k.

The induction base is obvious: H is holomorphic on $V_0 = B$.

Induction step. Let we have already shown that H is holomorphic on V_k and satisfies (6.9) on V_k:

$$H = \Lambda^{-1} H \circ F. \tag{6.18}$$

Let us prove that it extends holomorphically to V_{k+1} and satisfies (6.18) there. The latter composition $\Lambda^{-1} H \circ F$ is well-defined and holomorphic on V_{k+1}, since $F(V_{k+1}) \subset V_k$, H is holomorphic on V_k (induction hypothesis) and Λ is invertible. It coincides with H on V_k (induction hypothesis: equality (6.18) on V_k). Therefore, it yields a holomorphic extension of the mapping H to V_{k+1}, and equation (6.18) holds on V_{k+1}, since it holds on V_k and by uniqueness of analytic extension. The induction step is over. Thus, H is holomorphic on all of V and satisties (6.18) there. It follows by construction that $H : V \to H(V)$ is a biholomorphism. One has $H(V) = \mathbb{C}^2$, since $H(V_k) = \Lambda^{-k} H(V_0)$, $H(V_0)$ contains a ball $\widetilde{B}$ centered at the origin, and the images $\Lambda^{-k}(\widetilde{B})$ exhaust all of $\mathbb{C}^2$. Theorem 6.29 is proved. □

Corollary 6.30 *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be an injective holomorphic mapping (e.g., biholomorphic) that has a fixed point p whose linear part is diagonal non-resonant and has all the eigenvalues nonzero and lying in the unit disk. Then its attractive basin is biholomorphic to $\mathbb{C}^2$.*

Definition 6.31 A *Fatou–Bieberbach domain* is a domain in $\mathbb{C}^n$ different from $\mathbb{C}^n$ that is biholomorphically equivalent to $\mathbb{C}^n$. (These domains exist only for $n \geq 2$.)

Remark 6.32 In the case, when, e.g., F has an additional fixed point $q \neq p$, the attractive basin is different from all of $\mathbb{C}^2$, and hence, is a Fatou–Bieberbach domain.

Seminar material, March 19, 2024. Let us construct a polynomial automorphism with an attractive basin being a Fatou–Bieberbach domain. Take polynomial automorphisms

$$f : (z_1, z_2) \mapsto (z_1 + z_2, z_2); \;\; g : (z_1, z_2) \mapsto (z_1, z_2 + z_1^2).$$

Let us choose a non-resonant diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2), \;\; \lambda_1 \neq \lambda_2, \;\; 0 < |\lambda_1|, |\lambda_2| < 1.$$

Set

$$F(z) = \Lambda g \circ f(z) = \begin{pmatrix} \lambda_1(z_1 + z_2) \\ \lambda_2(z_2 + (z_1 + z_2)^2) \end{pmatrix}.$$

Proposition 6.33 *The attractive basin V of the fixed point 0 of the automorphism F is biholomorphically equivalent to $\mathbb{C}^2$. The automorphism F has an additional fixed point $q \neq 0$, hence $V \neq \mathbb{C}^2$ is a Fatou–Bieberbach domain.*

Proof The differential $dF(0)$ has distinct eigenvalues λ_1, λ_2, and hence, is conjugated to the diagonal matrix. Therefore, F is linearizable on V (Theorems 6.23 and 6.29). The system of equations on fixed points has the form

$$\begin{cases} z_1 = \lambda_1(z_1 + z_2) \\ z_2 = \lambda_2(z_2 + (z_1 + z_2)^2) \end{cases} \tag{6.19}$$

The first equation of the system is equivalent to each one of the two following equations:

$$z_1 = \frac{\lambda_1 z_2}{1 - \lambda_1}, \ z_1 + z_2 = z_2(1 + \frac{\lambda_1}{1 - \lambda_1}) = \frac{z_2}{1 - \lambda_1}.$$

Substituting the latter expression for $z_1 + z_2$ to the second equation in (6.19) and dividing it by z_2 yields

$$1 + \frac{z_2}{(1 - \lambda_1)^2} = \lambda_2^{-1}.$$

This yield a solution

$$z_2 = (1 - \lambda_1)^2(\lambda_2^{-1} - 1), \ z_1 = \frac{\lambda_1 z_2}{1 - \lambda_1} = \frac{\lambda_1}{\lambda_2}(1 - \lambda_1)(1 - \lambda_2)$$

of system (6.19), and hence, an additional fixed point of the mapping F. The proposition is proved. $\square$

7 Domains of holomorphy. Holomorphic convexity. Pseudoconvexity. Riemann domains.

Here we introduce the notion of domain of holomorphy: a domain that admits a holomorphic function "everywhere non-extendable" through the boundary. We prove Oka's Theorem, which says that being domain of holomorphy is equivalent to holomorphic convexity. Then we study local versions of convexity: *pseudo-convexity, Levi convexity,...* which appear to be equivalent to the global holomorphic convexity.

In the present section for every $r > 0$ the polydisk centered at z_0 with multiradius $(r, \dots, r)$ will be denoted by

$$\Delta(z_0, r) = \Delta_{r,\dots,r}(z_0) = \prod_{j=1}^{n} D_r(z_{0j}).$$

7.1 Domains of holomorphy and holomorphic convexity. Oka's Theorem

Let $D \subset \mathbb{C}^n$. For every $z_0 \in D$ set

$$r(z_0) = \max\{r > 0 \mid \Delta(z_0, r) \subset D\}$$

Definition 7.1 A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy*, if there exists a holomorphic function $f : D \to \mathbb{C}$ such that for every $z_0 \in D$ the function $f|_{\Delta(z_0, r(z_0))}$ cannot be extended holomorphically to a bigger polydisk $\Delta(z_0, R)$, $R > r(z_0)$.

Example 7.2 The unit disk $D_1 \subset \mathbb{C}$ is a domain of holomorphy. For example, the modular function $f : D_1 \to \mathbb{C}$ (providing the universal covering over $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ and obtained by reflecting ideal hyperbolic triangles) does not extend in the above sense, since it takes values arbitrarily close to 0, 1, ∞ in a neighborhood of every point of the boundary. One can show that *every domain in* $\mathbb{C}$ *is a domain of holomorphy.* Hartogs' figure $H \subset \Delta = \Delta_{1,1} \subset \mathbb{C}^2$ is not a domain of holomorphy, since every holomorphic function on H extends holomorphically to all of Δ (Hartogs' Theorem).

Everywhere below for a domain $D \subset \mathbb{C}^n$ by $H(D) = \mathcal{O}(D)$ we denote the space of holomorphic functions on D.

The following definition generalizes the notion of convexity.

Definition 7.3 Let $D \subset \mathbb{C}^n$ be a domain, $K \subset D$ be a subset. Fix a class of functions $F \subset H(D)$. Let us define the *F-convex hull*

$$\widehat{K}_F = \{z \in D \mid |f(z)| \leq \sup_{x \in K} |f(x)| \text{ for every } f \in F\}.$$

The subset K is called *F-convex*, if $\widehat{K}_F = K_F$ (then it is automatically closed). The *domain D is called F-convex,* if the F-hull $\widehat{K}_F$ of every compact subset $K \Subset D$ is compact. In the case, when $F = H(D)$ we call the above F-convex objects *holomorphically convex* and denote $\widehat{K} = \widehat{K}_{H(D)}$.

Remark 7.4 For every $F \subset H(D)$ and every $K \subset D$ the F-convex hull $\widehat{K}_F$ is a closed subset in D containing K. The intersection of arbitrary family of F-convex subsets in D is F-convex.

Remark 7.5 Let $D \subset \mathbb{C}^n$, $F_1 \subset F_2 \subset H(D)$. Then one has $\widehat{K}_{F_1} \supset \widehat{K}_{F_2} \supset \widehat{K} \supset K$. This implies that every F_1-convex subset $K \subset D$ is always F_2- and $H(D)$-convex. Similarly, if D is F_1-convex, then it is F_2-and $H(D)$-convex.

Example 7.6 A *holomorphic polyhedron* in a domain $D \subset \mathbb{C}^n$ is a (finite or infinite) intersection of its subsets $K_j := \{|f_j(z)| < c_j\}$ ($j \in J$ is some index), where $f_j \in H(D)$. The subsets K_j are $\{f_j\}$-convex, and hence F-convex, where $F = \{f_j \mid j \in J\}$. Therefore, their intersection is also F-convex, and hence, holomorphically convex.

Remark 7.7 Every closed convex subset in $\mathbb{C}^n$ in the usual geometric sense (e.g., a ball, or a polydisk) is convex with respect to the class $F_{\text{lin}} \subset H(\mathbb{C}^n)$ consisting of the exponents of the $\mathbb{C}$-linear functionals. Hence, it is holomorphically convex (Remark 7.5). Indeed, recall that any closed convex subset $K \subset \mathbb{R}^m$ is an intersection of closed half-spaces. For example, for each point $p \in \partial K$ we can take a hyperplane H_p through p which does not cross $Int(K)$. Then K is the intersection through $p \in \partial K$ of appropriate half-spaced bounded by H_p. Each closed half-space in $\mathbb{C}^n$ is defined by the inequality $\mathrm{Re}\, l \leq c$, where l is a $\mathbb{C}$-linear functional on $\mathbb{C}^n$, as in the proof of Theorem 6.2. Or equivalently, by the inequality $|f_l| \leq e^c$, $f_l(z) = e^{l(z)}$. Therefore, it is $\{f_l\}$-convex, and hence, F_{lin}-convex. Together with Remark 7.4, this implies that each closed convex subset in $\mathbb{C}^n$ (being an intersection of closed half-spaces) is F_{lin}-convex.

Remark 7.8 The $H(D)$-hull of a bounded subset K is bounded, since the modules of the coordinate functions cannot achieve values on $\widehat{K}$ greater than their suprema on K.

One of the main results in the theory is the following theorem.

Theorem 7.9 (Oka). *A domain $D \subset \mathbb{C}^n$ is a domain of holomorphy, if and only if it is holomorphically convex.*

Corollary 7.10 *The notion of domain of holomorphy is invariant under biholomorphisms, as is the notion of holomorphic convexity.*

First we prove Oka's Theorem. Afterwards (during the seminar) using it we prove a logarithmic convexity criterium characterizing convergence domains of power series.

The first step in the proof of Oka's Theorem is the next theorem.

Theorem 7.11 *Let a domain D be holomorphically convex. Then it is a domain of holomorphy.*

Proof In the proof of Theorem 7.11 we use the following proposition.

Proposition 7.12 *Let $D \subset \mathbb{C}^n$ be an F- convex domain. Then it admits a compact F-convex exhaustion*[3]

$$K_1 \Subset K_2 \Subset \dots = D, \ \widehat{K}_{j,F} = K_j.$$

Proof Consider an arbitrary compact exhaustion $B_1 \Subset B_2 \Subset B_3 \Subset \dots = D$. Set

$$j_1 = 1, \ K_1 = \widehat{B}_{1,F} = \widehat{(B_1)}_F, \ j_2 = \min\{j \mid B_j \Supset K_1\}, \ K_2 = \widehat{B}_{j_2,F}, \dots .$$

The sets $K_1 \Subset K_2 \Subset \dots$ form a compact F-convex exhaustion of the domain D. The proposition is proved. □

Fix an $H(D)$-convex exhaustion $K_1 \Subset K_2 \Subset \dots = D$ and a sequence of points $w_j \in K_{j+1} \setminus K_j$ accumulating to the boundary ∂D so that each open set intersecting the boundary ∂D contains a limit point of the sequence w_j: such a sequence w_j exists, since K_j form a compact exhaustion of the domain D. We will construct a function $f \in H(D)$ such that $f(w_j) \to \infty$, as $j \to \infty$. This will imply that f is non-extendable to polydisks $\Delta(z_0, R)$, $z_0 \in D$, $R > r(z_0)$: such a polydisk intersects the boundary, and hence, would contain a limit point of the sequence w_j; thus, f cannot extend holomorphically there. To do this, we construct functions $f_j \in H(D)$, set

$$F_k = \sum_{j=1}^{k} f_j,$$

with the following properties:

$$|f_j||_{K_j} < 2^{-j}, \ |F_j(w_j)| > 2^j. \tag{7.1}$$

[3] Recall that for any two subsets $A, B \subset \mathbb{C}^n$ the "compact inclusion" $A \Subset B$ means that $\overline{A}$ is a compact subset in $IntB$.

and prove the above statements for the function

$$f = \sum_{j=1}^{+\infty} f_j.$$

We construct the functions f_j inductively, taking $f_0 = 0$ as the induction base. Let we have already constructed f_j for $j \leq l-1$. Let us construct f_l. The compact K_l is holomorphically convex, and $w_l \in K_{l+1} \setminus K_l$. This implies that there exists a holomorphic function $g : D \to \mathbb{C}$ such that

$$g(w_l) = 1, |g||_{K_l} < \delta < 1 \quad \text{for some } \delta \in (0,1).$$

Set

$$f_l = (\frac{g}{\sqrt{\delta}})^N,$$

where N is chosen large enough so that

$$\delta^{\frac{N}{2}} < 2^{-l},\ |f_l(w_l)| = \delta^{-\frac{N}{2}} > 2^l + |F_{l-1}(w_l)|.$$

The first inequality implies that $|f_l||_{K_l} < 2^{-l}$. The second one implies that

$$|F_l(w_l)| \geq |f_l(w_l)| - |F_{l-1}(w_l)| > 2^l.$$

The induction step is over. The functions f_j satisfying (7.1) are constructed. The first inequality in (7.1) together with Weierstrass Convergence Theorem imply that the series $f = \sum_{j=1}^{+\infty} f_j$ converges uniformly on compact subsets in D, and the limit f is holomorphic on D. For every $l \in \mathbb{N}$ one has

$$F_l(w_l) > 2^l,\ f_j(w_l) < 2^{-j} \text{ for every } j \geq l+1.$$

The first inequality follows from (7.1). The second one follows from the first inequality in (7.1) and the inclusion $w_l \in K_{l+1}$. Therefore,

$$|f(w_l)| \geq |F_l(w_l)| - \sum_{j \geq l+1} |f_j(w_l)| \geq 2^{l-1},\ f(w_l) \to \infty, \text{ as } l \to \infty.$$

This proves Theorem 7.11. $\square$

7.2 Material of seminar. Logarithmic convexity characterization of convergence domains of power series

Definition 7.13 A *Reinhardt domain* in $\mathbb{C}^n$ (centered at the origin) is a domain invariant under the $\mathbb{T}^n$-action by coordinatewise rotations around the origin.

We consider the map

$$\lambda : \mathbb{C}^n \to \mathbb{R}^n, \ z \mapsto (\ln |z_1|, \dots, \ln |z_n|).$$

Definition 7.14 A Reinhardt domain is *logarithmically convex,* if the image $\lambda(D) \subset \mathbb{R}^n$ is convex.

Example 7.15 We already know that the convergence domain of a power series based at the origin is a Reinhardt domain containing the origin.

On the seminar we have proved the following theorem.

Theorem 7.16 *A Reinhardt domain is a convergence domain for a power series, if and only if it contains the origin and is logarithmically convex.*

Sketch of proof. Step 1. A convergence domain is a logarithmically convex Reinhardt domain.

Step 2. A logarithmically convex Reinhardt domain containing the origin is a union of polydisks centered at the origin. Or equivalently, its logarithmic image (which is a convex subset in $\mathbb{R}^n$ containing at least one negative quadrant) is a union of negative quadrants.

Step 3. A logarithmically convex Reinhardt domain D containing the origin is holomorphically convex. Hence, it is a domain of holomorphy of a function f, by the proved part of Oka's Theorem. This together with Step 2 and Abel's Lemma implies that this is the convergence domain of the Taylor series of the function f.

Proof of Step 3. Consider a closed convex subset in $\mathbb{R}^n_{x_1,\dots,x_n}$ whose interior is a union of negative quadrants. Then it is the intersection of half-spaces defined by inequalities of the type $\sum_{j=1}^n a_j x_j \leq c$ with $a_j \geq 0$, $a_j \in \mathbb{Q}$. Multiplying the latter inequality by a natural number (product of denominators of the rational numbers a_j) and substituting $x_j = \ln |z_j|$, we get an equivalent inequality of the type $|z^m| = |z_1^{m_1} \dots z_n^{m_n}| \leq C$.

First consider the case, when D is bounded. Set $D_\varepsilon := (1-\varepsilon)D$, where $\varepsilon \in (0,1)$ is small enough. One has $\overline{D}_\varepsilon \Subset D$, and $\overline{D}_\varepsilon$ is F-convex, where F is the class of all the monomials z^m, by the above discussion, Remark 7.4 and the arguments from Remark 7.7. Any compact subset $K \Subset D$ is contained in $\overline{D}_\varepsilon$ for every ε small enough. Hence, its F-convex hull is also contained there, and thus, is compact. This implies that D is F-convex, and hence, holomorphically convex (Remark 7.5).

Consider now the case, when D is unbounded. Fix an arbitrary compact subset $K \Subset D$ and a polydisk Δ_r containing K. The intersection $D \cap \Delta_r$ is a logarithmically convex bounded Reinhardt domain. Hence, it is F-convex, by the above discussion. Therefore, the F-convex hull of the set K is a compact subset in $D \cap \Delta_r$. This proves F-convexity of the domain D, and hence, its holomorphic convexity.

7.3 End of proof of Oka's Theorem: holomorphic convexity implies being domain of holomorphy

Theorem 7.17 *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy. Then it is holomorphically convex.*

In the proof of Theorem 7.17 we use the following notation and theorem. For every subset $K \subset D$ set

$$\rho(K, \partial D) = \inf\{r(z_0) \mid z_0 \in K\}.$$

Theorem 7.18 (Cartan–Thullen). *Let $D \subset \mathbb{C}^n$, $K \Subset D$ be a compact subset, $\sigma = \rho(K, \partial D)$. Then for every $z_0 \in \widehat{K}$ every function $f \in H(D)$ extends holomorphically from z_0 to the polydisk $\Delta(z_0, \sigma)$.*

Proof Fix a $z_0 \in \widehat{K}$. To show that the function f is holomorphic on $\Delta(z_0, \sigma)$, we show that its Taylor series converges uniformly on compact subsets in $\Delta(z_0, \sigma)$. To do this, we estimate its Taylor coefficients at z_0 by using Cauchy Inequality and convexity inequality. Fix a $0 < \delta < \sigma$ and a function $f \in H(D)$. For every point $t \in D$ and every $k \in \mathbb{Z}^n_{\geq 0}$ let

$$c_k(t) = \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|}}{\partial z^k} f(t)$$

denote the Taylor coefficient at $(z - t)^k$ of the function f at t. One has

$$\left|\frac{\partial^{|k|}}{\partial z^k} f(z_0)\right| \leq \sup_K \left|\frac{\partial^{|k|}}{\partial z^k} f\right|,$$

since $z_0 \in \widehat{K}$ and by the $H(D)$-convex hull inequality applied to the above partial derivative instead of the function f. This implies that

$$|c_k(z_0)| \leq \sup_{t \in K} |c_k(t)|, \tag{7.2}$$

For every $t \in K$ one has $r(t) \geq \sigma > \delta$, by definition. Therefore,

$$K^\delta := \overline{\cup_{t\in K}\Delta(t,\delta)} \subset D$$

is a compact subset. Set $M = \sup_{K^\delta} |f|$. One has

$$|c_k(t)| \leq \frac{M}{\delta^{|k|}} \text{ for every } t \in K,$$

by Cauchy Inequality. Hence, $c_k(z_0) \leq \frac{M}{\delta^{|k|}}$, by (7.2). This together with Abel's Lemma implies that the Taylor series at z_0 of the function f converges in $\Delta(z_0, \delta)$ uniformly on compact subsets. The above convergence takes place in the polydisk $\Delta(z_0, \sigma)$, since δ can be chosen arbitrarily close to σ. Hence, f extends holomorphically there. Theorem 7.18 is proved. □

Proof of Theorem 7.17. Let D be a domain of holomorphy of a function f. Let $K \Subset D$ be an arbitrary compact set, $\sigma = \rho(K, \partial D)$. For every $z_0 \in \widehat{K}$ the function f extends holomorphically to $\Delta(z_0, \sigma)$, by Theorem 7.18. This implies that $\sigma \leq r(z_0)$, by the definition of domain of holomorphy. Or equivalently,

$$r(z_0) \geq \rho(K, \partial D) \text{ for every } z_0 \in \widehat{K}. \tag{7.3}$$

Finally, the gap between the subset $\widehat{K} \subset D$ and ∂D is bounded from below by $\rho(K, \partial D)$, and $\widehat{K}$ is a bounded closed subset in D, see Remark 7.8. Hence, $\widehat{K}$ is compact. Theorem 7.17 is proved. □

Proof of Theorem 7.9. Theorem 7.9 follows immediately from Theorems 7.11 and 7.17. □

Corollary 7.19 *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy (or equivalently, holomorphically convex). Then for every compact subset $K \Subset D$ one has*

$$\rho(\widehat{K}, \partial D) = \rho(K, \partial D).$$

Proof The corollary follows immediately from inequality (7.3) and the obvious inequality $\rho(\widehat{K}, \partial D) \leq \rho(K, \partial D)$, which follows from the inclusion $K \subset \widehat{K}$. □

7.4 Continuity Principle. Levi convexity

Definition 7.20 A domain $D \subset \mathbb{C}^n$ is *locally connected* at its boundary point $\zeta \in \partial D$, if there exists an arbitrarily small neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ such that the intersection $U \cap D$ is connected. If this holds for every point

$\zeta \in \partial D$, then D is said to be *locally connected at the boundary.* A domain D locally connected at a point $\zeta \in \partial D$ is said to be *holomorphically non-extendable* at ζ, if there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ as above and a holomorphic function $f : U \cap D \to \mathbb{C}$ that does not extend holomorphically to ζ. If D is locally connected at the boundary, we say that D is *holomorphically non-extendable,* if it is holomorphically non-extendable at each point of its boundary.

Remark 7.21 A domain of holomorphy locally connected at the boundary is obviously holomorphically non-extendable. The converse statement was a problem stated by Levy and solved by Oka.

Theorem 7.22 (Oka). *A domain in $\mathbb{C}^n$ locally connected at the boundary is a domain of holomorphy, if and only if it is holomorphically non-extendable.*

Example 7.23 The complement $V = \mathbb{C}^2 \setminus \overline{B}_R$ to the closed ball $\overline{B}_R$ is holomorphically extendable at each point $\zeta \in \partial V = \partial B_R$. Indeed, without loss of generality let us consider that ζ is the north pole $(0, 1)$ (making a rotation). Take a polydisk $\Delta = \Delta_{R_1,R_2}(\zeta) \subset \mathbb{C}^2$ centered at ζ where R_1 is much bigger than R_2 so that there exists a $r_1 \in (0, R_1)$ for which $A := \{r_1 < |z_1| < R_1\} \times D_{R_2}(1) \subset \Delta$ lies outside the ball $\overline{B}_R$, i.e., in $V \cap \Delta$. The union $H := A \cup B$ with $B := D_{R_1} \times \{|z_2| > 1\} \subset \Delta \cap V$ is a Hartogs figure in Δ. Hence, every function holomorphic on H extends holomorphically to all of Δ, including ζ.

The above extendability is due to the fact that there is a sequence of embedded "disks" limiting to a disk S (in the above example $S = \{0\} \times \overline{D}_\sigma$) whose boundary is contained in the domain V under consideration and whose some interior point lies in ∂V. We will prove a theorem stating that presence of limiting embedded "disks" as above is basically the only reason for holomorphic extendability of a domain to its boundary point.

The next theorem states that presence of limiting embedded "disks" as above is basically the only reason for holomorphic extendability of a domain to its boundary point. To state it, let us introduce the following notions.

Definition 7.24 Let $n > r \geq 1$. Let $W \subset \mathbb{C}^r$ be a domain with compact closure, $\phi : \overline{W} \to \mathbb{C}^n$ be an injective holomorphic mapping, whose differential has maximal rank r at each point. The image

$$S = \phi(\overline{W})$$

is called a *compact holomorphic surface.*

Recall that for every subset $K \subset \mathbb{C}^n$ and each $\delta > 0$ we set

$$K^\delta = \cup_{t \in K} \Delta(t, \delta).$$

Definition 7.25 A sequence of subsets $M_k \subset \mathbb{C}^n$ *converges* to a closed subset $M \subset \mathbb{C}^n$, if for every $\varepsilon > 0$ there exists a $N > 0$ such that for every $k > N$ one has

$$M_k \subset M^\varepsilon \text{ and } M \subset M_k^\varepsilon.$$

Theorem 7.26 (Behnke–Sommer Continuity Principle). *Let $D \subset \mathbb{C}^n$, $S_k \subset D$ be a sequence of compact holomorphic surfaces converging to a subset $S \subset \mathbb{C}^n$ whose boundaries ∂S_k converge to a compact subset $\Gamma \Subset D$. Then for every point $z_0 \in S$ there exists a $\sigma > 0$ such that for every sequence of points $z_k \in S_k \cap \Delta(z_0, \sigma)$ converging to z_0, as $k \to \infty$, the germ at z_k of every holomorphic function $f : D \to \mathbb{C}$ extends holomorphically to $\Delta(z_0, \sigma)$. See Fig. 2.*

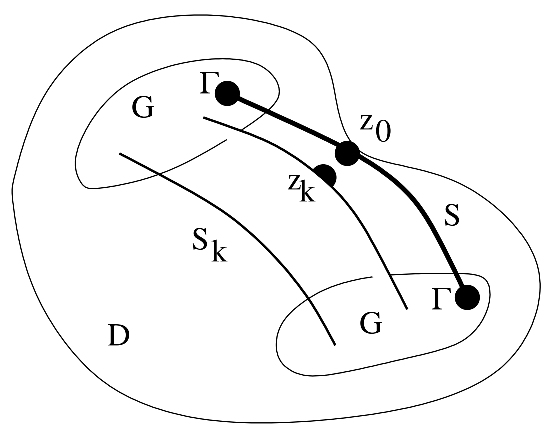

Figure 2: Converging compact holomorphic surfaces S_k and their limit S.

Proof Let us choose an auxiliary open subset G, $\Gamma \Subset G \Subset D$: the closure $\overline{G}$ is a compact subset in D and Γ is a compact subset in G. Set

$$r = \rho(G, \partial D).$$

There exists a $N > 1$ such that for every $k > N$ one has $\partial S_k \subset G$. Therefore, for those k every holomorphic function $f : D \to \mathbb{C}$ satisfies the inequality

$$\sup_{S_k} |f(z)| = \sup_{\partial S_k} |f(z)| \le \sup_{G} |f(z)|.$$

The first equality follows from the Maximum Principle applied to the restriction of the function f to the surface S_k. The latter inequality implies that $S_k \subset \widehat{G} = \widehat{G}_{H(D)}$ for every $k > N$. Therefore, for those k for every $z_k \in S_k$ the function f extends holomorphically to $\Delta_r(z_k)$ (Cartan–Thullen Theorem 7.18.) For every k large enough one has $S \subset S_k^{\frac{r}{2}}$, hence $S^{\frac{r}{2}} \subset S_k^r$, by convergence. This implies the statement of the theorem for $\sigma = \frac{r}{2}$. $\square$

Remark 7.27 One can show (slightly modifying the above proof) that the statement of the theorem holds for every $\sigma < \rho(\Gamma, \partial D)$.

Corollary 7.28 *1) Let in the condition of Theorem 7.26 the domain D be locally connected at the boundary, and let S contain a point $z_0 \in \partial D$. Then D is not a domain of holomorphy.*

2) Let in addition to the above, z_0 be an isolated point of intersection $S \cap \partial D$. Then D is holomorphically extendable to z_0. See Fig. 2.

Definition 7.29 Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. We say that D is *Levy-* (or *L-*) convex at ζ, if for every compact holomorphic surface $S \subset \mathbb{C}^n$ through ζ with $\partial S \subset D$ for every sequence S_k of compact holomorphic surfaces converging to S with $\partial S_k \to \partial S$ one has $S_k \setminus D \neq \emptyset$, whenever k is large enough. We say that D is *locally L-convex* at ζ, if there exists an arbitrarily small neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ such that the above statement holds with D replaced by $D \cap U$. We say that D is *(locally) L-convex,* if so it is at each its boundary point.

Remark 7.30 L-convexity at a point $\zeta \in \partial D$ obviously implies local L-convexity at ζ.

Proposition 7.31 *1) Every domain of holomorphy is L-convex.*

2) If D is holomorphically non-extendable at a point $\zeta \in \partial D$, then D is locally L-convex there.

Proof Suppose the contrary: a domain of holomorphy D is not L-convex at a $\zeta \in \partial D$. Then there exist a compact holomorphic surface $S \subset \mathbb{C}^n$, $\zeta \in S$, $\partial S \subset D$ and a sequence $S_k \to S$ of compact holomorphic surfaces $S_k \subset D$ converging to S with boundaries. Then each holomorphic function on D extends holomorphically to ζ (Continuity Principle). Thus, D is not a domain of holomorphy. The contradiction thus obtained proves Statement 1) of the proposition. The proof of Statement 2) is analogous, with D replaced by $D \cap U$, where $U = U(\zeta)$ is a small enough neighborhood from the above definition such that the function f that does not extend holomorphically to ζ is holomorphic on $U \cap D$. $\square$

The next theorem provides a global converse statement.

Theorem 7.32 (Oka). *A domain $D \subset \mathbb{C}^n$ is holomorphically non-extendable (at all the points of its boundary), if and only if it is L-convex.*

Theorem 7.33 (Sufficient condition for local L-convexity). *Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. Let there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a function f holomorphic on U such that*

$$f(\zeta) = 0, \;\; f|_{D \cap U} \not\equiv 0.$$

Then D is holomorphically non-extendable (hence, locally L-convex) at ζ.

Proof The function $f^{-1} = \frac{1}{f}$ is holomorphic on $U \cap D$, $f^{-1}(\zeta) = \infty$. This implies that the function $f^{-1}(w)$ does not extend holomorphically to ζ, by definition. Therefore, D is not holomorphically extendable to ζ, and hence, it is L-convex there. The theorem is proved. $\square$

7.5 Material of seminar

As an application of the Continuity Principle, let us prove the following lemma on erasing real singularities of holomorphic functions in two complex variables.

Lemma 7.34 *Let $D \subset \mathbb{C}^2$ be a domain intersecting $\mathbb{R}^2$. Each holomorphic function $f : D \setminus \mathbb{R}^2 \to \mathbb{C}$ extends holomorphically to all of D.*

Proof It suffices to treat the case, when $D = \Delta_{\delta,\delta} = \Delta(0, \delta)$, and prove that each holomorphic function $f : D \setminus \mathbb{R}^2$ extends holomorphically to the origin. To do this, consider the family of parabolas

$$S_t = \{w = i(z^2 + t)\} \cap \{|z| \leq \frac{\delta}{4}\}, \; 0 \leq t \leq \frac{\delta}{4}, \; S := S_0.$$

The sets S_t are one-dimensional compact surfaces.

Claim. *One has $S_t \cap \mathbb{R}^2 = \emptyset$ for $t > 0$; $S_0 \cap \mathbb{R}^2 = \{0\}$.*

Proof Let $(z, w) \in S_t \cap \mathbb{R}^2$. Then $z^2 + t \geq 0$, hence $w \in \mathbb{R} \cap i\mathbb{R} = \{0\}$, $w = 0 = z^2 + t$. The latter equality holds only for $z = t = 0$. This proves the claim. $\square$

The surfaces S_t with $t > 0$ are contained in $D \setminus \mathbb{R}^2$ and converge to the surface $S = S_0$ passing through $0 \in \partial(D \setminus \mathbb{R}^2)$ with boundaries, and $\partial S \subset D \setminus \mathbb{R}^2$. Therefore, each holomorphic function on $D \setminus \mathbb{R}^2$ extends holomorphically to a neighborhood of the surface S, and hence, to the origin (Continuity Principle). This proves the lemma. □

Remark 7.35 One can prove the lemma by extending the functions to S as Cauchy integrals along the surfaces S_t, without using the Continuity Principle. That is, consider the new coordinates $(z, \widetilde{w})$, $\widetilde{w} = w - iz^2$, in which the parabolas S_t are discs $\widetilde{S}_t = \{\widetilde{w} = it\}$. Then the Cauchy formula for a function f written via integrating along the boundaries ∂S_t depends holomorphically on $\widetilde{w}$ and defines a holomorphic extension of the function f to $\widetilde{S}_0$.

Exercise. Prove higher-dimensional analogue of Lemma 7.34.

7.6 Levi form. Necessary and sufficient Levi convexity conditions for domains with C^2-smooth boundary

Here we consider a domain $D \subset \mathbb{C}^n$ and a point $\zeta \in \partial D$ where the boundary is C^2-smooth. That is, there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a C^2-function $\phi : U \to \mathbb{R}$ such that

$$\phi(\zeta) = 0, \ D \cap U = \{\phi < 0\}, \ d\phi(\zeta) \neq 0. \tag{7.4}$$

We give necessary and sufficient conditions for L-convexity of the domain D at ζ in terms of the Hessian of the function ϕ: positive (non-negative) definiteness of an appropriate Hermitian form called the Levi form.

The *extended Levi form* is the Hermitian quadratic form on $T_\zeta\mathbb{C}^n$ defined by the formula

$$\widetilde{L} = \sum_{j,s=1}^{n} \frac{\partial^2 \phi}{\partial \bar{z}_s \partial z_j}(\zeta) dz_j \overline{dz_s}. \tag{7.5}$$

By definition, this is the quadratic form associated to the Hermitian inner product

$$\widetilde{L}(v_1, v_2) = \sum_{j,s=1}^{n} \frac{\partial^2 \phi}{\partial \bar{z}_s \partial z_j}(\zeta) v_{1,j} \bar{v}_{2,s}, \tag{7.6}$$

where $v_j = (v_{j,1} \dots, v_{j,n})$. It is defined by the (1,1)-Hessian matrix $(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_s})$, whose Hermitianity will be proved below.

Let $H \subset T_\zeta \partial D$ denote the maximal complex vector subspace. The *Levi form* is the restriction

$$L = \widetilde{L}|_H.$$

Theorem 7.36 (Levi–Krzoska). *Let D, U, ζ, ϕ be the same, as in (7.4). Let H and L be the same, as above.*

1) Let L be positive definite. Then D is holomorphically non-extendable at ζ.

2) Let D be locally L-convex at ζ. Then L is non-negatively definite.

Before the proof of Theorem 7.36 we recall some preparatory material and show that the Levi form is indeed a Hermitian form.

The differential of every complex-valued function g on a domain $U \subset \mathbb{C}^n$ is the sum of its $\mathbb{C}$-linear part and its $\mathbb{C}$-antilinear part:

$$dg = \partial g + \bar{\partial} g;$$

$$\partial g(z) : T_z\mathbb{C}^n \to \mathbb{C} \text{ is } \mathbb{C}- \text{ linear}, \qquad \bar{\partial} g(z) : T_z\mathbb{C}^n \to \mathbb{C} \text{ is } \mathbb{C}- \text{ antilinear},$$

$$\partial g = \sum_{j=1}^{n} \frac{\partial g}{\partial z_j} \partial z_j, \ \bar{\partial} g = \sum_{j=1}^{n} \frac{\bar{\partial} g}{\partial \bar{z}_j} \overline{\partial z_j}.$$

Proposition 7.37 *Let ϕ be a real-valued function. Then*

$$(\bar{\partial}\phi(\zeta))(v) = \overline{(\partial\phi(\zeta))(v)} \text{ for every } v \in T_\zeta\mathbb{C}^n. \tag{7.7}$$

If $d\phi(\zeta) \neq 0$, then one has $H = \operatorname{Ker} \partial\phi(\zeta)$.

Proof The differential $d\phi$ is real-valued, as is ϕ. Therefore, the forms $\bar{\partial}\phi(\zeta)$ and $\partial\phi(\zeta)$ have opposite imaginary parts. Thus, the difference $\overline{\bar{\partial}\phi(\zeta)} - \partial\phi(\zeta)$ is a $\mathbb{C}$-linear form on $T_\zeta\mathbb{C}^n$ with identically zero imaginary part. Hence, it is identically zero. This proves (7.7), which in its turn implies that $d\phi(\zeta) = 2 \operatorname{Re} \partial\phi(\zeta)$. Let now $d\phi(\zeta) \neq 0$. Then the differential $\partial\phi(\zeta)$ is non-zero and hence, its kernel is a complex hyperplane. Its real part $\frac{1}{2} d\phi(\zeta)$ vanishes identically on the complex hyperplane H. Hence, $\partial\phi(\zeta) \equiv 0$ on H and $H = \operatorname{Ker} \partial\phi(\zeta)$. The claim is proved. □

Proposition 7.38 *Let ϕ be a real-valued function on a neighborhood of a point $\zeta \in \mathbb{C}^n$ that is C^2-smooth at ζ. Consider a biholomorphic system of coordinates $(z_1, \dots, z_n)$ centered at ζ. The asymptotic Taylor formula for the function ϕ at ζ takes the form*

$$\phi(z) = 2 \operatorname{Re}(\partial\phi(\zeta)z + Q^{2,0}(z)) + \widetilde{L}(z, z) + o(||z||^2). \tag{7.8}$$

Here $\widetilde{L}(z,z)=\sum_{j,s=1}^{n}\frac{\partial^2\phi}{\partial\bar{z}_s\partial z_j}(\zeta)z_j\bar{z}_s$ is the extended Levi form (7.5) written in the coordinates z and evaluated on the Euler vector field $z=(z_1,\dots,z_n)$. It is real-valued and obtained from a Hermitian bilinear form: its matrix $(\frac{\partial^2\phi}{\partial\bar{z}_s\partial z_j}(\zeta))$ is Hermitian. The form $Q^{2,0}$ is a quadratic form defined by a $\mathbb{C}$-bilinear form.

Proof The linear and the quadratic Taylor polynomials of the function $\phi(z)$ are real-valued, as is ϕ. The linear terms are given by $d\phi(\zeta)(z)=2\operatorname{Re}(\partial\phi(\zeta)z)$, by (7.7). Thus, its homogeneous quadratic part is also real-valued. It is the sum of three components: a $\mathbb{C}$-bilinear quadratic form $Q^{2,0}$ (a linear combination of products z_jz_k), a $\mathbb{C}$-bi-antilinear quadratic form $Q^{0,2}$ (a linear combination of $\bar{z}_j\bar{z}_k$) and a so-called $(1,1)$-quadratic form $Q^{1,1}$ that is a linear combination $z_j\bar{z}_k$. This decomposition is unique. Complex conjugation preserves the homogeneous quadratic part, sends $Q^{0,2}$ to a $\mathbb{C}$-bilinear quadratic form, $Q^{2,0}$ to $\mathbb{C}$-bi-antilinear quadratic form and $Q^{1,1}$ to a $(1,1)$-quadratic form. Therefore, it fixes $Q^{1,1}$ and permutes $Q^{2,0}$ and $Q^{0,2}$. The asymptotic Taylor formula shows that $Q^{1,1}(z)=\widetilde{L}(z,z)$. This proves (7.8). Hermitianity of the matrix $(\frac{\partial^2\phi}{\partial\bar{z}_s\partial z_j}(\zeta))$ of the form $\widetilde{L}$ follows from the latter statement and invariance of the form $Q^{1,1}(z)$ under conjugacy. The proposition is proved. □

Let us give a coordinate-independent equivalent definition of the extended Levi form. Given a C^2-function $\phi:U\to\mathbb{R}$ and a $\zeta\in U$, we define a Hermitian form $\widetilde{L}(v_1,v_2)$ on $T_\zeta\mathbb{C}^n$ as follows. For given $v_1,v_2\in T_\zeta\mathbb{C}^n$ let us take two arbitrary germs at ζ of holomorphic vector fields $u_1(z)$, $u_2(z)$ such that $u_j(\zeta)=v_j$. Set

$$g(z):=(\partial\phi(z))(u_1(z)),\ \psi(z):=(\bar{\partial}g(z))(u_2(z)),\quad \widetilde{L}(v_1,v_2):=\psi(\zeta). \quad (7.9)$$

Proposition 7.39 *The value $\widetilde{L}(v_1,v_2)$ is well-defined: it depends only on $v_1,v_2\in T_\zeta\mathbb{C}^n$ and does not depend on the choice of vector fields u_j. It is given by an Hermitian form $\widetilde{L}$ on $T_\zeta\mathbb{C}^n$. In local holomorphic coordinates $(z_1,\dots,z_n)$ centered at ζ the latter form is given by (7.5), (7.6).*

Proof It suffices to prove the coordinate presentation (7.5): the well-definedness then follows immediately, and Hermitianity follows from Proposition 7.38. One has

$$g(z)=\sum_{j=1}^{n}\frac{\partial\phi}{\partial z_j}(z)u_j(z).$$

Taking $\bar{\partial}$-differential of the latter right-hand side results in differentiating only the partial derivatives of the function ϕ: $\bar{\partial} u_j = 0$, since u_j are holomorphic. This implies that $\psi(z) = (\bar{\partial} g(z))(u_2(z))$ equals the value of the Hermitian form (7.5) on the pair of vector fields $(u_1(z), u_2(z))$. Taking the value $\psi(\zeta) = \widetilde{L}(v_1, v_2)$ yields (7.6). The proposition is proved. $\Box$

In what follows we will use the invariance of the (extended) Levi form under holomorphic mappings.

Proposition 7.40 *The extended Levi form associated to a function ϕ is invariant under holomorphic mappings. That is, let $W \subset \mathbb{C}^r$, $h : W \to D \subset \mathbb{C}^n$ be a holomorphic mapping, $\phi : D \to \mathbb{R}$ be a C^2-function. Let $\widetilde{L}_\phi$ and $\widetilde{L}_{\phi\circ h}$ be respectively the extended Levi forms associated to the functions ϕ and $\phi \circ h$. Then for every $z \in W$ and vectors $v_1, v_2 \in T_z\mathbb{C}^n$ one has*

$$\widetilde{L}_{\phi\circ h}(v_1, v_2) = \widetilde{L}_\phi((dh(z))(v_1), (dh(z))(v_2)). \tag{7.10}$$

Proof Consider the invariant definition (7.9) of the extended Levi form $\widetilde{L}_{\phi\circ h}$ at $\zeta \in W$ with u_j being holomorphic vector fields on a neighborhood of ζ, $u_j(\zeta) = v_j$. One has

$$g(z) = (\partial\phi(h(z)))(dh(z)u_1(z)), \ \psi(z) = (\bar{\partial} g(z))(u_2(z)).$$

The function $g(z)$ is a linear combination of partial derivatives of the function ϕ with coefficients being holomorphic functions: the components of the holomorphic vector function $(dh(z))u_1(z)$. Taking its $\bar{\partial}$-derivative along the field $u_2(z)$ results in differentiating the derivatives of the function ϕ only and subsequent multiplying them by the components of the vector function $\overline{(dh(z))u_2(z)}$, by holomorphicity. This implies (7.10) and proves the proposition. $\Box$

Proposition 7.41 *Let ϕ be a germ of real-valued C^2-function on a neighborhood of the origin in $\mathbb{C}^n$. Let $\phi(0) = 0$, $d\phi(0) \neq 0$. Let $\widetilde{L}$ denote the corresponding extended Levi form on $T_0\mathbb{C}^n$. Then in appropriate local biholomorphic coordinates $z = (z_1, \dots, z_n)$ centered at 0 the function ϕ takes the form*

$$\phi(z) = \operatorname{Re} z_1 + \widetilde{L}(z, z) + o(|z|^2), \ \text{as } z \to 0. \tag{7.11}$$

Here we take the value of the extended Levi form on the Euler vector field $z = (z_1, \dots, z_n)$.

Proof Let us introduce coordinates $(z_1, \dots, z_n)$ such that $H = \{z_1 = 0\}$. We normalize z_1 so that $\partial\phi(0) = \frac{1}{2}dz_1$. Then

$$\phi(z) = \mathrm{Re}(z_1 + 2Q^{2,0}(z)) + \widetilde{L}(z, z) + o(||z||^2),$$

by (7.8). Replacing the coordinate z_1 by $\widetilde{z}_1 := z_1 + 2Q^{2,0}(z)$ yields (7.11). $\square$

Proof of Theorem 7.36. Let us prove Statement 1). Let $L > 0$. Consider the local coordinates z centered at ζ, $z(\zeta) = 0$ satisfying (7.11).

Claim. *There exists a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ such that $z_1 \neq 0$ on $D \cap U$.*

Proof Set $w = (z_2, \dots, z_n)$. We have to show that the intersection of the domain D with the coordinate w-subspace $H_{\mathbb{C}}$ does not accumulate to $0 = \zeta$. Or equivalently, $\phi|_{H_{\mathbb{C}}} \geq 0$ on a neighborhood of the origin in $H_{\mathbb{C}}$. One has

$$\phi|_{H_{\mathbb{C}}} = L(w, \bar{w}) + o(|w|^2), \tag{7.12}$$

by (7.11). This together with positive definiteness of the Levi form L on $H_{\mathbb{C}}$ implies non-negativity of the latter right-hand side on a neighborhood of the origin in $H_{\mathbb{C}}$. The claim is proved. $\square$

The function z_1 vanishes at $\zeta \in \partial D$ and does not vanish on $D \cap U$. This together with Theorem 7.33 implies holomorphic non-extendability of the domain D at ζ. Statement 1) is proved.

Let us now prove Statement 2). Suppose the contrary: $L(v, \bar{v}) = -c < 0$ for some $v \in H_{\mathbb{C}}$. Let us show that D is not L-convex. Without loss of generality we consider that $v = (0, 1, 0, \dots, 0)$ (applying a linear change of coordinates $w = (z_2, \dots, z_n)$, which does not change (7.11)). For a small $\delta > 0$ set

$$S = \{|z_2| \leq \delta, \ z_1 = z_3 = \dots, z_n = 0\}.$$

We show that $\partial S \subset D$ and the compact holomorphic curve S is the limit of a family of compact holomorphic curves S_k on which $\phi < 0$, hence $S_k \subset D$. This will imply that D is not L-convex.

For every δ small enough one has $\phi|_S \leq 0$ and $\phi|_{\partial S} < 0$, by (7.12), as in the above proof of Statement 1). Hence, $\partial S \subset D$. For every $k \in \mathbb{N}$ set

$$S_k = S - (\frac{1}{k}, 0, \dots, 0): \text{ the curve } S \text{ shifted by the vector } (-\frac{1}{k}, 0, \dots, 0).$$

The Levi form $\widetilde{L}(z, z)$ restricted to S is equal to $-c|z_2|^2$ for appropriate $c > 0$. One has

$$\phi|_{S_k} = -\frac{1}{k} - c|z_2|^2 + O(\frac{|z_2|}{k}) + o(|z_2|^2) + o(\frac{1}{k^2}) = -\frac{1}{k}(1 + O(z_2)) - c|z_2|^2(1 + o(1)),$$

by definition and (7.11). The latter right-hand side is negative for $|z_2| < \delta$, whenever δ is small enough and k is big enough. Therefore, $\phi|_{S_k} < 0$, hence $S_k \subset D$. This together with the previous discussion proves Statement 2) and the Theorem. $\square$

7.7 Subharmonic functions and L-convexity

Levi–Krzoska Theorem gives a sufficient condition for L-convexity of a domain with C^2-smooth boundary: strict positivity of the Levi form. Here we show that a domain is automatically L-convex (and hence, a domain of holomorphy), if it is a sublevel set of a function from a specific class: the plurisubharmonic functions. The corresponding Levi forms are nonnegative definite but not necessarily strictly positive definite. The plurisubharmonic functions are natural generalizations of the subharmonic functions in one complex variable. They have important applications. For example, the proof of one of the most fundamental theorems of geometry, the Poincaré–Köbe Uniformization Theorem, is based on use of subharmonic functions.

Definition 7.42 A C^2-function $\phi : V \to \mathbb{R}$ on a domain $V \subset \mathbb{C}$ is *harmonic (subharmonic)*, if for every $z_0 \in V$ and every $r > 0$ small enough (depending on z_0, in particular, such that $\overline{D_r(z_0)} \subset V$) one has

$$\phi(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta, \tag{7.13}$$

respectively,

$$\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta. \tag{7.14}$$

Recall that the Laplacian of a function in one complex variable is expressed in terms of ∂ and $\overline{\partial}$ operators as follows:

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

Proposition 7.43 *A C^2-function of one complex variable is harmonic, if and only if it satisfies the Laplace equation $\Delta\phi = 0$, or equivalently,*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \phi = 0.$$

A C^2-function of one complex variable is subharmonic, if and only if its Laplacian is nonnegative:

$$\frac{\partial^2}{\partial z \partial \bar{z}} \phi \geq 0. \tag{7.15}$$

A C^2-function on a domain $V \subset \mathbb{C}$ is (sub)harmonic, if and only if for every point $z_0 \in V$ equality (7.13) (inequality (7.14)) holds for every $r > 0$ such that $\overline{D_r(z_0)} \subset V$.

Proof Let us prove the second statement, on the subharmonic functions. It will imply the statement on harmonic functions, being applied to $\pm\phi$.

Step 1): subharmonicity implies non-negativity of the Laplacian. Fix an arbitrary $z_0 \in V$, and let us prove that $\frac{\partial^2}{\partial z \partial \bar{z}} \phi(z_0) \geq 0$. Let us choose the affine coordinate z centered at z_0 and write Taylor expansion of the function ϕ at $z_0 = 0$:

$$\phi(z) = \phi(0) + az + \overline{az} + cz^2 + \overline{cz^2} + dz\bar{z} + o(|z|^2), \ a, c \in \mathbb{C}, \ d = \frac{\partial^2}{\partial z \partial \bar{z}} \phi(0) \in \mathbb{R}.$$

It suffices to show that $d \geq 0$. The non-negative difference of the right- and left-hand sides in inequality (7.14) is equal to the integral

$$\frac{1}{2\pi} \int_0^{2\pi} (\psi(z) + dz\bar{z} + o(|z|^2)) d\theta, \ z = e^{i\theta}, \ \psi(z) = 2 \operatorname{Re}(az + cz^2).$$

The integral of the function $\psi(z)$ vanishes, since ψ is a linear combination of the exponents $e^{\pm i\theta}$, $e^{\pm 2i\theta}$. The integral of the function $dz\bar{z}$ equals dr^2, and it dominates the integral of the third function. Therefore, if $d < 0$, then the total integral is negative, – a contradiction. Hence, $d \geq 0$.

Step 2): non-negativity of the Laplacian implies subharmonicity. Let $\frac{\partial^2}{\partial z \partial \bar{z}} \phi \geq 0$. Let us prove inequality (7.14) at a given point z_0. We choose affine coordinate z centered at z_0, thus we consider that $z_0 = 0$. Set

$$g(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(re^{i\theta}) d\theta, \ \ \overline{D_r(z_0)} \subset V.$$

Claim. *The function $g(r)$ is non-decreasing.*

Proof One has

$$\frac{\partial g}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial \phi}{\partial z} e^{i\theta} + \frac{\partial \phi}{\partial \bar{z}} e^{-i\theta} \right) d\theta = \frac{1}{2\pi i r} \oint_{|z|=r} \left(\frac{\partial \phi}{\partial z} dz - \frac{\partial \phi}{\partial \bar{z}} \overline{dz} \right)$$

$$= -\frac{1}{\pi i r} \int_{|z|<r} \frac{\partial^2}{\partial z \partial \bar{z}} \phi dz \wedge \overline{dz} = \frac{2}{\pi r} \int_{|z|<r} \frac{\partial^2}{\partial z \partial \bar{z}} \phi dx \wedge dy \geq 0.$$

This proves the claim. $\square$

One has $g(0) = \phi(0)$, hence $g(r) \geq \phi(0)$. This proves inequality (7.14) and the second step, which in its turn implies the third statement of the proposition and finishes its proof. $\square$

Remark 7.44 The general definition of subharmonic function does not require even continuity: only upper semicontinuity and inequality (7.14) are required. They are defined as functions with values in $\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. For example, the function $\ln|z|$ is harmonic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and subharmonic on $\mathbb{C}$: the mean inequality (7.14) holds at the origin, where the function equals minus infinity. This is a continuous $\widehat{\mathbb{R}}$-valued function. The series

$$\sum_{k=1}^{+\infty} \frac{1}{k^3} \ln|z - \frac{1}{k}|$$

defines a subharmonic function on $\mathbb{C}$ that is discontinuous at the origin.

Remark 7.45 The motivation of the term "subharmonic" is the following. Consider the Dirichlet problem to find a harmonic function f on a domain $V \subset \mathbb{C}$ that is continuous on its closure and satisfies the boundary condition $f|_{\partial V} = \psi$, where $\psi : \partial V \to \mathbb{R}$ is a given continuous function. Let now ϕ be a subharmonic function on V that satisfies the same boundary condition $\phi|_{\partial V} = \psi$. Then $\phi \leq f$ on V. In the case, when V is a disk, the above inequality at the center of the disk V follows immediately from the mean inequality and equality of the boundary values of the harmonic and harmonic functions in question. For every other point $z_0 \in V$ the inequality $\phi(z_0) \leq f(z_0)$ follows from the same inequality at the center of the disk by applying conformal automorphism of the disk V sending z_0 to its center: the notions of harmonic and subharmonic functions are invariant under pre-compositions with holomorphic maps. Vice versa, take an arbitrary continuous function ϕ on a domain $W \subset \mathbb{C}$. Take an arbitrary disk $D = D_r(z_0)$, $\overline{D} \subset W$, and the harmonic extension $f : D \to \mathbb{R}$ of the restriction $\phi|_{\partial D}$. Let for every D as above one have $\phi \leq f$ in D. Then ϕ is subharmonic.

Theorem 7.46 (Maximum Principle for subharmonic functions). *Let $V \subset \mathbb{C}$ be a bounded connected domain, $\phi : V \to \mathbb{R}$ be a subharmonic function continuous on $\overline{V}$. Then*

$$\max_{\overline{V}} \phi = \max_{\partial V} \phi.$$

If ϕ achieves its maximum at an interior point $z_0 \in V$, then it is constant.

Proof Let ϕ take a maximum at $z_0 \in V$. Then inequality (7.14) implies that $\phi \equiv \phi(z_0)$ on a neighborhood of the point z_0. Applying this statement to any point of the level subset $\{\phi = \phi(z_0))\} \subset V$, we get that it is open and closed. Hence, it coincides with V (connectivity). Theorem 7.46 is proved. □

Definition 7.47 Let $D \subset \mathbb{C}^n$ be a domain. A C^2 function $\phi : D \to \mathbb{R}$ is *pluri(sub)harmonic,* if for every complex line $\Lambda \subset \mathbb{C}^n$ the restriction to $\Lambda \cap D$ of the function ϕ is (sub)harmonic.

Proposition 7.48 *A C^2-function $\phi : D \to \mathbb{R}$ is pluriharmonic, if and only if the corresponding extended Levi form $\widetilde{L}$ vanishes identically. A C^2-function $\phi : D \to \mathbb{R}$ is plurisubharmonic, if and only if its extended Levi form $\widetilde{L}$ is non-negative definite at each point in D.*

Proof Let $\Lambda \subset \mathbb{C}^n$ be an arbitrary complex line. Consider a system of affine coordinates $(z_1, \dots, z_n)$ such that Λ is the z_1-axis. Let $\zeta \in \Lambda$, $v_\zeta = \frac{\partial}{\partial z_1} \in T_\zeta\Lambda$ denote the unit vector directing the z_1-axis. One has

$$\frac{\partial^2 \phi}{\partial z_1 \partial \bar{z}_1}(\zeta) = \widetilde{L}(v_\zeta, v_\zeta),$$

by definition. Therefore, the latter derivative is zero (non-negative) for all Λ and $\zeta \in \Lambda \cap D$, if and only if the extended Levi form vanishes identically (respectively, non-negative definite) at each point in D. The proposition is proved. □

Corollary 7.49 *The notion of pluri(sub)harmonicity is invariant under holomorphic mappings. Namely, the composition $\phi \circ h$ of a pluri(sub)harmonic function ϕ with a holomorphic mapping h is pluri(sub)harmonic.*

The corollary follows from Propositions 7.48 and 7.40.

Corollary 7.50 *The restriction of a pluri(sub)harmonic function to a compact holomorphic surface is pluri(sub)harmonic.*

Remark 7.51 In fact, the notion of plurisubharmonic function is more general than in the above definition: it includes discontinuous functions. The general definition requires only upper semicontinuity on the definition domain and subharmonicity of restrictions to complex lines.

Theorem 7.52 (Maximum Principle for plurisubharmonic functions). *Let $\phi : D \to \mathbb{R}$ be a plurisubharmonic function, and let $S \subset D$ be a compact holomorphic surface parametrized by closure of a connected domain in $\mathbb{C}^r$. Then*

$$\max_S \phi = \max_{\partial S} \phi.$$

Or equivalently, if the function $\phi|_S$ achieves its maximum in the interior of the surface S, then it is constant.

Proof Let $h : \overline{W} \to S$ be a holomorphic parametrization by a connected domain $W \subset \mathbb{C}^r$ with compact closure. The function $g = \phi \circ h$ is plurisubharmonic. Suppose the contrary: it achieves a maximum at a point $\zeta \in W$. Then its restriction to each line through ζ is a subharmonic function achieving a local maximum at ζ and hence, is equal to the same constant $g(\zeta)$ for all lines. Finally, the function g is constant on a neighborhood of the point ζ. The above argument shows that the level set $\{g = g(\zeta)\}$ is open, and it is closed by continuity. This together with connectivity implies that the latter level set coincides with all of W, hence $g \equiv const$. The theorem is proved. □

Theorem 7.53 *Let $D \subset \mathbb{C}^n$ be a domain, $\zeta \in \partial D$. Let there exist a neighborhood $U = U(\zeta) \subset \mathbb{C}^n$ and a plurisubharmonic function $\phi : U \to \mathbb{R}$ such that $\phi(\zeta) = 0$ and $D \cap U = \{\phi < 0\}$. Then D is locally L-convex at ζ.*

Proof Suppose the contrary: D is not locally L-convex at ζ. Then there exists a compact holomorphic surface $S \subset U$ through ζ such that $\partial S \subset D \cap U$. Thus, $\phi|_S$ is a plurisubharmonic function such that $\phi|_{\partial S} < 0$ and $\phi(\zeta) = 0$, – a contradiction to the Maximum Principle. The theorem is proved. □

Corollary 7.54 *Let $\phi : V \to \mathbb{R}$ be a plurisubharmonic function. Let $D = \{\phi < 0\} \Subset V$ (i.e., $\overline{D} \subset V$). Then D is L-convex, and hence, a domain of holomorphy.*

Proof The domain D is locally L-convex by Theorem 7.53. Its proof applied for $U = V$ implies (global) L-convexity of the domain D. This together with Theorems 7.22 and 7.32 implies that D is a domain of holomorphy. □

8 Stein manifolds

8.1 Stein manifolds: definition and main properties

Definition 8.1 A complex manifold M is said to be *holomorphically convex*, if the holomorphically convex hull ($H(M)$-hull) of each its compact subset is a compact subset. We say that the holomorphic functions on M *separate points,* if for every two distinct points $x \neq y$ in M there exists a holomorphic function $f : M \to \mathbb{C}$ such that $f(x) \neq f(y)$.

Definition 8.2 A complex manifold M, set $n = dimM$, is said to be a *Stein manifold,* if it satisfies the following conditions:

1) M is holomorphically convex;

2) the holomorphic functions on M separate points;

3) for every $z \in M$ there exist n holomorphic functions $f_1, \dots, f_n$ on M whose differentials at z are linearly independent: that is, the holomorphic vector function $(f_1, \dots, f_n) : M \to \mathbb{C}^n$ is a local biholomorphism at z.

Example 8.3 Every domain of holomorphy in $\mathbb{C}^n$ is obviously Stein. Let now $M \subset \mathbb{C}^N$ be a holomorphic submanifold. Then it is Stein. Indeed, let H denote the collection of projections $M \to \mathbb{C}$ to the coordinate axes, which are holomorphic functions. The functions from class H obviously separate points. Condition 3) holds even for functions from class H. Finally, the H-convex hull of each compact subset in M is a bounded closed subset in $\mathbb{C}^N$ contained in M. Hence, it is compact. The next big theorem states the converse.

Theorem 8.4 (Embedding Theorem). *Each Stein manifold can be embedded as a submanifold in $\mathbb{C}^N$ for appropriate N. This is true for $N = 2n + 2$, where n is the dimension of the manifold.*

Corollary 8.5 *Every domain of holomorphy in $\mathbb{C}^n$ can be embedded as a submanifold in $\mathbb{C}^{2n+2}$. In particular, every geometrically convex domain in $\mathbb{C}^n$ can be embedded as a submanifold in $\mathbb{C}^{2n+2}$.*

Example 8.6 The unit disk $D_1 \subset \mathbb{C}$ is Stein. It admits an embedding as a submanifold in $\mathbb{C}^N$. The proof of this statement is non-trivial. In Problems 5 and 6 from Task 3 it is asked to prove the existence of its embedding to $\mathbb{C}^2$ by using polynomial automorphisms with Fatou–Bieberbach domains. Namely, consider a Fatou–Bieberbach domain $B \subset \mathbb{C}^2$ that is an attracting basin of a linearizable attractive fixed point of a polynomial automorphism $F : \mathbb{C}^2 \to$

$\mathbb{C}^2$. We choose F so that $B \neq \mathbb{C}^2$. Recall that B is always biholomorphic to $\mathbb{C}^2$. Consider a complex line L intersecting B and passing through its boundary point. One can show that each connected component U of the intersection $L \cap B$ is simply connected, applying the Maximum Principle to the iterates F^m. Therefore, $U \subset L \simeq \mathbb{C}$ is conformally equivalent to the unit disk. This together with biholomorphism $B \simeq \mathbb{C}^2$ yields an embedding $U \to \mathbb{C}^2$ as a submanifold.

9 Dolbeault cohomology

9.1 Basic definitions and $\bar{\partial}$-Poincaré Lemma

Here we introduce the $\bar{\partial}$-complex of differential forms, analogous to de Rham complex.

Recall that each $\mathbb{R}$-linear $\mathbb{C}$-valued 1-form on a complex space $\mathbb{C}^n$ is a linear combination of the forms dz_j and their complex conjugates $\overline{dz_j}$ with constant complex coefficients. Thus, each $\mathbb{R}$-linear 1-form is a sum of two 1-forms: a $\mathbb{C}$-linear form (linear combination of dz_j) and a $\mathbb{C}$-antilinear form (linear combination of $\overline{dz_j}$). Therefore, the space of $\mathbb{R}$-linear $\mathbb{C}$-valued 1-forms is the direct sum $\Lambda^{1,0} \oplus \Lambda^{0,1}$, where $\Lambda^{1,0}$ is the space of $\mathbb{C}$-linear forms and $\Lambda^{0,1}$ (its complex conjugate) is the space of $\mathbb{C}$-antilinear forms.

Passing to exterior powers, we get that the space of $\mathbb{R}$-polylinear $\mathbb{C}$-valued skew-symmetric d-forms is the direct sum

$$\oplus_{p+q=d}\Lambda^{p,q}, \ \Lambda^{p,q} = (\wedge_{j=1}^{p}\Lambda^{1,0}) \wedge (\wedge_{s=1}^{q}\Lambda^{0,1}).$$

Passing to differential forms on a complex manifold M, we get that the space Ω^d of C^∞-smooth differential d-forms is the direct sum

$$\Omega^d = \oplus_{p+q=d}\Omega^{p,q},$$

$$\Omega^{p,q} = \{\text{forms whose restriction to each tangent space } T_xM \text{ lies in } \Lambda^{p,q}\}.$$

Recall that for a function $g(z)$ on a complex manifold M, we denote by $\bar{\partial}g$ the $\mathbb{C}$-antilinear part of its differential at each point:

$$dg(x) = \partial g(x) + \bar{\partial}g(x).$$

Thus the image of the $\bar{\partial}$-operator on functions lies in the space $\Omega^{0,1}$. The $\bar{\partial}$-operator acting on differential forms is defined analogously. Namely, for every $\mathbb{R}$-linear $\mathbb{C}$-valued d-form $\omega \in \Omega^{p,q}$ one has

$$d\omega \in \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \tag{9.1}$$

Indeed, each term of the form ω is locally a linear combination of forms

$$dz_{i_1} \wedge \ldots dz_{i_p} \wedge \overline{dz_{j_1}} \wedge \cdots \wedge \overline{dz_{j_q}}.$$

Taking differential transforms ω to sum of the above forms wedge-multiplied by 1-forms: differentials of functional coefficients. Each 1-form multiplier clearly lies in $\Omega^{1,0} + \Omega^{0,1}$. This implies (9.1). Set

$$\bar{\partial}\omega := \text{ the } \Omega^{p,q+1} - \text{component of } d\omega; \ \partial\omega := \text{ its } \Omega^{p+1,q} - \text{component:}$$

$$d\omega = \partial\omega + \bar{\partial}\omega.$$

This yields the *Dolbeault $\bar{\partial}$-complex*

$$\bar{\partial} : \Omega^{p,q} \mapsto \Omega^{p,q+1}.$$

Proposition 9.1 *One has $\bar{\partial}^2 = 0$.*

Proof One has $d = \partial + \bar{\partial}$. Hence,

$$d^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

For every (p,q)-form ω the three forms $\partial^2\omega$, $(\partial\bar{\partial}+\bar{\partial}\partial)\omega$, $\bar{\partial}^2\omega$ lie in the vector subspaces $\Omega^{p+2,q}$, $\Omega^{p+1,q+1}$ and $\Omega^{p,q+2}$ respectively. Therefore, $\bar{\partial}^2\omega$ is the $\Omega^{p,q+2}$-component of $d^2\omega$. Hence, it vanishes, since $d^2\omega = 0$. $\Box$

Definition 9.2 A differential form ω is *$\bar{\partial}$-closed,* if $\bar{\partial}\omega = 0$. A form is *exact,* if it is the image of another form under the $\bar{\partial}$-operator. The (p,q)-th *Dolbeault cohomology* is the group

$$H^{p,q} := \operatorname{Ker} \bar{\partial}|_{\Omega^{p,q}} / \operatorname{Im}(\bar{\partial}(\Omega^{p,q-1}).$$

Remark 9.3 Each $\bar{\partial}$-exact form is $\bar{\partial}$-closed, since $\bar{\partial}^2 = 0$.

Remark 9.4 Every holomorphic mapping of complex manifolds $G : M \to N$ induces the pullback mapping $G^* : H^{p,q}(N) \to H^{p,q}(M)$. The Dolbeault cohomology of biholomorphically equivalent manifolds are isomorphic.

The $\bar{\partial}$-problem has the following versions:

- Given a function $g(z)$ of one variable, find a function f such that

$$\frac{\bar{\partial} f}{\partial \bar{z}} = g. \tag{9.2}$$

- Given a $\bar\partial$-closed differential (p,q)–form ω, find an $(p,q-1)$-form α such that

$$\bar\partial\alpha = \omega. \tag{9.3}$$

The obstruction to solve the $\bar\partial$-problem is non-triviality of the Dolbeault cohomology class of the form ω.

The results on $\bar\partial$-problem form an important base for many famous results in complex analysis and related topics such as quasiconformal mappings, Teichmüller theory, moduli spaces of Riemann surfaces, algebraic geometry. It has many applications in the above-mentioned domains and complex dynamics.

Theorem 9.5 ($\bar\partial$-Poincaré Lemma). *For every $(p,q)\in\mathbb{Z}^2_{\geq 0}$ with $q>0$ the (p,q)-cohomology of polydisk is trivial: each $\bar\partial$-closed (p,q)-form is exact.*

Remark 9.6 The $\bar\partial$-Poincaré Lemma has important corollaries in complex geometry. For example, it allows to prove that every holomorphic hypersurface in a polydisk Δ (i.e., an analytic subset in Δ given locally as zero locus of one local holomorphic function) is a zero locus of a global holomorphic function $\Delta\to\mathbb{C}$. The proof of this statement (omitted in the present lectures) is based on solution of holomorphic additive Cousin problem. See the next subsection.

Theorem 9.7 (Generalized $\bar\partial$-Poincaré Lemma). *Each Stein manifold has trivial (p,q)-cohomology for every $(p,q)\in\mathbb{Z}^2_{\geq 0}$ with $q>0$.*

We will not prove Theorem 9.7 in this lectures.

9.2 Proof of $\bar\partial$-Poincaré Lemma in one variable

First we treat the $\bar\partial$-problem for functions. Then we introduce $\bar\partial$-operator acting on differential forms, which defines Dolbeault complex and cohomology. We solve the corresponding $\bar\partial$-problem on polydisk by proving triviality of Dolbeault cohomology ($\bar\partial$-Poincaré Lemma). Afterwards we apply this result together with elements of sheaf theory to show that each hypersurface in a polydisk is the zero locus of a global holomorphic function.

Theorem 9.8 *For every C^∞ function $g: D_1\to\mathbb{C}$ on the unit disk $D_1\subset\mathbb{C}$ there exists a C^∞ function $f: D_1\to\mathbb{C}$ satisfying (9.2).*

Addendum. *Let the function $g=g(z,w,s)$ depend on additional parameters $(w,s)\in V$, where V is a domain in $\mathbb{C}^n\times S$. Let g be C^∞-smooth* *in (z, w, s) and holomorphic in w. Then the corresponding function f can be chosen from the same class: C^∞-smooth in (z, w, s) and holomorphic in w.*

The addendum will be further applied to prove the above-mentioned $\bar\partial$-Poincaré Lemma. The proof of the theorem and the addendum will be split into two steps:

- case of a function f with finite support in D_1;
- the general case.

In the deduction of the general case from the finite support case we use the following obvious remark.

Remark 9.9 If a C^∞-smooth function f solving (9.2) exists, then it is unique up to addition of a holomorphic function.

Proposition 9.10 *1) For every C^∞ function $g : D_1 \to \mathbb{C}$ with finite support the function*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta}. \tag{9.4}$$

is C^∞-smooth in $z \in \mathbb{C}$ and satisfies (9.2). Here we extend g to a C^∞-function on all of $\mathbb{C}$ by setting it zero outside the unit disk D_1.

2) In the case, when g depends on additional parameters as in the addendum, the function f satisfies the statements of the addendum.

Proof The above integral converges and is a well-defined continuous function, being an integral of a function $O(\frac{1}{\zeta - z})$ over a real two-dimensional domain. One has

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(u+z)}{u} du \wedge \overline{du}, \ u = \zeta - z. \tag{9.5}$$

Taking the $\frac{\bar\partial}{\partial \bar z}$-derivative yields

$$\frac{\bar\partial f}{\partial \bar z}(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\bar\partial g}{\partial \bar z}(u+z)\frac{du}{u} \wedge \overline{du}$$

$$= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\bar\partial g}{\partial \bar z}(u+z)\overline{du} \wedge \frac{du}{u} = -\frac{1}{2\pi i}\int_{\mathbb{C}} d(g(u+z)\frac{du}{u})$$

The latter integral coincides with the same integral but taken over the disk D_2, since the subintegral form vanishes outside it, as does g. It equals the integral of the form $g(u+z)\frac{du}{u}$ over the boundary ∂D_2 minus the limit of its integral over the circle ∂D_δ, by Stokes formula. Taking into account that the former integral vanishes, as does $g(u+z)$ on ∂D_2, one has

$$\frac{\bar\partial f}{\partial \bar z}(z) = \lim_{\delta\to 0}\left(\frac{1}{2\pi i}\oint_{\partial D_\delta} g(u+z)\frac{du}{u}\right).$$

The expression under the limit is equal to the mean value of the function g over the circle of radius δ centered at z; this follows by substitution $u = \delta e^{i\theta}$. Therefore, the limit equals $g(z)$. This proves (9.2). The function f is C^∞, since the integral in (9.5) can be differentiated: the derivatives of the subintegral expression have converging integrals, since the differentiations do not affect the denominator u. The same argument proves smoothness (holomorphicity) of the function f in the additional parameters of the function g. The proposition is proved. $\square$

Proof of Theorem 9.8. Set

$$r_n = 1 - \frac{1}{2^n}.$$

Consider a sequence of "hat functions":

$$\rho_n : \mathbb{C} \to \mathbb{R}_{\geq 0} \text{ is a } C^\infty - \text{function supported in } D_1,\ \rho_n|_{D_{r_{n+1}}} \equiv 1, \tag{9.6}$$

Set

$$g_n(z) := g(z)\rho_n(z), \quad f_n(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_n(\zeta)}{\zeta - z} d\zeta \wedge \overline{d\zeta}. \tag{9.7}$$

One has

$$\frac{\bar\partial f_n}{\partial \bar z} = g \text{ on } D_{r_{n+1}}, \tag{9.8}$$

by Proposition 9.10. In the case, when the functions f_n converge with all the derivatives uniformly on compact subsets in D_1 to a function f, the latter satisfies the statements of the theorem. Let us show that one can modify the functions f_n inductively, replacing them by

$$\widetilde{f_n} = f_n - P_n,\ P_n(z) \text{ being a polynomial},$$

so that the new functions $\widetilde{f_n}$ converge. One has

$$\bar\partial \widetilde{f_n} = \bar\partial f_n = g \text{ on } D_{r_{n+1}}, \tag{9.9}$$

by construction. This together with the above argument will prove the theorem.

Induction base: $n = 1$. Set $\widetilde{f_1} = f_1$.

Induction step. Let $\widetilde{f}_n$ be already constructed. Let us construct $\widetilde{f}_{n+1}$ so that

$$|\widetilde{f}_{n+1} - \widetilde{f}_n| < \frac{1}{2^n} \text{ on } \overline{D}_{r_n}. \tag{9.10}$$

The difference $\psi_n = f_{n+1} - \widetilde{f}_n$ is holomorphic on $D_{r_{n+1}}$, by (9.9). Therefore, its Taylor series converges to it uniformly on $\overline{D}_{r_n} \Subset D_{r_{n+1}}$. Fix a Taylor polynomial P_n such that

$$|\psi_n - P_n| < \frac{1}{2^n} \text{ on } \overline{D}_{r_n}. \tag{9.11}$$

Set $\widetilde{f}_{n+1} = f_{n+1} - P_n$. Then inequality (9.10) holds by construction. The induction step is over. The functions $\widetilde{f}_n$ are constructed.

For every compact subset $K \Subset D_1$ there exists an $N > 0$ such that the functions $\widetilde{f}_n$ with $n > N$ are well-defined on K and converge uniformly on K, as $n \to \infty$. This follows from inequality (9.10), which implies that for every $n, m > N$ the difference $\widetilde{f}_n - \widetilde{f}_m$ has module less than $\frac{1}{2^l}$, $l = \min\{m, n\} - 1$ on K. This together with the above discussion proves the theorem. □

Proof of the addendum. The functions f_n given by (9.7) have the same regularity in parameters, as g, by Proposition 9.10. We show that one can choose the above polynomials P_n so that the functions $\widetilde{f}_n$ converge uniformly with derivatives on compact subsets in $D_1 \times V$, and hence, the limit function f has the same regularity. To do this, we use the following proposition.

Proposition 9.11 *Let a function $f(z, w, s)$ be C^∞-smooth on a domain $D_1 \times V$, where V is a domain in $\mathbb{C}^n \times S$, S being a manifold. Let f be holomorphic in (z, w). Then the finite sums of its Taylor series in powers of z with coefficients depending on $(w, s) \in V$ are C^∞-smooth in (z, w, s), holomorphic in (z, w) and converge to f uniformly on compact sets in $D_1 \times V$ together with all the derivatives in (z, w, s).*

Proof Smoothness (holomorphicity in w) of Taylor coefficients follows from the same regularity of the function and the Cauchy integral formulas for Taylor coefficients. Let us first prove uniform convergence of the Taylor series. Fix $0 < \mu < \nu < 1$ and a compact subset $K \Subset V$. Set

$$M = \max_{\overline{D_\nu} \times K} |f|.$$

Let $c_k(w, s)$ denote the Taylor coefficient of the function f at z^k. One has

$$|c_k(w, s)| \leq \frac{M}{\nu^k} \text{ for every } s \in K,$$

by the Cauchy Inequalities. Therefore, for every $z \in \overline{D_\mu}$ one has

$$|c_k(w,s)z^k| \leq M\beta^{|k|}, \ \beta = \frac{\mu}{\nu} < 1.$$

The right-hand sides of the above inequality form a series converging to $M(1-\beta)^{-n}$. This implies the uniform convergence of the Taylor series on $\overline{D_\mu} \times K$. The convergence of derivatives is proved by the same argument with f being replaced by its derivatives. The proposition is proved. $\square$

Fix a compact exhaustion $K_1 \Subset K_2 \Subset \cdots = V$ of the domain V. In the above construction of the functions $\widetilde{f}_n$ let us choose polynomials P_n so that inequality (9.11) holds on the product $\overline{D_{r_n}} \times K_n$ for the function $\psi_n - P_n$ and all its mixed derivatives in the variable z and the parameters up to order n. Then the functions $\widetilde{f}_n$ thus constructed converge with derivatives uniformly on compact sets. The limit is C^∞-smooth and is holomorphic in (z,w), by Weierstrass Theorem. The addendum is proved. $\square$

9.3 Proof of $\bar{\partial}$-Poincaré Lemma in higher dimensions

Let $\omega \in \Omega^{p,q}$, $q>0$, $\bar{\partial}\omega = 0$. Let us show that there exists a form $\alpha \in \Omega^{p,q-1}$ such that $\bar{\partial}\alpha = \omega$. This will prove the theorem.

1) Reduction to the case $(0,q)$. The form ω can be written as a sum

$$\omega = \sum_I dz_I \wedge \omega_I, \ \omega_I = \sum_J g_{IJ}\overline{dz_J} \in \Omega^{0,q}, \quad |I| = p,$$

$$\bar{\partial}\omega = (-1)^{p(q+1)} \sum_I dz_I \wedge \bar{\partial}\omega_I = 0.$$

The p-forms dz_I are linearly independent. This together with the latter equality implies that $\bar{\partial}\omega_I = 0$. Suppose that we have proved the theorem for $(p,q) = (0,q)$. Then we can find $(0,q-1)$-forms α_I such that $\bar{\partial}\alpha_I = \omega_I$. Set

$$\alpha = (-1)^{pq} \sum_I dz_I \wedge \alpha_I.$$

One has $\bar{\partial}\alpha = \omega$, by definition. This proves the theorem in the general case.

2) Case $(0,q)$: $p=0$. Then

$$\omega = \sum_J g_J \overline{dz_J}, \ \bar{\partial}\omega = 0, \quad |J| = q > 0.$$

Set

$$K = \cup_{J,\ g_J \not\equiv 0}\{j_1, \dots, j_q\} \subset \{1,\dots,n\}; \quad |K| \geq q.$$

We prove the existence of a form α such that $\bar{\partial}\alpha = \omega$ by induction in $|K|$.

Induction base: $|K| = q$. Then up to permutation of coordinates we can and will consider that

$$\omega = g(z)\overline{dz_1} \wedge \cdots \wedge \overline{dz_q}.$$

The equality $\bar{\partial}\omega = 0$ is equivalent to the statement that the function g is holomorphic in $z_{q+1}, \dots, z_n$. There exists a C^∞ function $f : \Delta \to \mathbb{C}$ holomorphic in the same variables such that

$$\frac{\bar{\partial} f}{\partial z_1} = g,$$

by Theorem 9.8 and its addendum. Set

$$\alpha = f\overline{dz_2} \wedge \cdots \wedge \overline{dz_q}.$$

One has $\bar{\partial}\alpha = \omega$, by definition. This proves the induction base.

Induction step. Let we have proved the existence of the above form α in the case, when $|K| < l$, $l > q$. Let us prove its existence for a form ω with $|K| = l$. Up to permutation of coordinates, we can and will consider that $1 \in K$. Then

$$\omega = \sum_I g_I \overline{dz_1} \wedge \overline{dz_I} + \sum_J g_J \overline{dz_J},\ I = (i_1, \dots, i_{q-1}),\ J = (j_1, \dots, j_q),$$

$$i_s, j_r \in K' = K \setminus \{1\}.$$

The functions g_I are holomorphic in variables z_t, $t \notin K$. This follows from the equality $\bar{\partial}\omega = 0$ and linear independence of the collection of 1-forms $\overline{dz_1} \wedge \overline{dz_I}$ and $\overline{dz_J}$. Therefore, for every I there exists a C^∞ function $f_I : \Delta \to \mathbb{C}$ holomorphic in the same variables such that $\frac{\bar{\partial} f_I}{\partial z_1} = g_I$. Set

$$\beta = \sum_I f_I \overline{dz_I}.$$

The difference $\omega - \bar{\partial}\beta$ is a $\bar{\partial}$-closed form, and it contains only products $\overline{dz_S}$ with $S = (s_1, \dots, s_q)$, $s_j \in K' = K \setminus \{1\}$, by construction. One has $|K'| < l$. Therefore, there exists a form α such that $\bar{\partial}\alpha = \omega - \bar{\partial}\beta$, by the induction hypothesis. Thus,

$$\omega = \bar{\partial}(\alpha + \beta).$$

The induction step is over. The proof of the $\bar{\partial}$-Poincaré Lemma is complete.

9.4 Case of $(p, 0)$-forms

Proposition 9.12 *For every $p \in \mathbb{Z}_{\geq 0}$ the $(p, 0)$-cohomology of every complex manifold is isomorphic to the space of holomorphic $(p, 0)$-forms.*

Proof The space of $(p, -1)$-form being trivial, is its $\bar{\partial}$-image in $\Omega^{p,0}$ is also trivial. Therefore, the $(p, 0)$-cohomology coincides with the kernel of the operator $\bar{\partial}$ in $\Omega^{p,0}$. A form $\omega \in \Omega^{p,0}$ with local coordinate presentation $\sum_J a_J(z) dz_J$ lies in its kernel, if and only if $\frac{\bar{\partial} a_J}{\partial \bar{z}_s} = 0$ for all J and s, i.e., the coefficients $a_J(z)$ are holomorphic; or equivalently, so is ω. This follows from linear independence of the forms $\overline{dz_s}$, dz_J. The proposition is proved. □

9.5 Holomorphic hypersurfaces. Existence of defining holomorphic functions. Cousin problems

Definition 9.13 A *holomorphic hypersurface* in a complex manifold M is an analytic subset $A \subset M$ such that each its point has a neighborhood U in M where the set $A \cap U$ is zero locus of just one holomorphic function on U.

Here we study the question of the existence of global holomorphic function vanishing exactly on A.

Proposition 9.14 *An analytic subset is a hypersurface, if and only if it has pure codimension one, i.e., each connected component of its regular part has codimension one.*

Proof Let $A \subset M$ be a hypersurface. Fix an arbitrary point $x \in A$ and its neighborhood $U = U(x) \subset M$ where there exists a holomorphic function $f : U \to \mathbb{C}$ vanishing exactly on $A \cap U$. Without loss of generality we consider that in local coordinates $(z_1, \dots, z_n)$, $n = \dim M$, centered at x the domain U is a Weierstrass polydisk $\Delta^{n-1} \times D_\delta$ for the function f, and f is a Weierstrass polynomial. Here Δ^{n-1} is a polydisk in the coordinate $(z_1, \dots, z_{n-1})$-hyperplane. Let $\pi : U \to \Delta^{n-1}$ denote the projection. Then there exists an analytic subset $B \subset \Delta^{n-1}$ such that the intersection $A^o := (A \cap U) \cap \pi^{-1}(\Delta^{n-1} \setminus B)$ lies in the regular part A_{reg} and the restriction $\pi : A^o \to \Delta^{n-1} \setminus B$ is a holomorphic covering. This immediately implies that A is of pure codimension one.

Conversely, let A be of pure codimension one. Fix an arbitrary point $x \in A$ and its neighborhood U where there exists a collection of (not identically zero) holomorphic functions $f_1, \dots, f_k : U \to \mathbb{C}$ such that $A \cap U = \{f_1 =$

$\cdots = f_k = 0\}$. Set $f = f_1$. Again without loss of generality we can and will consider that U is its Weierstrass polydisk $\Delta^{n-1} \times D_\delta$ and f is a Weierstrass polynomial. Set

$$\widetilde{A} := \{f = 0\} \subset U.$$

Let $B \subset \Delta^{n-1}$ be the above analytic subset constructed for the zero locus $\widetilde{A}$: $\widetilde{A}^o := (\widetilde{A} \cap U) \cap \pi^{-1}(\Delta^{n-1} \setminus B) \subset \widetilde{A}_{reg}$ and the projection $\pi : \widetilde{A}^o \to \Delta^{n-1} \setminus B$ is a covering. One has $(A \cap U) \subset \widetilde{A}$. Therefore, $\widetilde{A}^o \cap A$ is an open subset in A. The intersection of the set A with a connected component of the latter covering should be either emply, or the whole component. Indeed, otherwise the latter intersection would be a non-trivial analytic subset in the component (thus, of codimension at least two in $\mathbb{C}^n$) that is simultaneously an open subset in A. Therefore, A is not of pure codimension one (density of A_{reg} in A), – a contradiction. The set $A \cap U$ cannot contain an open subset lying entirely in $\pi^{-1}(B)$ for the same reason. Therefore, the intersection $A \cap \widetilde{A}^0$ is open and dense subset in $A \cap U$, and it is a union of connected components of the covering $\widetilde{A}^0$. Hence, $A \cap U$ is the closure of their union. Each component contained in $A \cap U$ corresponds to an irreducible factor g_j of the function f in U. Therefore, $A \cap U = \{\prod_j g_j = 0\}$. Hence, A is a hypersurface. The proposition is proved. □

Proposition 9.15 *Let $A \subset M$ be a hypersurface. For every $x \in A$ let $I_A(x)$ denote the ideal in the local ring $\mathcal{O}_n$ of germs of holomorphic functions at x consisting of those germs that vanish on A. The ideal $I_A(x)$ is principal: it has one generator given by the above product $g := \prod_j g_j$. A generator g is completely characterized (up to multiplication by unity) by the property that $dg(y) \neq 0$ for an open and dense subset of points $y \in A$ close to x. For every $y \in A$ close enough to x the function g is a generator of the corresponding ideal $I_A(y)$.*

Proof The property of the function g to be a generator of ideal and the second statement of the proposition (on differentials) follow immediately from the fact that each component of the above covering is the zero locus of an irreducible holomorphic function (a unique Weierstrass polynomial): an irreducible factor in every function vanishing on A. See Subsection 5.7. Its third statement follows from its second statement. □

Definition 9.16 Let $A \subset M$ be a holomorphic hypersurface in a complex manifold M. A holomorphic function $f : M \to \mathbb{C}$ is a *global defining function* for the hypersurface A, if $A = \{f = 0\}$ and for every point $x \in A$ the germ of the function f at x generates the corresponding ideal $I_A(x)$.

Theorem 9.17 *Every holomorphic hypersurface in a polydisk has a global defining function.*

We will see that existence of a global defining function is equivalent to solvability of the so-called multiplicative holomorphic Cousin problem (vanishing of appropriate cohomology). Then we make a first step towards its proof for polydisk: we solve additive holomorphic Cousin problem. We discuss how its solution together with a topological fact (stated without proof) imply solvability of the multiplicative problem and thus, existence of global defining function.

Definition 9.18 Let M be a complex manifold. Consider its locally finite covering $\cup_j U_j$ by open subsets. An *additive holomorphic (C^∞-smooth) covering 1-cocycle* is a collection h_{ij} of functions holomorphic (C^∞-smooth) on $U_i \cap U_j$ satisfying the *cocycle identities:*

$$h_{ij} = -h_{ji}; \quad h_{ij} + h_{jk} + h_{ki} = 0 \text{ on } U_i \cap U_j \cap U_k. \tag{9.12}$$

A cocycle is a *holomorphic (C^∞-smooth) coboundary,* if there exists a collection of holomorphic (C^∞-smooth) functions $h_j : U_j \to \mathbb{C}$ such that

$$h_{ij} = h_j - h_i \text{ on } U_i \cap U_j. \tag{9.13}$$

A multiplicative 1-cocycle (coboundary) is defined in the same way, but with all functions being nowhere vanishing and with relations (9.12), (9.13) written in the multiplicative group: $h_{ij} = h_{ji}^{-1}$; $h_{ij}h_{jk}h_{ki} = 1$; $h_{ij} = \frac{h_j}{h_i}$.

Remark 9.19 Each holomorphic coboundary is a holomorphic 1-cocycle.

The Holomorphic (C^∞) Multiplivative (Additive) Cousin Problem. Is the converse true? Is it true that each holomorphic (C^∞) multiplicative (additive) 1-cocycle is a holomorphic coboundary?

Let $A \subset M$ be a holomorphic hypersurface. Let us consider its locally finite covering by open subsets $U_j \subset M$ such that there exist holomorphic functions $g_j : U_j \to \mathbb{C}$ for which $A \cap U_j = \{g_j = 0\}$ and g_j generate the ideals $I_A(y)$ for every $y \in A \cap U_j$. Then

$$h_{ij} = \frac{g_j}{g_i} \tag{9.14}$$

are nonvanishing holomorphic functions on $U_i \cap U_j$, by Proposition 9.15.

Let us consider the covering of the whole manifold by the above neighborhoods U_j and the complement

$$U_0 = M \setminus A.$$

Set

$$g_0 = 1, \ h_{0j} = \frac{g_j}{g_0} = g_j \text{ on } U_0 \cap U_j. \tag{9.15}$$

The collection of the functions h_{ij} on $U_i \cap U_j$ is obviously a holomorphic multiplicative 1-cocycle.

Proposition 9.20 *An analytic hypersurface $A \subset M$ admits a global defining function, if and only if the Multiplicative Cousin Problem for the above cocycle is solvable: that is, if and only if there exist nonvanishing holomorphic functions $f_i : U_i \to \mathbb{C}^*$ such that*

$$\frac{f_j}{f_i} = h_{ij}. \tag{9.16}$$

Proof Let f be a defining function. Then the functions $f_j = \frac{g_j}{f}$ are holomorphic and nonvanishing on U_j and satisfy (9.16). Conversely, let $f_j : U_j \to \mathbb{C}^*$ be nonvanishing holomorphic functions satisfying (9.16). Then

$$f = \frac{g_j}{f_j}$$

is a global defining function: the latter fractions are holomorphic on the domains U_j, coincide on their intersections, by (9.16), vanish exactly on A and generate the corresponding ideals. The proposition is proved. □

It appears that solvability of Multiplicative Cousin Problem is closely related to solvability of the additive one and to the singular integer homology of the ambient manifold. Namely, consider the case, when all the covering domains U_j are simply connected and their pairwise intersections are connected and simply connected. Then for a given multiplicative one-cocycle h_{ij} we can take the logarithms $g_{ij} = -g_{ji} := \frac{\ln h_{ij}}{2\pi i}$, which are well-defined on the intersections $U_i \cap U_j$. However each g_{ij} is well-defined up to additive integer constant. The cocycle relation

$$G_{ijk} := g_{ij} + g_{jk} + g_{ki} = 0$$

is not necessarily true anymore, but it holds modulo $\mathbb{Z}$. At the same time, the left-hand sides G_{ijk} are well-defined constant integer-valued functions on the triple intersections $U_i \cap U_j \cap U_k$. They satisfy the *2-cocycle relations:*

G_{ijk} is (anti) invariant under (odd) even permutation of indices i, j, k;

$$G_{jk\ell} - G_{ik\ell} + G_{ij\ell} - G_{ijk} = 0 \quad \text{on} \;\; U_i \cap U_j \cap U_k \cap U_\ell.$$

Proposition 9.21 *Let an integer 2-cocycle G_{ijk} be an integer coboundary: there exist integers ϕ_{ij} such that $G_{ijk} = \phi_{ij} + \phi_{jk} + \phi_{kj}$. Then*

$$\widetilde{g}_{ij} := g_{ij} - \phi_{ij}$$

is an additive holomorphic cocycle.

The proposition follows from definition.

Remark 9.22 It appears that for a covering as above each integer 2-cocycle is an integer coboundary, if and only if the manifold M has trivial second integer singular cohomology $H^2(M, \mathbb{Z})$. For example, this holds in the case, when M is contractible, e.g., a polydisk. But we will not prove this in the present lectures.

Corollary 9.23 *Let M have trivial second integer cohomology. Consider its covering as above. Then each holomorphic multiplicative cocycle can be transformed to a holomorphic additive cocycle by taken appropriate branch of* log *as above. For these M and covering solvability of every holomorphic Additive Cousin Problem implies solvability of every holomorphic Multiplicative Cousin Problem.*

Theorem 9.24 *On every polydisk Δ each Holomorphic* **Additive** *Cousin Problem can be solved: for every covering each holomorphic 1-cocycle is a coboundary. In general, this is true on every complex manifold with trivial Dolbeault* $(0,1)$ *$\bar{\partial}$-cohomology.*

Proof Consider a locally finite covering $\Delta = \cup_j U_j$ and a holomorphic 1-cocycle $h_{ij} : U_i \cap U_j \to \mathbb{C}$. Let us prove that there exist holomorphic functions $h_j : U_j \to \mathbb{C}$ such that $h_{ij} = h_j - h_i$.

Step 1. Realizing the cocycle as a smooth coboundary. Construction of C^∞-smooth functions h_j satisfying the coboundary identity (9.13): $h_{ij} = h_j - h_i$. Take a partition of unity ρ_j subordinated to the covering U_j: $\rho_j : \Delta \to \mathbb{R}$ are C^∞-smooth functions vanishing on $\Delta \setminus U_j$ and also on neighborhoods of the boundaries ∂U_j, and such that $\sum_j \rho_j = 1$. For every j set

$$h_j = \sum_s \rho_s h_{sj} : U_j \to \mathbb{C}.$$

These are well-defined C^∞-smooth functions on U_j. Indeed, each ρ_s vanishes on a neighborhood of the boundary ∂U_s in Δ. Hence, the function equal to

$\rho_s h_{sj}$ on $U_j \cap U_s$ and zero on $U_j \setminus U_s$ is well-defined and C^∞-smooth on U_j. Therefore, each h_j is well-defined and C^∞-smooth on U_j, by local finiteness of covering. One has

$$h_j - h_i = \sum_s \rho_s (h_{sj} - h_{si}).$$

In the latter sum each ρ_s vanishes everywhere on $U_i \cap U_j$, except for the intersection $U_i \cap U_j \cap U_s$. On the latter intersection one has $h_{sj} - h_{si} = h_{ij}$, by the cocycle identity (9.12). Therefore, $h_j - h_i = h_{ij} \sum_s \rho_s = h_{ij}$. Thus, the coboundary identity holds.

Step 2. Correcting h_j to make them holomorphic, keeping h_{ij} the same. Set

$$\omega := \bar{\partial} h_j \text{ on } U_j.$$

This is a well-defined C^∞-smooth form of type $(0,1)$ on the whole manifold Δ. Indeed, this definition matches on the intersections $U_i \cap U_j$: $\bar{\partial} h_j - \bar{\partial} h_i = \bar{\partial} h_{ij} = 0$, since h_{ij} are holomorphic. The form ω is closed, since $\bar{\partial}^2 = 0$. Therefore, it is exact, since the manifold has trivial Dolbeault $(0,1)$ $\bar{\partial}$-cohomology. Hence, there exists a C^∞-smooth function $f : \Delta \to \mathbb{C}$ such that $\bar{\partial} f = \omega$. Set now

$$\widetilde{h}_j := h_j - f.$$

The functions $\widetilde{h}_j$ are holomorphic on U_j, since $\bar{\partial}\widetilde{h}_j = \bar{\partial} h_j - \bar{\partial} f = \omega - \omega = 0$. One has $\widetilde{h}_j - \widetilde{h}_i = h_{ij}$ on $U_i \cap U_j$, by construction. Therefore, the 1-cocycle in question is a holomorphic coboundary. Theorem 9.24 is proved. $\square$

Theorem 9.24 together with Corollary 9.23 imply Theorem 9.17.