Exercises on symmetric functions 16.01.2024

These exercises are due by January 23rd. This is a general rule: the due date is one week after the assignment. The final grade for the course is calculated as 0.1 of the percentage of completely solved problems. There will be about 100 problems in total. You may submit e.g. the high quality scans of your handwritten solutions in the natural order. I will grade neither poor quality scans nor randomly ordered scans. You may also submit your handwritten solutions as a hardcopy or solutions typeset in TeX.

1. For a box $x = (i, j)$ in a Young diagram λ of length $\ell(\lambda) = n$ we define its hooklength as $h(x) = h(i, j) := \lambda_i + \lambda^t_j - i - j + 1$ (here λ^t stands for the transposed Young diagram). Also, we set $\mu_i := \lambda_i + n - i,\ 1 \le i \le n$. Prove $\displaystyle\sum_{j=1}^{\lambda_1} t^{\lambda^t_j+\lambda_1-j} + \sum_{j=1}^{n} t^{\lambda_1-1+j-\lambda_j} = \sum_{j=0}^{\lambda_1+n-1} t^j$.

2. Prove $\displaystyle\sum_{j=1}^{\lambda_1} t^{h(1,j)} + \sum_{j=2}^{n} t^{\mu_1-\mu_j} = \sum_{j=1}^{\mu_1} t^j$.

3. Prove $\displaystyle\sum_{x\in\lambda} t^{h(x)} + \sum_{i<j} t^{\mu_i-\mu_j} = \sum_{i\ge1}\sum_{j=1}^{\mu_i} t^j$.

4. Prove $\displaystyle\prod_{x\in\lambda}(1 - t^{h(x)}) = \frac{\prod_{i\ge1}\prod_{j=1}^{\mu_i}(1 - t^j)}{\prod_{i<j}(1 - t^{\mu_i-\mu_j})}$.

5. Prove $\displaystyle\prod_{x\in\lambda} h(x) = \frac{\prod_{i\ge1}\mu_i!}{\prod_{i<j}(\mu_i - \mu_j)}$.

Exercises on symmetric functions 23.01.2024

1. Prove that the sum of all hooklengths of a diagram λ is $\sum_{x\in\lambda} h(x) = n(\lambda) + n(\lambda^t) + |\lambda|$, where $n(\lambda) := \sum_{i\ge1}(i - 1)\lambda_i = \sum_{i\ge1}\binom{\lambda^t_i}{2}$.

2. For a box $x = (i, j) \in \lambda$ we define its *content* as $c(x) := j - i$. Prove that $\sum_{x\in\lambda} c(x) = n(\lambda^t) - n(\lambda)$.

3. For $n \ge \ell(\lambda)$ prove $\displaystyle\prod_{x\in\lambda}(1 - t^{n+c(x)}) = \prod_{i\ge1}\frac{\varphi_{\lambda_i+n-i}(t)}{\varphi_{n-i}(t)}$, where $\varphi_r(t) := (1-t)(1-t^2)\cdots(1-t^r)$.

4. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ can be written in the Frobenius notation $(\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$, where $\alpha_i := \lambda_i - i;\ \beta_i := \lambda^t_i - i,\ 1 \le i \le r$, and r is the length of the intersection of the diagram λ with the diagonal $i = j$. Prove $\displaystyle\sum_{i=1}^{n} t^i(1 - t^{-\lambda_i}) = \sum_{j=1}^{r}(t^{\beta_j+1} - t^{-\alpha_j})$.

5. Prove $\displaystyle\sum_{x\in\lambda}(h(x)^2 - c(x)^2) = |\lambda|^2$.

Exercises on symmetric functions 30.01.2024

1. Let $\lambda, \mu \in \mathcal{P}(n)$ be two partitions of n. Prove that $\lambda \geq \mu$ (i.e. $\lambda_1 \geq \mu_1,\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \ldots$) if and only if there is a double stochastic $n \times n$-matrix M (i.e. $m_{ij} \in \mathbb{R}^{\geq 0}$, and the sums of the matrix entries in every column and in every row are equal to 1) such that $M\lambda = \mu$.

2. We specialize $x_i = 1/n$ for $1 \leq i \leq n$, and $x_i = 0$ for $i > n$. (a) Prove that $e_r = n^{-r}\binom{n}{r},\ h_r = n^{-r}\binom{n+r-1}{r}$. (b) Let us take the limit as $n \to \infty$. Prove that $e_r = h_r = \frac{1}{r!},\ p_1 = 1$ and $p_r = 0$ for $r > 1$; moreover, $m_\lambda = 0$ unless $\lambda = (1^r)$.

3. We specialize $x_i = q^{i-1}$ for $1 \leq i \leq n$, and $x_i = 0$ for $i > n$.

(a) Prove $E(t) = \displaystyle\prod_{i=0}^{n-1}(1 + q^i t) = \sum_{r=0}^{n} q^{r(r-1)/2} \begin{bmatrix} n \\ r \end{bmatrix} t^r$, where $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ is the Gaussian q-binomial coefficient $\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$.

(b) Prove $H(t) = \displaystyle\prod_{i=0}^{n-1}(1 - q^i t)^{-1} = \sum_{r=0}^{\infty} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} t^r$.

4. Let us take the limit as $n \to \infty$.

(a) Prove $E(t) = \displaystyle\prod_{i=0}^{\infty}(1 + q^i t) = \sum_{r=0}^{\infty} q^{r(r-1)/2} t^r/\varphi_r(q)$, where $\varphi_r(q) = (1-q)(1-q^2)\cdots(1-q^r)$.

(b) Prove $H(t) = \displaystyle\prod_{i=0}^{\infty}(1 - q^i t)^{-1} = \sum_{r=0}^{\infty} t^r/\varphi_r(q)$, and $p_r = (1 - q^r)^{-1}$.

5. Since the functions h_r are algebraically independent, we can specialize their values in an arbitrary way. For instance, we may take $H(t) = \displaystyle\prod_{i=0}^{\infty} \frac{1 - bq^i t}{1 - aq^i t}$.

Prove $h_r = \displaystyle\prod_{i=1}^{r} \frac{a - bq^{i-1}}{1 - q^i},\ e_r = \prod_{i=1}^{r} \frac{aq^{i-1} - b}{1 - q^i},\ p_r = \frac{a^r - b^r}{1 - q^r}$.

Exercises on symmetric functions 06.02.2024

1. We set $x_i = q^{i-1},\ 1 \leq i \leq n$. Prove

(a) $a_{\lambda+\rho} = q^{n(\lambda)+\binom{n}{3}} \displaystyle\prod_{i<j}(1 - q^{\lambda_i - \lambda_j - i + j}) = q^{n(\lambda)+\binom{n}{3}} \frac{\prod_{i\geq 1} \varphi_{\lambda_i + n - i}(q)}{\prod_{x\in\lambda}(1 - q^{h(x)})}$.

(b) $s_\lambda = q^{n(\lambda)} \displaystyle\prod_{x\in\lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}}$ (notations of the previous problem sets). In other words, $s_\lambda(1, q, \ldots, q^{n-1}) = q^{n(\lambda)} \left[\begin{smallmatrix} n \\ \lambda^t \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right] := \displaystyle\prod_{x\in\lambda} \frac{1 - q^{n-c(x)}}{1 - q^{h(x)}}$.

2. Let us take the limit as $n \to \infty$, so that $H(t) = \prod_{i=0}^{\infty}(1-q^it)^{-1}$.
Prove $s_\lambda = q^{n(\lambda)} \prod_{x\in\lambda}(1-q^{h(x)})^{-1} = q^{n(\lambda)}H_\lambda(q)^{-1}$, where
$H_\lambda(q) = \prod_{x\in\lambda}(1-q^{h(x)})$ is the hook polynomial.

3. If we set $H(t) = \prod_{i=0}^{\infty} \frac{1-bq^it}{1-aq^it}$, prove $s_\lambda = q^{n(\lambda)} \prod_{x\in\lambda} \frac{a-bq^{c(x)}}{1-q^{h(x)}}$.

4. (a) If we set $x_i = 1,\ 1 \le i \le n$, and $x_i = 0,\ i > n$, then prove $E(t) = (1+t)^n$ and $s_\lambda = \prod_{x\in\lambda} \frac{n+c(x)}{h(x)}$.
(b) If we set $E(t) = (1+t)^a$ for arbitrary a (not necessarily a positive integer), then prove $s_\lambda = \prod_{x\in\lambda} \frac{a+c(x)}{h(x)}$. Let $\binom{a}{\lambda} := \prod_{x\in\lambda} \frac{a-c(x)}{h(x)}$.
(c) Prove $\binom{a}{\lambda} = \det\left(\binom{a}{\lambda_i - i + j}\right)$ and $\binom{-a}{\lambda} = (-1)^{|\lambda|}\binom{a}{\lambda^t}$.

5. Let us specialize $x_i = 1/n,\ 1 \le i \le n;\ x_i = 0,\ i > n$, and take the limit as $n \to \infty$. Prove (a) $E(t) = H(t) = e^t$.
(b) $s_\lambda = \lim_{n\to\infty} n^{-|\lambda|} \prod_{x\in\lambda} \frac{n+c(x)}{h(x)} = \prod_{x\in\lambda} h(x)^{-1}$.

Exercises on symmetric functions 13.02.2024

1. Prove (a) $\prod_{i,j}(1+x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda^t}(y)$.
(b) $E(t)^n = \sum_\lambda s_\lambda(x)s_{\lambda^t}(y) = \sum_\lambda \binom{n}{\lambda} s_\lambda(x)t^{|\lambda|}$ (where we set $y_1 = \ldots = y_n = t$, and $0 = y_{n+1} = y_{n+2} = \ldots$)
(c) $E(t)^a = \sum_\lambda \binom{a}{\lambda} s_\lambda t^{|\lambda|}$.
(d) $H(t)^a = \sum_\lambda \binom{a}{\lambda^t} s_\lambda t^{|\lambda|}$.

2. We set $y_i = q^{i-1},\ 1 \le i \le n$, and $y_i = 0,\ i > n$. Prove
(a) $\prod_{i=1}^{n} E(q^{i-1}) = \sum_\lambda q^{n(\lambda^t)} \begin{bmatrix} n \\ \lambda \end{bmatrix} s_\lambda$.
(b) $\prod_{i=1}^{n} H(q^{i-1}) = \sum_\lambda q^{n(\lambda)} \begin{bmatrix} n \\ \lambda^t \end{bmatrix} s_\lambda$.

(c) $\displaystyle\prod_{i,j\geq 1}(1+x_jq^{i-1})=\sum_\lambda \frac{q^{n(\lambda^t)}}{H_\lambda(q)}s_\lambda(x).$

(d) $\displaystyle\prod_{i,j\geq 1}(1-x_jq^{i-1})^{-1}=\sum_\lambda \frac{q^{n(\lambda)}}{H_\lambda(q)}s_\lambda(x),$

where $H_\lambda(q)=\displaystyle\prod_{x\in\lambda}(1-q^{h(x)})$ is the hook polynomial.

3. We set $y_1=\ldots=y_n=t/n,\ y_i=0,\ i>n$, and take the limit as $n\to\infty$. Prove (a) $\displaystyle\frac{1}{n^{|\lambda|}}\binom{n}{\lambda}\to\prod_{x\in\lambda}h(x)^{-1}=:h(\lambda)^{-1}.$

(b) $\displaystyle\prod_i\left(1+\frac{x_it}{n}\right)^n\to\prod_i\exp(x_it)=\exp(e_1t)=\sum_\lambda\frac{s_\lambda}{h(\lambda)}t^{|\lambda|}.$

(c) $\displaystyle e_1^n=\sum_{|\lambda|=n}\frac{n!}{h(\lambda)}s_\lambda\Leftrightarrow\langle e_1^n,s_\lambda\rangle=n!/h(\lambda).$

4. Prove that the number of standard tableaux of shape $\lambda\in\mathcal{P}(n)$ equals $K_{\lambda,(1^n)}=\langle s_\lambda,h_1^n\rangle=n!/h(\lambda)$.

5. Prove that $\langle h_n,p_\lambda\rangle=1$ and $\langle e_n,p_\lambda\rangle=\varepsilon_\lambda:=(-1)^{|\lambda|+\ell(\lambda)}$ for any $\lambda\in\mathcal{P}(n)$.

Exercises on symmetric functions 20.02.2024

1. Let $H(t)=(1-t^r)/(1-t)^r,\ r\geq 2$. Prove (a) $h_n=\binom{n+r-1}{r-1}-\binom{n-1}{r-1}$.

(b) $p_n=0$ if $n\equiv 0 \pmod r$, and $p_n=r$ if $n\not\equiv 0\pmod r$.

(c) $\displaystyle\sum_\lambda z_\lambda^{-1}r^{\ell(\lambda)}=\binom{n+r-1}{r-1}-\binom{n-1}{r-1}$, where the sum is taken over the set of partitions of n whose parts are not divisible by r. In particular, for $r=2$ we get $\displaystyle\sum_\lambda z_\lambda^{-1}2^{\ell(\lambda)}=2$, where the sum is taken over the set of partitions of n all of whose parts are odd.

2. Let $p_n=an^n/n!,\ n\geq 1$. Prove (a) if $t=x\exp(-x)$, then $P(t)=a\exp(x)/(1-x)$.

(b) $\displaystyle h_n=\frac{a(a+n)^{n-1}}{n!},\ e_n=\frac{a(a-n)^{n-1}}{n!}.$

3. Let $h_n=n,\ n\geq 1$. Prove that

(a) the sequence (p_n) is periodic with period 6.

(b) the sequence (e_n) is periodic with period 3.

4. Prove that $\displaystyle\sum_\theta z_\theta^{-1}=\sum_\sigma z_\sigma^{-1}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$, where the first sum is taken over all the partitions of $2n$ into even parts, while the second sum is taken over all the partitions of $2n$ into odd parts.

5. For any $\lambda \in \mathcal{P}(n)$ we set $M_\lambda(x) := \dfrac{1}{n!} \sum_{w \in S_n} w(x^\lambda)$, where $x = (x_1, \ldots, x_n)$. Prove that the following statements are equivalent:
(a) $\lambda \geq \mu$;
(b) $M_\lambda(x) \geq M_\mu(x)$ for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

Exercises on symmetric functions 27.02.2024

1. For any $f \in \Lambda$ we define $D(f) \colon \Lambda \to \Lambda$ by $\langle D(f)u, v\rangle = \langle u, fv\rangle$ for any $u, v \in \Lambda$. Then $D \colon \Lambda \to \mathrm{End}(\Lambda)$ is a ring homomorphism. We denote $D(s_\mu)$ by D_μ. Prove that
(a) for any $f \in \Lambda$, $f(x, y) = \sum_\mu D_\mu f(x) \cdot s_\mu(y)$.
(b) $D(h_\lambda)m_\mu = 0$ unless $\mu = \lambda \cup \nu$ (that is, μ is the union of reordered parts of λ and ν), in which case $D(h_\lambda)m_\mu = m_\nu$.
(c) For any $f(x_0, x_1, \ldots) \in \Lambda$, $(D(h_n)f)(x_1, x_2, \ldots)$ is the coefficient of x_0^n in f.

2. Prove that (a) $D(p_n)h_N = h_{N-n}$, that is $D(p_n) = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}$, where we view the symmetric functions as polynomials in h_i, $i \geq 0$.
(b) $D(p_n) = (-1)^{n-1} \sum_{r \geq 0} e_r \frac{\partial}{\partial e_{n+r}}$.
(c) $D(p_n) = n \frac{\partial}{\partial p_n}$. In other words, if $f \in \Lambda$ is written as a polynomial $f = \varphi(p_1, p_2, \ldots)$, then $D(f) = \varphi(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots)$ is a linear differential operator with constant coefficients.

3. Prove that (a) for $f \in \Lambda^m$, $g \in \Lambda^n$, we have $\omega(f \circ g) = f \circ (\omega g)$ if n is even, and $\omega(f \circ g) = (\omega f) \circ (\omega g)$ if n is odd.
(b) $f \circ (-g) = (-1)^m (\omega f) \circ g$.
(c) $p_\lambda \circ p_\mu = p_\mu \circ p_\lambda = p_{\lambda \circ \mu}$, where $\lambda \circ \mu$ is the partition with parts $\lambda_i \cdot \mu_j$.
(d) $\omega(h_r \circ p_s) = (-1)^{r(s-1)} e_r \circ p_s$.

4. Prove that $f \circ (g + h) = \sum_\mu ((D_\mu f) \circ g)(s_\mu \circ h)$ for any $f, g, h \in \Lambda$.

5. Prove that $h_n \circ (gf) = \sum_{|\lambda|=n} (s_\lambda \circ g)(s_\lambda \circ f)$, and $e_n \circ (gf) = \sum_{|\lambda|=n} (s_\lambda \circ g)(s_{\lambda^t} \circ f)$.

Exercises on symmetric functions 05.03.2024

1. Let $\Delta = \det\left((1 - x_i y_j)^{-1}\right)_{1 \leq i,j \leq n}$ (Cauchy determinant). Prove that $\Delta = a_\rho(x) a_\rho(y) \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1} = \sum_\lambda a_{\lambda+\rho}(x) a_{\lambda+\rho}(y)$ (the sum is taken over all partitions of length at most n).

2. Prove that for a partition $\lambda = (i^{m_i}) = (\lambda_1 \geq \lambda_2 \geq \ldots)$ we have
$$q^{|\lambda| + 2n(\lambda)} \prod \varphi_{m_i}(\lambda)(q^{-1}) = \prod q^{\lambda_1^t + \ldots + \lambda_r^t} (1 - q^{\nu_r^t - \lambda_r^t}),$$
where the second product is taken over $r = \lambda_1, \lambda_2, \ldots$, and $\nu = (\lambda_1, \ldots, \lambda_{k-1})$ for $r = \lambda_k$. Furthermore, λ^t stands for the dual partition (corresponding to the transposed Young diagram), and $\varphi_m(t) := (1-t) \cdots (1-t^m)$.

3. Construct a bijection between the set of partitions λ whose Young diagram is contained in the $k\times\ell$-box and the set of sequences of nonnegative integers $(a_1,\ldots,a_m;b_0,\ldots,b_m)$ such that $\sum a_i=k$, $\sum b_j=\ell$, and $a_1,\ldots,a_m,b_1,\ldots,b_{m-1}$ are all positive, but b_0 and b_m can possibly vanish.

4. Fix a complete flag $0=V_0\subset V_1\subset\ldots\subset V_{k+\ell}=\mathbb{C}^{k+\ell}$. We define the *Schubert cell* $X_\lambda\subset \mathrm{Gr}(k,k+\ell)$ as the set of all k-dimensional subspaces $U\subset\mathbb{C}^{k+\ell}$ such that

$$\dim(U\cap V_{b_0})=0,\ \dim(U\cap V_{b_0+a_1})=a_1,$$

$$\vdots$$

$$\dim(U\cap V_{b_0+a_1+\ldots+b_{i-2}+a_{i-1}+b_{i-1}})=a_1+\ldots+a_{i-1},\ \dim(U\cap V_{b_0+a_1+\ldots+b_{i-1}+a_i})=a_1+\ldots+a_i,$$

$$\vdots$$

$$\dim(U\cap V_{b_0+a_1+\ldots+b_{m-2}+a_{m-1}+b_{m-1}})=a_1+\ldots+a_{m-1},\ \dim(U\cap V_{b_0+a_1+\ldots+b_{m-1}+a_m})=a_1+\ldots+a_m.$$

Prove that a) $\mathrm{Gr}(k,k+\ell)=\bigsqcup_\lambda X_\lambda$.

b) X_λ is an orbit in $\mathrm{Gr}(k,k+\ell)$ of the Borel subgroup of $\mathrm{GL}(k+\ell,\mathbb{C})$ preserving the above complete flag.

c) $X_\mu\subset\overline{X}_\lambda$ iff $\mu\subset\lambda$, i.e. the Young diagram of μ is contained in the Young diagram of λ.

5. Construct an isomorphism $X_\lambda\simeq\mathbb{C}^{|\lambda|}$.

Exercises on symmetric functions 12.03.2024

1. Let $\ell(\lambda)\leq n$. We set $d:=(n-1)|\lambda|$. Prove
(a) $(s_\lambda\circ s_{(n-1)})(x_1,x_2)=s_\lambda(x_1^{n-1},x_1^{n-2}x_2,\ldots,x_2^{n-1})=x_2^{(n-1)|\lambda|}\cdot s_\lambda(q^{n-1},q^{n-2},\ldots,q,1)=:$ $\sum_{n_1+n_2=d}c_{n_1,n_2}s_{(n_1,n_2)}(x_1,x_2)$, where $q=x_1x_2^{-1}$, and $c_{n_1,n_2}\in\mathbb{N}$.
(b) $s_\lambda(q^{n-1},q^{n-2},\ldots,q,1)=:\sum_{i=0}^d a_iq^i$ is a unimodal palindromic polynomial in q, that is $0\leq a_0\leq a_1\leq\ldots\leq a_{\lfloor\frac{d}{2}\rfloor}$, and $a_{d-i}=a_i$.
(c) $\begin{bmatrix}n\\ \lambda\end{bmatrix}$ is a unimodal palindromic polynomial in q for any n,λ.

2. We set $\Phi(x_1,\ldots,x_n):=\sum_\lambda s_\lambda(x_1,\ldots,x_n)$ (the sum is taken over all the partitions of length $\leq n$). Prove that
(a) $\Phi(x_1,\ldots,x_n,y)=\sum_{\lambda,\mu}y^{|\lambda-\mu|}s_\mu(x_1,\ldots,x_n)$ (the sum is taken over all the pairs of partitions $\lambda\supset\mu$ such that $\ell(\mu)\leq n$, and $\lambda-\mu$ is a horizontal strip).
(b) $\sum_{\lambda,\mu}y^{|\lambda-\mu|}s_\mu(x_1,\ldots,x_n)=\sum_{\mu,\nu}y^{|\mu-\nu|}(1-y)^{-1}s_\mu(x_1,\ldots,x_n)$ (the right hand sum is taken over all the pairs of partitions $\mu\supset\nu$ such that $\ell(\mu)\leq n$, and $\mu-\nu$ is a horizontal strip).
(c) $\sum_{\mu,\nu}y^{|\mu-\nu|}(1-y)^{-1}s_\mu(x_1,\ldots,x_n)=$
$\sum_{\nu,r}y^r(1-y)^{-1}h_r(x_1,\ldots,x_n)s_\nu(x_1,\ldots,x_n)=$
$(1-y)^{-1}\prod_{i=1}^n(1-x_iy)^{-1}\Phi(x_1,\ldots,x_n)$ (the middle sum is taken over all the partitions ν of length $\leq n$ and all $r\geq 0$).
(d) $\sum_\lambda s_\lambda=\prod_i(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}$ (the sum is taken over all the partitions). In

other words, for a vector space V, $\mathrm{Sym}^\bullet(V \oplus \Lambda^2 V)$ is a direct sum of all the irreducible polynomial representations of $GL(V)$ with multiplicities 1 (a *model*).

3. Prove (a) $(\sum_\mu s_\mu)(\sum_0^\infty e_r) = \sum_\lambda s_\lambda$ (the left hand sum is taken over all the *even* partitions μ, i.e. such that all the parts are even; while the right hand sum is taken over all the partitions λ).
(b) $\sum_{\mu \text{ even}} s_\mu = \prod_i (1-x_i^2)^{-1} \prod_{i<j}(1-x_ix_j)^{-1}$.
(c) $(\sum_{\nu^t \text{ even}} s_\nu)(\sum_0^\infty h_r) = \sum_\lambda s_\lambda$.
(d) $\sum_{\nu^t \text{ even}} s_\nu = \prod_{i<j}(1-x_ix_j)^{-1}$.

4. Prove (a) $\prod_{i<j}(1+x_ix_j)^{-1} = \sum_\nu (-1)^{|\nu|/2} s_\nu$ (the sum is taken over all the diagrams ν with columns of even heights).
(b) $\prod_i (1-x_i)^{-1} \prod_{i<j}(1+x_ix_j)^{-1} = (\sum_0^\infty h_r)(\sum_\nu (-1)^{|\nu|/2} s_\nu)$.
(c) $(\sum_0^\infty h_r)(\sum_\nu (-1)^{|\nu|/2} s_\nu) = \sum_\lambda (-1)^{f(\lambda)} s_\lambda$, where $f(\lambda) = \sum_i \lfloor \frac{\lambda_i^t}{2} \rfloor$.
(d) $\prod_i (1-x_i)^{-1} \prod_{i<j}(1+x_ix_j)^{-1} = \sum_\lambda (-1)^{n(\lambda)} s_\lambda$.

5. Prove (a) $\sum_\lambda t^{c(\lambda)} s_\lambda = \prod_i (1-tx_i)^{-1} \prod_{i<j}(1-x_ix_j)^{-1}$ (the sum is taken over all the partitions λ, and $c(\lambda)$ is the number of columns of odd height in λ).
(b) $\displaystyle\sum_\lambda t^{r(\lambda)} s_\lambda = \prod_i \frac{1+tx_i}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_ix_j}$ (the sum is taken over all the partitions λ, and $r(\lambda)$ is the number of rows of odd length in λ).

Exercises on symmetric functions 19.03.2024

1. Recall that $K = M(s,m)$ is the Kostka transition matrix from the monomial base to the Schur base. Prove that $(K^{-1})_{\lambda,(1^n)} = \varepsilon_\lambda \cdot \ell(\lambda)!/\prod_i m_i!$, where $\lambda = (i^{m_i})$ is a partition of n.

2. Let $X := M(p,s)$ (the character table of the symmetric group S_n), and $L := M(p,m)$. Prove that
(a) $XX^t = z$, where $z_{\lambda\mu} = \delta_{\lambda\mu} z_\lambda$.
(b) $XJ = \varepsilon X$, where $\varepsilon_{\lambda\mu} = \delta_{\lambda\mu}\varepsilon_\lambda$.
(c) $L \in U_-$ (i.e. $L_{\lambda\mu} = 0$ unless $\lambda \leq \mu$).
(d) $L_{\mu\mu} = \prod_i m_i!$, where $\mu = (i^{m_i})$.
(e) $L_{\lambda\mu}/L_{\mu\mu} \in \mathbb{Z}$.
(f) $X = LK^{-1},\ K^{-1}JK = L^{-1}\varepsilon L,\ K^tK = L^t z^{-1} L$.
(g) $M(p,e) = \varepsilon z L^*,\ M(p,h) = zL^*$, where $L^* = (L^t)^{-1}$.

3. We denote by $\Lambda_+^n \subset \Lambda^n$ the submonoid $\mathbb{N}\langle s_\lambda \rangle_{\lambda \in \mathcal{P}(n)}$, and $\Lambda_+ = \bigoplus_n \Lambda_+^n$. We have $h_\lambda, e_\lambda, s_{\nu/\mu} \in \Lambda_+$ for any λ, μ, ν. For $f, g \in \Lambda^n$ we say $f \geq g$ when $f - g \in \Lambda_+^n$. For $\lambda, \mu \in \mathcal{P}(n)$ prove that the following statements are equivalent:
(a) $\lambda \geq \mu$.
(b) $s_\lambda \leq h_\mu$.
(c) $s_{\lambda^t} \leq e_\mu$.
(d) $h_\lambda \leq h_\mu$.

(e) $e_\lambda \leq e_\mu$.
(f) $M(e,m)_{\lambda^t\mu} > 0$.
(g) $K_{\lambda\mu} > 0$.
(Hint: the key claim is (a)$\Rightarrow$(d). Then we may assume $\lambda = R_{ij}\mu$ (a raising operator) and apply the Jacobi-Trudi identity $s_{(\mu_i,\mu_j)} = h_{\mu_i}h_{\mu_j} - h_{\mu_i+1}h_{\mu_j-1}$).

4. Let $\mathbb{O}_\lambda \subset \mathrm{Mat}_{n\times n}$ denote the set (conjugacy class) of nilpotent matrices with Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots$ Prove that $\overline{\mathbb{O}}_\lambda \supset \mathbb{O}_\mu$ if and only if $\lambda \geq \mu$.

5. Let Gr_N be the *positive affine Grassmannian*: the set of *sublattices* (i.e. $\mathbb{C}[\![z]\!]$-submodules of finite codimension) in $\mathbb{C}[\![z]\!]^N$. For a partition λ let Gr_N^λ be the set of sublattices $L \subset \mathbb{C}[\![z]\!]^N$ such that the nilpotent operator z on the quotient $\mathbb{C}[\![z]\!]^N/L$ has Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots$ Prove that $\overline{\mathrm{Gr}_N^\lambda} \supset \mathrm{Gr}_N^\mu$ if and only if $\lambda \geq \mu$.

Exercises on symmetric functions 26.03.2024

1. (a) For partitions λ, μ, we denote by $\lambda\mu$ (resp. $\lambda \otimes \mu$) a partition with parts $\lambda_i\mu_i$ (resp. $\min(\lambda_i, \mu_j)$). Prove that $(\lambda\mu)^t = \lambda^t \otimes \mu^t$.
(b) Let M, N be finite $\mathcal{O}$-modules of types μ, ν. Prove that the type of $M \oplus N$ (resp. $M \otimes N$) is $\mu \cup \nu$ (resp. $\mu \otimes \nu$).

2. Prove that the structure constant in the Hall algebra $H(\mathcal{O})$
$G^\lambda_{\mu(1^m)}(q) = q^{n(\lambda)-n(\mu)-n(1^m)} \prod_{i\geq 1} \begin{bmatrix} \lambda_i^t - \lambda_{i+1}^t \\ \lambda_i^t - \mu_i^t \end{bmatrix}_{q^{-1}}$.

3. Prove (a) $R_\lambda(1, t, \ldots, t^{n-1}; t) = t^{n(\lambda)} v_n(t)$, where $v_n(t) = \prod_{i=1}^n \frac{1-t^i}{1-t}$.
(b) $Q_\lambda(1, t, \ldots, t^{n-1}; t) = t^{n(\lambda)} \varphi_n(t)/\varphi_{m_0}(t)$, where $m_0 = n - \ell(\lambda)$, and $\varphi_n(t) = v_n(t)(1-t)^n$. As $n \to \infty$, we get in the limit $Q_\lambda(1, t, t^2, \ldots; t) = t^{n(\lambda)}$.

4. Prove
(a) $P_\lambda(x_1, \ldots, x_n; t) = v_\lambda(t)^{-1} \prod_{i<j}(1 - tR_{ji}) s_\lambda(x_1, \ldots, x_n) = \prod_{\lambda_i > \lambda_j}(1 - tR_{ji}) s_\lambda(x_1, \ldots, x_n)$, where $v_\lambda(t) = \prod_{i\geq 0} v_{m_i}(t)$ for $\lambda = (i^{m_i})$ (starting from $i = 0$, so that $m_0 = n - \ell(\lambda)$).
(b) $P_{(n)} = \sum_{r=0}^{n-1} (-t)^r s_{(n-r,1^r)}$.

5. Prove (a) $\displaystyle\sum_{i=1}^n \prod_{j\neq i} \frac{x_j - tx_i}{x_j - x_i} = \frac{v_n(t)}{v_{n-1}(t)} = \frac{1-t^n}{1-t}$.
(b) $\displaystyle\sum_{i=1}^n \prod_{j\neq i} \left(1 - \frac{x_i}{x_j}\right)^{-1} = 1$.
(c) Let $(a_1, \ldots, a_n) \in \mathbb{N}^n$. We define $c(a_1, \ldots, a_n)$ as the constant term of $\displaystyle\prod_{1\leq i\neq j\leq n} \left(1 - \frac{x_j}{x_i}\right)^{a_j}$.
Then $c(a_1, \ldots, a_n) = \displaystyle\sum_{i=1}^n c(a_1, \ldots, a_i - 1, \ldots, a_n)$.
(d) $c(a_1, \ldots, a_n) = (a_1 + \ldots + a_n)!/a_1! \cdots a_n!$.

Exercises on symmetric functions 02.04.2024

1. Prove the following formulas for the dimension $\chi^\lambda(1)$ of an irreducible S_n-module:
(a) $\chi^\lambda(1)$ is the coefficient of x^μ in $(\sum x_i)^n \sum_{w\in S_n} \varepsilon(w)x^{w\rho}$, where $\mu = \lambda + \rho$.
(b) $\chi^\lambda(1) = n! \det\left(1/(\mu_i - n + j)!\right)$.
(c) $\chi^\lambda(1) = \frac{n!}{\mu!}\Delta(\mu)$, where $\mu! = \prod_i \mu_i!$, and $\Delta(\mu) = \prod_{i<j}(\mu_i - \mu_j)$.

2. Let $\nu = (r, 1^{n-r})$. Prove the following formulas for the character value χ^λ_ν on an r-cycle:
(a) χ^λ_ν is the coefficient of x^μ in $(\sum x_i^r)(\sum x_i)^{n-r} \sum_{w\in S_n} \varepsilon(w)x^{w\rho}$.
(b) $\chi^\lambda_\nu = \sum_i \dfrac{(n-r)!\Delta(\mu_1,\dots,\mu_i - r,\dots,\mu_n)}{\mu_1!\dots(\mu_i - r)!\dots\mu_n!}$.
(c) $\dfrac{\chi^\lambda_\nu}{\chi^\lambda(1)} = \dfrac{(n-r)!}{n!}\sum_{i=1}^n \dfrac{\mu_i!}{(\mu_i - r)!}\prod_{j\neq i}\dfrac{\mu_i - \mu_j - r}{\mu_i - \mu_j}$.
(d) $\dfrac{-r^2 h_\nu \chi^\lambda_\nu}{\chi^\lambda(1)} = \sum_{i=1}^n \mu_i(\mu_i - 1)\dots(\mu_i - r + 1)\varphi(\mu_i - r)/\varphi'(\mu_i)$, where $\varphi(t) = \prod_i (t - \mu_i)$, and $h_\nu = n!/z_\nu = n!/r(n-r)!$.
(e) $\frac{-r^2 h_\nu \chi^\lambda_\nu}{\chi^\lambda(1)}$ is the coefficient of t^{-1} in the Taylor expansion of $t(t-1)\cdots(t-r+1)\varphi(t-r)/\varphi(t)$ in powers of t^{-1}.
(f) If $r = 2$, then $\frac{h_\nu \chi^\lambda_\nu}{\chi^\lambda(1)} = n(\lambda^t) - n(\lambda)$.

3. Prove (a) $\sum_\lambda (\chi^\lambda_\nu)^2 = z_\nu$.
(b) $s_\lambda * s_\lambda = \sum_\nu z_\nu^{-1}(\chi^\lambda_\nu)^2 p_\nu$ (*inner* product).
(c) $\sum_{|\lambda|=n} s_\lambda * s_\lambda = \sum_{|\nu|=n} p_\nu$.
(d) $\sum_\lambda s_\lambda * s_\lambda = \prod_{k\geq 1}(1 - p_k)^{-1}$.
(e) $\prod_{i,j,k}(1 - x_i y_j z_k)^{-1} = \sum_{\lambda,\mu} s_\lambda(x)s_\mu(y)(s_\lambda * s_\mu)(z)$.
(f) $\prod_{i,j,k}(1 + x_i y_j z_k) = \sum_{\lambda,\mu} s_\lambda(x)s_{\mu^t}(y)(s_\lambda * s_\mu)(z)$.

4. Recall (Problem 2, 12.03.2024) that $\Phi = \sum_\lambda s_\lambda$.
(a) Prove $\Phi = \prod_{n \text{ odd}} \exp\left(\dfrac{p_n}{n} + \dfrac{p_n^2}{2n}\right)\prod_{n \text{ even}} \exp\left(\dfrac{p_n^2}{2n}\right)$.
(b) Set $\varphi := \sum_{|\lambda|=n}\chi^\lambda$. Prove that the value at a permutation of the cycle type ν is given by $\varphi(\nu) = \langle \Phi, p_\nu\rangle = \prod_{i\geq 1} a_i^{(m_i(\nu))}$, where $a_i^{(m)}/m!$ is the coefficient of t^m in the series $\exp(t + it^2/2)$ or $\exp(it^2/2)$ depending on whether i is odd or even.
(c) Prove that $\varphi(\nu) = 0$ if the cycle type ν has an odd number of $2r$-cycles for some r.

5. Set $\psi := \sum \chi^\mu$, where the sum is taken over all the even partitions of $2n$ (into even parts). Prove that
(a) the value at a permutation of the cycle type ν is given by $\psi(\nu) = \prod_{i\geq 1} b_i^{(m_i(\nu))}$, where $b_i^{(m)}/m!$ is the coefficient of t^m in the series $\exp(t + it^2/2)$ or $\exp(it^2/2)$ depending on whether i is even or odd.
(b) $\psi(\nu) = 0$ if the cycle type ν has an odd number of $2r - 1$-cycles for some $r \geq 1$.

(c) $\psi(1) = \frac{(2n)!}{2^n n!}$.
(d) $\psi = \operatorname{Ind}_{B_n}^{S_{2n}}(1)$, where B_n is the centralizer in S_{2n} of a permutation of the cycle type (2^n).

Exercises on symmetric functions 09.04.2024

1. We identify $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with the ring of symmetric functions in variables x, y: $f \otimes g \mapsto f(x)g(y)$. We define a coproduct $\Delta\colon \Lambda \to \Lambda \otimes_{\mathbb{Z}} \Lambda$ by $(\Delta f)(x, y) = f(x, y)$. We define a counit $\varepsilon\colon \Lambda \to \mathbb{Z}$ requiring that $\varepsilon(\Lambda^n) = 0$ for $n > 0$, and $\varepsilon(1) = 1$. Prove that
(a) $\Delta h_n = \sum_{0 \le k \le n} h_k \otimes h_{n-k}$.
(b) $\Delta e_n = \sum_{0 \le k \le n} e_k \otimes e_{n-k}$.
(c) $\Delta p_n = p_n \otimes 1 + 1 \otimes p_n$ (i.e. p_n are primitive).
(d) $\Delta s_\lambda = \sum_\mu s_{\lambda/\mu} \otimes s_\mu$.

2. We equip $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with a scalar product such that $\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle \cdot \langle g_1, g_2 \rangle$. Prove that $\Delta\colon \Lambda \to \Lambda \otimes_{\mathbb{Z}} \Lambda$ is adjoint to the multiplication $\mathsf{m}\colon \Lambda \otimes_{\mathbb{Z}} \Lambda \to \Lambda$, and the counit $\varepsilon\colon \Lambda \to \mathbb{Z}$ is adjoint to the unit $e\colon \mathbb{Z} \to \Lambda$. In other words, the Hopf algebra Λ is selfdual.

3. Prove that (notation of 27.02.2024) $D(f)(gh) = \sum_i (D(f_i^{(1)})g) \cdot (D(f_i^{(2)})h)$, where $\Delta f = \sum_i f_i^{(1)} \otimes f_i^{(2)}$.

4. Prove that any primitive element $p \in \Lambda^n$ (i.e. $\Delta p = p \otimes 1 + 1 \otimes p$) is proportional to p_n.

5. Define an involution $\tilde{\omega} = (-1)^n \omega$ on Λ^n. Prove that $\tilde{\omega}$ is an antipode, i.e. $\mathsf{m} \circ (\tilde{\omega} \otimes \mathrm{Id}) \circ \Delta = \mathsf{m} \circ (\mathrm{Id} \otimes \tilde{\omega}) \circ \Delta = e \circ \varepsilon\colon \Lambda \to \Lambda$.

Exercises on symmetric functions 16.04.2024

1. Let $F = \mathbb{Q}(q, t)$, and $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$. Let $\Delta\colon \Lambda_F \to \Lambda_F \otimes \Lambda_F$ be the scalar extension of Δ of 09.04.2024. Let $\langle , \rangle$ be an F-valued nondegenerate symmetric bilinear form on Λ_F such that $\langle \Lambda_F^n, \Lambda_F^m \rangle = 0$ for $m \ne n$. Prove that the following conditions are equivalent:
a) $\langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle$ for any $f, g, h \in \Lambda_F$.
b) $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \zeta_\lambda$ for any λ, μ for a multiplicative family $\zeta_\lambda \in F^\times$ (i.e. $\zeta_\lambda = \zeta_{(\lambda_1)} \zeta_{(\lambda_2)} \cdots \zeta_{(\lambda_\ell)}$).
c) There is an algebra homomorphism $\chi\colon \Lambda_F \to F$ such that $\chi(p_n) \ne 0$ for $n \ge 1$ and $\langle f, g \rangle = \chi(f * g)$ (*inner* product).
d) The dual basis $\{m_\lambda^*\}$ of the monomial basis $\{m_\lambda\}$ is multiplicative (i.e. $m_\lambda^* = m_{(\lambda_1)}^* m_{(\lambda_2)}^* \cdots m_{(\lambda_\ell)}^*$).

2. We set $(a; q)_\mu = \prod_{i \ge 1} (a; q)_{\mu_i}$, where the Pochhammer symbol $(a; q)_m := (a; q)_\infty / (aq^m; q)_\infty = \prod_{r=0}^{m-1} (1 - aq^r)$. Prove that
a) $g_n(x; q, t) = \displaystyle\sum_{|\mu| = n} \frac{(t; q)_\mu}{(q; q)_\mu} m_\mu(x)$.
b) $(q; q)_n g_n(x; q, t) \in \Lambda[q, t]$.

3. Prove that $\omega_{q,t} g_r(x; 0, t^{-1}) = (-t)^{-r} g_r(x; 0, q)$.

4. Prove that the coefficient of m_μ in $D_n(X; q, t) m_\lambda$ equals
$a_{\lambda\mu}(X; q, t) = \sum (-1)^w K_{\pi\mu} \prod_{i=1}^n (1 + X q^{\alpha_i} t^{n-i})$, where
$D_n(X; q, t) = a_\rho(x)^{-1} \sum_{w \in \mathfrak{S}_n} (-1)^w x^{w\rho} \prod_{i=1}^n (1 + X t^{(w\rho)_i} T_{q, x_i})$, the sum runs over all the

triples (w, α, π) such that w is a permutation in $\mathfrak{S}_n$, and $\alpha \sim \lambda$ is a composition in $\mathbb{N}^n$, and π is a partition with $\alpha + \rho = w(\pi + \rho)$, and $K_{\pi\mu}$ is a Kostka number.

5. Prove that in case $\lambda_1 = \mu_1$, we have $a_{\lambda\mu}(X; q, t) = (1 + Xq^{\lambda_1}t^{n-1})a_{\bar\lambda\bar\mu}(X; q, t)$, where $\bar\lambda = (\lambda_2, \lambda_3, \ldots)$ and $\bar\mu = (\mu_2, \mu_3, \ldots)$.

Exercises on symmetric functions 23.04.2024

1. Prove that for $r \geq 0$ and $x = (x_1, \ldots, x_n)$ we have

$$(t-1)\sum_{i=1}^n A_i(x; t)x_i^r = t^n g_r(x; 0, t^{-1}) - \delta_{0r},$$

where $\sum_{i=1}^n A_i(x; t)T_{q,x_i} = D_n^1$, and hence $A_i(x; t) = \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j}$.

2. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\Pi = \Pi(x, y; q, t)$, and $\Pi_0 = \omega_{q,t}\Pi = \prod_{1\leq i,j\leq n}(1 + x_i y_j)$. Prove that

a) $\Pi^{-1}T_{q,x_i}\Pi = \sum_{r\geq 0} g_r(y; 0, t^{-1})t^r x_i^r$ and $\Pi_0^{-1}T_{q,x_i}\Pi_0 = \sum_{r\geq 0}(-1)^r g_r(y; 0, q)x_i^r$.

b) $\Pi^{-1}\tilde E\Pi = \sum_{r\geq 0} g_r(x; 0, t^{-1})g_r(y; 0, t^{-1})t^r$ and $\Pi_0^{-1}\tilde E\Pi_0 = \sum_{r\geq 0}(-1)^r g_r(x; 0, t^{-1})g_r(y; 0, q)$, where $\tilde E = \tilde E_{q,t} := t^{-n}(1 + (t-1)D_n^1)$ acts on symmetric functions in the x variables.

c) $\omega_{q,t}(\Pi^{-1}\tilde E_{q,t}\Pi) = \Pi_0^{-1}\tilde E_{t^{-1},q^{-1}}\Pi_0$ and $\omega_{q,t}\tilde E_{q,t} = \tilde E_{t^{-1},q^{-1}}\omega_{q,t}$, where $\omega_{q,t}$ acts on the x variables.

3. Prove that for any λ, the coefficient of $x_1^{\lambda_1}$ in $P_\lambda(x; q, t)$ equals $P_{\bar\lambda}(\bar x; q, t)$, where $\bar\lambda = (\lambda_2, \lambda_3, \ldots)$ and $\bar x = (x_2, x_3, \ldots)$.

4. For any λ set $f_\lambda(q, t) = (1 - t)\sum_{i\geq 1}(q^{\lambda_i} - 1)t^{i-1}$. Prove that

a) $f_\lambda(q, t) = f_{\lambda^T}(t, q)$.

b) The eigenvalues of the operator $(t-1)E$ (where E is the limit of $E_n = t^{-n}D_n^1 - \sum_{i=1}^n t^{-i}$) are $f_\lambda(q, t^{-1})$.

5. Here is an alternative approach to the computation of $\omega_{q,t}P_\lambda(x; q, t)$. We have $\tilde E = 1 + (t-1)E$. Prove that

a) $\tilde E_{t^{-1},q^{-1}}\omega_{q,t}P_\lambda(q, t) = (1 + f_{\lambda^T}(t^{-1}, q))\omega_{q,t}P_\lambda(q, t)$.

b) $\langle \omega P_\lambda(q, t), P_{\lambda^T}(t, q)\rangle = 1$.

c) $\langle \omega_{q,t}P_\lambda(q, t), P_{\lambda^T}(t, q)\rangle_{t,q} = 1$.

d) $\omega_{q,t}P_\lambda(x; q, t) = Q_{\lambda^T}(x; t, q)$.

Exercises on symmetric functions 30.04.2024

1. A connected skew diagram θ is called a border p-strip if $|\theta| = p$, and it does not contain a 2×2-square. Let λ, μ be partitions of length $\leq m$ such that $\lambda \supset \mu$, and the complement $\lambda - \mu$ is a border p-strip. Let $\xi = \lambda + \rho_m$, $\eta = \mu + \rho_m$, where $\rho_m = (m-1, m-2, \ldots, 1, 0)$. Prove that for certain $j \leq k \leq m$ we have $\xi_j = \eta_k + p$, $\xi_{j+r} = \eta_{j+r-1}$ $(1 \leq r \leq k - j)$, and $\xi_i = \eta_i$ for $i < j$ or $j > k$.

2. Assume moreover that for any $0 \le r \le p-1$, the partition ξ has m_r parts ξ_i congruent to r modulo p. Let us write $\xi_i = p\xi_j^{(r)} + r$ $(1 \le j \le m_r)$, where $\xi_1^{(r)} > \xi_2^{(r)} > \ldots > \xi_{m_r}^{(r)} \ge 0$. We set $\lambda_j^{(r)} = \xi_j^{(r)} - m_r + j$, so that $\lambda^{(r)}$ is a partition. The collection $\lambda^* := (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)})$ is called the p-quotient of λ. Prove that λ^* is independent of $m \ge \ell(\lambda)$ up to cyclic permutation.

3. Let us remove a border p-strip from λ so that what remains is the diagram of a partition. We repeat this procedure as long as it is possible. What remains is called the p-core $\tilde\lambda$ of λ. Prove that it is independent of the choices involved. We write $\lambda \underset{p}{\sim} \nu$ if $\tilde\lambda = \tilde\nu$. For example, 2-cores are exactly partitions $\rho_m,\ m \in \mathbb{N}$.

4. Prove that (a) $\lambda \underset{p}{\sim} \nu$ iff $\eta \equiv w\xi \pmod p$ for a permutation $w \in S_m$, where $\xi = \lambda + \rho_m,\ \eta = \nu + \rho_m$, and $m \ge \ell(\lambda), \ell(\nu)$.
(b) $\lambda \underset{p}{\sim} \nu$ iff $\lambda^t \underset{p}{\sim} \nu^t$.

5. Prove that a partition λ is uniquely defined by its p-quotient λ^* and its p-core $\tilde\lambda$.

Exercises on symmetric functions 07.05.2024

1. Prove that (a) The generating function for the partitions with a given p-core $\tilde\lambda$ is $\sum_{\tilde\mu=\tilde\lambda} t^{|\mu|} = t^{|\tilde\lambda|}P(t^p)^p$, where $P(t) = \prod_{n=1}^\infty (1-t^n)^{-1}$ is the generating function of all the partitions.
(b) The generating function for the p-cores is $\sum t^{|\tilde\lambda|} = P(t)/P(t^p)^p = \prod_{n=1}^\infty \frac{(1-t^{np})^p}{1-t^n}$.
(c) In particular, for $p=2$, we get $\sum_{m=1}^\infty t^{m(m-1)/2} = \prod_{n=1}^\infty \frac{1-t^{2n}}{1-t^{2n-1}}$ (a specialization of the Jacobi triple product identity).

2. Prove that (a) $h(\lambda) = p^{|\lambda^*|}h(\lambda^*)h'(\lambda)$, where $h(\lambda) = \prod_{x\in\lambda} h(x)$ is the product of the hook lengths, $h(\lambda^*) = \prod_{r=0}^{p-1} h(\lambda^{(r)})$, and $h'(\lambda)$ is the product of those hook lengths $h(x)$ which are not divisible by p.
(b) If p is a prime, then $h'(\lambda) \equiv \pm h(\tilde\lambda) \pmod p$.
(c) If p is a prime, then λ is a p-core iff $h(\lambda)$ is relatively prime to p.

3. The content polynomial $c_\lambda(t) := \prod_{x\in\lambda}(t + c(x))$. Prove that $\frac{c_\lambda(t+m)}{c_\lambda(t+m-1)} = \prod_{i=1}^m \frac{t+\xi_i}{t+m-i}$, where $m \ge \ell(\lambda)$, and $\xi_i = \lambda_i + m - i$.

4. Let p be a prime. Prove that $c_\lambda(t) \equiv c_{\tilde\lambda}(t)(t^p - t)^{|\lambda^*|} \pmod p$.

5. Let p be a prime. Let $|\lambda| = |\mu|$. Prove that $\lambda \underset{p}{\sim} \mu \ \Leftrightarrow\ c_\lambda(t) \equiv c_\mu(t) \pmod p$.

Exercises on symmetric functions 14.05.2024

1. Let $v_1, \ldots, v_n \in V$ be distinct vectors in a real vector space equipped with a positive definite scalar product $(,)$. Prove that $\det(q^{(v_i,v_j)}) \not\equiv 0$ as a function of q.

2. a) Let $\lambda \supset \mu$ be partitions such that $\lambda \setminus \mu$ is a horizontal strip. Prove that

$$L(\varphi_{\lambda/\mu}) = \frac{t-q}{1-q} \sum_{1\le i\le j\le \ell(\lambda)} (q^{\lambda_i-\lambda_j} - q^{\lambda_i-\mu_j} - q^{\mu_i-\lambda_{j+1}} + q^{\mu_i-\mu_{j+1}})t^{j-i}$$

and

$$\varphi_{\lambda/\mu} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{\lambda_i-\lambda_j}t^{j-i})f(q^{\mu_i-\mu_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})},$$

where $f(u) = (tu;q)_\infty/(qu;q)_\infty$.
b) Let $\lambda \supset \mu$ be partitions such that $\lambda \setminus \mu$ is a horizontal strip. Prove that

$$L(\psi_{\lambda/\mu}) = \frac{t-q}{1-q} \sum_{1\le i\le j\le \ell(\mu)} (q^{\mu_i-\mu_j} - q^{\lambda_i-\mu_j} - q^{\mu_i-\lambda_{j+1}} + q^{\lambda_i-\lambda_{j+1}})t^{j-i}$$

and

$$\psi_{\lambda/\mu} = \prod_{1\le i\le j\le \ell(\mu)} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})}.$$

c) Let $\lambda \supset \mu$ be partitions such that $\lambda \setminus \mu$ is a vertical strip. Prove that

$$L(\varphi'_{\lambda/\mu}) = (t-q) \sum_{1\le i<j<\infty}^{\lambda_i=\mu_i+1,\ \lambda_j=\mu_j} q^{\mu_i-\mu_j}(t^{j-i} - t^{j-i-1})$$

and

$$\varphi'_{\lambda/\mu} = \prod_{1\le i<j<\infty}^{\lambda_i=\mu_i+1,\ \lambda_j=\mu_j} \frac{(1-q^{\lambda_i-\lambda_j}t^{j-i-1})(1-q^{\mu_i-\mu_j}t^{j-i+1})}{(1-q^{\lambda_i-\lambda_j}t^{j-i})(1-q^{\mu_i-\mu_j}t^{j-i})}.$$

d) Recall that

$$L(\psi'_{\lambda/\mu}) = \frac{(t-q)(t-1)}{qt} \sum_{1\le i<j<\infty}^{\lambda_i=\mu_i,\ \lambda_j=\mu_j+1} q^{\mu_i-\mu_j}t^{j-i}$$

and

$$\psi'_{\lambda/\mu} = \prod_{1\le i<j<\infty}^{\lambda_i=\mu_i,\ \lambda_j=\mu_j+1} \frac{(1-q^{\mu_i-\mu_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j}t^{j-i+1})}{(1-q^{\mu_i-\mu_j}t^{j-i})(1-q^{\lambda_i-\lambda_j}t^{j-i})}.$$

3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, 0, 0, \ldots)$ be a partition viewed as an infinite sequence. Set $G_\lambda(q,t) = (1-t)^2 \sum_{1\le i<j<\infty}(1-q^{\lambda_i-\lambda_j}t^{j-i-1})$. Prove that $\frac{G_\lambda(q,t)}{(1-q)(1-t)} = \sum_{x\in\lambda} q^{a(s)}t^{l(s)}$ and $G_\lambda(q,t) = G_{\lambda^T}(t,q)$.
4. For a partition λ of length $\ell(\lambda) \le n$, set $t = q^k$ and

$$v_\lambda(q,t) = \prod_{1\le i<j\le n} (q^{\lambda_i-\lambda_j}t^{j-i};q)_k,\ v'_\lambda(q,t) = \prod_{1\le i<j\le n} (q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_k.$$

Prove that

$$\varepsilon_{u,t}(P_\lambda) = t^{n(\lambda)} v_\lambda(q,t) \prod_{i=1}^n \frac{(ut^{1-i};q)_{\lambda_i}}{(t;q)_{\lambda_i+k(n-i)}},$$

$$\varepsilon_{u,t}(Q_\lambda) = t^{n(\lambda)} v'_\lambda(q,t) \prod_{i=1}^n \frac{(ut^{1-i};q)_{\lambda_i}}{(q;q)_{\lambda_i+k(n-i)}}.$$

5. For a partition λ of length $\ell(\lambda) \le n$, prove that

$$\varepsilon_{t^n,t}(P_\lambda) = t^{n(\lambda)} \prod_{1\le i<j\le n} \frac{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(t^{j-i};q)_{\lambda_i-\lambda_j}},$$

$$\varepsilon_{qt^{n-1},t}(Q_\lambda) = t^{n(\lambda)} \prod_{1\le i<j\le n} \frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}.$$

Exercises on symmetric functions 21.05.2024

1. Recall that $H(t) = \sum_{r\ge 0} h_r t^r$ is the generating function of complete symmetric functions. Let $H(t) = \prod_{i=0}^\infty \frac{1-bq^it}{1-aq^it}$. Recall from Problem 5 of 30.01.2024 that $h_r = \prod_{i=1}^r \frac{a-bq^{i-1}}{1-q^i}$, $e_r = \prod_{i=1}^r \frac{aq^{i-1}-b}{1-q^i}$, $p_r = \frac{a^r-b^r}{1-q^r}$. Prove that $\displaystyle\sum_{m\ge 0} a^m \frac{(q^{m+1};q)_\infty}{(a^{-1}bq^m;q)_\infty} = \frac{(b;q)_\infty(q;q)_\infty}{(a;q)_\infty(a^{-1}b;q)_\infty}$.

2. Let $0<q<1$ and let f be a function on $[0,1]$. We define the q-integral $\int_0^1 f(x)d_q(x) := (1-q)\sum_{r=0}^\infty q^r f(q^r)$ (assuming that the RHS converges). Similarly, if $f(x) = f(x_1,\dots,x_n)$ is a function on the cube $C^n := [0,1]^n$, we define $\int_{C^n} f(x)d_q(x) := (1-q)^n \sum_{\alpha\in\mathbb{N}^n} q^{|\alpha|} f(q^\alpha)$, where $\alpha = (\alpha_1,\dots,\alpha_n)$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, and $f(q^\alpha) = f(q^{\alpha_1},\dots,q^{\alpha_n})$. Prove the following q-analogue of the Euler formula for his beta-integral (r,s are positive integers):

$$B_q(r,s) := \int_0^1 x^{r-1}(qx;q)_{s-1} d_q x = \Gamma_q(r)\Gamma_q(s)/\Gamma_q(r+s),$$

where $\Gamma_q(r) := (q;q)_{r-1}/(1-q)^{r-1}$ is the q-gamma function.

3. Let $x = (x_1,\dots,x_n)$, and $t = q^k$. We set

$$\Delta^*(x;q,t) := \prod_{1\le i<j\le n}\prod_{r=0}^{k-1}(x_i - q^r x_j)(x_i - q^{-r}x_j) = (-1)^A q^{-B}(x_1\cdots x_n)^{k(n-1)}\Delta(x;q,t),$$

$$\Delta^*_{r,s}(x;q,t) := \Delta^*(x;q,t)\prod_{i=1}^n x_i^{r-1}(qx_i;q)_{s-1}, \quad I_\lambda := \frac{1}{n!}\int_{C^n} P_\lambda(x;q,t)\Delta^*_{r,s}(x;q,t)d_q x,$$

where $A = kn(n-1)/2$, and $B = k(k-1)n(n-1)/4$. For instance, I_0 is a q-analogue of the famous Selberg integral. We define the coefficients $c_{\lambda\beta}$ so that $P_\lambda \Delta^* = \sum_\beta c_{\lambda\beta} x^\beta$ (the sum over $\beta \in \mathbb{N}^n$ such that $|\beta| = |\lambda| + kn(n-1)$). Prove that

$$n! I_\lambda = \sum_\beta c_{\lambda\beta} \int_{C^n} \prod_{i=1}^n x^{r+\beta_i-1}(qx_i;q)_{s-1} d_q x = \sum_\beta c_{\lambda\beta} \prod_{i=1}^n \frac{\Gamma_q(r+\beta_i)\Gamma_q(s)}{\Gamma_q(r+s+\beta_i)} = B_q(r,s)^n \sum_\beta \frac{c_{\lambda\beta}(q^r;q)_\beta}{(q^{r+s};q)_\beta},$$

where $(a;q)_\beta := \prod (a;q)_{\beta_i}$.

4. Prove that a) $\displaystyle\sum_\mu \varepsilon_{u,t}(Q_\mu) P_\mu(x) = \prod_{i=1}^n \frac{(ux_i;q)_\infty}{(x_i;q)_\infty}$.

b) $\varepsilon_{u,t}(Q_\mu)\langle P_\mu, P_\mu\rangle_n'$ equals

$$\left\langle \prod_{i=1}^n \frac{(ux_i;q)_\infty}{(x_i;q)_\infty}, P_\mu \right\rangle_n' = \left[\frac{1}{n!} P_\lambda(x;q,t)(x_1\cdots x_n)^{(n-1)k+a} \Delta(x;q,t) \prod_{i=1}^n \frac{(ux_i^{-1};q)_\infty}{(x_i^{-1};q)_\infty} \right]_1$$
$$= \left[\frac{1}{n!}(-1)^A q^B (x_1\cdots x_n)^a \sum_\beta c_{\lambda\beta} x^\beta \prod_{i=1}^n \frac{(ux_i^{-1};q)_\infty}{(x_i^{-1};q)_\infty} \right]_1,$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is the partition defined by $\mu_i = \lambda_i + (n-1)k + a$ for a positive integer a.

5. Prove that a) $\displaystyle\left[x_i^{a+\beta_i}(ux_i^{-1};q)_\infty/(x_i^{-1};q)_\infty \right]_1 = \frac{(u;q)_{a+\beta_i}}{(q;q)_{a+\beta_i}} = \frac{(u;q)_a}{(q;q)_a} \frac{(uq^a;q)_{\beta_i}}{(q^{a+1};q)_{\beta_i}}$.

b) $\displaystyle\varepsilon_{u,t}(Q_\mu)\langle P_\mu, P_\mu\rangle_n' = \frac{1}{n!}(-1)^A q^B \frac{(u;q)_a^n}{(q;q)_a^n} \sum_\beta c_{\lambda\beta} \frac{(uq^a;q)_\beta}{(q^{a+1};q)_\beta}$.

Exercises on symmetric functions 28.05.2024

1. Let $\Delta'(x;q,t) = \Delta(x;q,t) \prod_{1\le i<j\le n} \frac{1-tx_ix_j^{-1}}{1-x_ix_j^{-1}} = \prod_{1\le i<j\le n} \prod_{r=1}^k (1-q^r x_i x_j^{-1})(1-q^{r-1}x_i^{-1}x_j)$ for $t = q^k$. Prove that

a) $[\Delta'(x;q,t)]_1 = \prod_{i=2}^n \begin{bmatrix} ik \\ k \end{bmatrix}_q$ (or you may just assume this fact in what follows).

b) $\displaystyle\sum_{w\in\mathfrak{S}_n} w \prod_{1\le i<j\le n} \frac{1-tx_ix_j^{-1}}{1-x_ix_j^{-1}} = \prod_{i=2}^n \frac{1-t^i}{1-t}$.

c) $c_n := \frac{1}{n!}[\Delta(x;q,t)]_1 = \prod_{i=2}^n \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}_q$.

d) $\displaystyle\langle P_\lambda, P_\lambda\rangle_n' = \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty} = \prod_{1\le i<j\le n} \prod_{r=1}^{k-1} \frac{1-q^{\lambda_i-\lambda_j+r}t^{j-i}}{1-q^{\lambda_i-\lambda_j-r}t^{j-i}}$.

2. Prove that $\displaystyle\frac{1}{n!}\sum_\beta c_{\lambda\beta}\frac{(uq^a;q)_\beta}{(q^{a+1};q)_\beta} = (-1)^A q^{-B+kn(\mu)} v_\lambda(q,t) \prod_{i=1}^n \frac{(q;q)_a (uq^{k(1-i)};q)_{\mu_i}}{(q;q)_{\mu_i+k(n-i)}(u;q)_a}$, where $v_\lambda(q,t)$ was introduced in Problem 4 of 14.05.2024.

3. Prove that $\displaystyle\frac{1}{n!}\sum_{\beta} c_{\lambda\beta}\frac{(uq^a;q)_\beta}{(q^{a+1};q)_\beta} = u^A q^E v_\lambda(q,t)\prod_{i=1}^{n}\frac{(qu^{-1};q)_{k(i-1)}(uq^a;a)_{\lambda_i+k(n-i)}}{(q^{a+1};q)_{\lambda_i+k(2n-i-1)}}$, where $E = -B + kn(\mu) - \sum_{i=1}^{n} C_i$, and $C_i = (i-1)k(k(i-1)+1)/2$, so that $E = 2k^2\binom{n}{3} + k\left(n(\lambda) + a\binom{n}{2}\right)$.

4. Now take $a = r+s-1$ and $u = q^{1-t}$ in the previous Problem and prove that

$$I_\lambda = q^F\prod_{i=1}^{n}\frac{\Gamma_q(\lambda_i+r+k(n-i))\Gamma_q(s+k(i-1))}{\Gamma_q(\lambda_i+r+s+k(2n-i-1))}\prod_{1\le i<j\le n}\frac{\Gamma_q(\lambda_i-\lambda_j+k(j-i+1))}{\Gamma_q(\lambda_i-\lambda_j+k(j-i))},$$

where $F = k(n(\lambda) + \frac{1}{2}m(n-1)) + \frac{1}{3}k^2 n(n-1)(n-2)$.

5. Prove that $I_\lambda/I_0 = \varepsilon_{u,t}(P_\lambda)\varepsilon_{v,t}(P_\lambda)/\varepsilon_{w,t}(P_\lambda)$, where $u = q^r t^{n-1}$, $v = t^n$, and $w = q^{r+s}t^{2n-2}$.