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Multiparametric R-matrices and Link Invariants

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Abstract

We classify all solutions to the Yang–Baxter equation $R_1R_2R_1 = R_2R_1R_2$ in three dimensions under a specific ansatz, which reduces the problem to solving the equation for several smaller blocks. Then we consider multiparametric (at least four different eigenvalues) R-matrices, obtained through this classification, and explore the link/knot invariants associated with them.

Introduction

Braid Groups

A braid is a homotopy class of a set of n intertwined non-intersecting strands (curves in $\mathbb{R}^3$), such that a horizontal plane cross-section of any strand at any place is exactly one point. The strands are ordered (enumerated).

We define the multiplication operation on the set of braids with a fixed number of strands: if we take two braids a, b and connect the lower ends of a with the upper ends of b which have the same numbers, we get a braid ab.

A set of braids from n strands with multiplication operation forms a braid group B_n.

Artin presentation of the braid group

Consider a set of generators $b_1, b_2, \dots, b_{n-1}$ and their inverses $b_1^{-1}, b_2^{-1}, \dots, b_{n-1}^{-1}$,

$$b_k b_k^{-1} = b_k^{-1} b_k = 1.$$

We impose two more sets of relations on the generators,

1. $b_k b_m = b_m b_k$; $|k-m| \geq 2$, k, m $\in 1, \dots n-1$, commutativity
2. $b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1}$; k $\in 1, \dots n-1$, braid relation.

We obtain the Artin presentation of the braid group B_n. The generator b_i corresponds to a braid where the i-th strand passes under the (i+1)-th strand, and all other strands match the strands of the trivial braid. Similarly, the generator b_i^{-1} corresponds to a braid where the i-th strand passes over the (i+1)-th strand, and all other strands match the strands of the trivial braid.

R-matrix representations of the braid group

Consider automorphisms of the space $V^{\otimes 3}$ $R_1 = R \otimes Id_V$ and $R_2 = Id_V \otimes R$, where $R \in Aut(V \otimes V), Id_V -$ identity operator acting on V.

We call an R-matrix a matrix R which satisfies the Yang-Baxter equation

$$R_1R_2R_1 = R_2R_1R_2. \tag{1}$$

Using R-matrices, we get a representation ρ_R of the braid group $B_n \longrightarrow Aut(V^{\otimes n})$, such that $b_i \longrightarrow R_i$, where R_i is a matrix acting as the identity operator on all components of $V^{\otimes n}$ except V_i and V_{i+1}, and as R on components $V_i \otimes V_{i+1}$.

Finding R-matrices

Ansatz: degree-conserving R-matrices

Consider the basis $\{v_i\}_{i=1,2,...n}$ in space V, $dimV = n$.

We define *degree* of a basis vector $\{v_i\}$ as its index i:

$$deg\{v_i\} := i.$$

We extend the definition of the degree to the basic vectors in the tensor product spaces $V^{\otimes k}$:

$$deg(v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_k}) := \sum_{a=1}^{k} i_a.$$

That gives us decomposition of the space $V^{\otimes k}$ imto a direct sum of its fixed degree subspaces

$$V_d^{(k)} := \mathrm{Span}\{v_{i_1} \otimes ... \otimes v_{i_k} | i_1 + ... + i_k = d\}$$

$$V^{\otimes k} = \bigoplus_{d=k}^{kn} V_d^{(k)}$$

We define a *degree-conserving* R-matrix[1] as an R-matrix for which

$$R\, V_d^{(2)} \subset V_d^{(2)} \quad \forall\, d = 2, \dots, 2n.$$

A degree-conserving R-matrix is a block-diagonal matrix – every block acts on a corresponding subspace $V_d^{(2)}$.

This ansatz has several advantages: a) both the R-matrix and the matrix equation (1) in this ansatz are block-diagonal; b) some of the cubic relations arising in the blocks can be factorized.

R-matrix blocks for dim V = 3

$$1)\ d = i_1 + i_2 = 2, \text{i.e. } (i_1 i_2) = (1,1), \quad A = (q)$$

$$2)\ d = 3, \text{i.e. } (i_1 i_2) \in \{(1,2),(2,1)\}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$3)\ d = 4, \text{i.e. } (i_1 i_2) \in \{(1,3),(2,2),(3,1)\}, \quad C = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

$$4)\ d = 5, \text{i.e. } (i_1 i_2) \in \{(2,3),(3,2)\}, \quad D = \begin{pmatrix} x & p \\ z & y \end{pmatrix},$$

$$5)\ d = 6, \text{i.e. } (i_1 i_2) = (3,3), \quad F = (m).$$

[1] The degree conservation property is called 'Additive Charge Conservation' in [2]. We use a different notation here to avoid unnecessary physical associations.

An R-matrix consisting of these blocks:

$$R = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & u_2 & 0 & u_3 & 0 & 0 \\ 0 & c & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 0 & v_2 & 0 & v_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & p & 0 \\ 0 & 0 & w_1 & 0 & w_2 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m \end{pmatrix}.$$

braid relation blocks for dim V = 3

The braid relation for such a matrix is also block-diagonal, and the blocks are indexed by the values of the quantity $\sum\limits_3 = i_1 + i_2 + i_3$.

$$\sum_3 = i_1 + i_2 + i_3.$$

$$1) \sum_3 = 3, (i_1 i_2 i_3) = (1,1,1),\ K_1 = K_2 = (q)$$

$$2) \sum_3 = 4, (i_1 i_2 i_3) \in \{(1,1,2),(1,2,1),(2,1,1)\}$$

$$L_1 = \begin{pmatrix} q & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, L_2 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & q \end{pmatrix}$$

$$L = L_1 \cdot L_2 \cdot L_1 - L_2 \cdot L_1 \cdot L_2 = 0$$

$$3) \sum_3 = 5, (i_1 i_2 i_3) \in \{(1,1,3),(1,2,2),(1,3,1),(2,1,2),(2,2,1),(3,1,1)\}$$

$$M_1 = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & u_1 & 0 & u_2 & u_3 \\ 0 & c & 0 & d & 0 & 0 \\ 0 & 0 & v_1 & 0 & v_2 & v_3 \\ 0 & 0 & w_1 & 0 & w_2 & w_3 \end{pmatrix}, M_2 = \begin{pmatrix} u_1 & u_2 & u_3 & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & 0 & 0 & 0 \\ w_1 & w_2 & w_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

$$M = M_1 \cdot M_2 \cdot M_1 - M_2 \cdot M_1 \cdot M_2 = 0$$

$$4) \sum_3 = 6, (i_1 i_2 i_3) \in \{(1,2,3),(1,3,2),(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1)\}$$

$$N_1 = \begin{pmatrix} a & 0 & b & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & u_2 & 0 & u_3 & 0 \\ c & 0 & d & 0 & 0 & 0 & 0 \\ 0 & v_1 & 0 & v_2 & 0 & v_3 & 0 \\ 0 & 0 & 0 & 0 & a' & 0 & b' \\ 0 & w_1 & 0 & w_2 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & c' & 0 & d' \end{pmatrix}, N_2 = \begin{pmatrix} a' & b' & 0 & 0 & 0 & 0 & 0 \\ c' & d' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & u_2 & u_3 & 0 & 0 \\ 0 & 0 & v_1 & v_2 & v_3 & 0 & 0 \\ 0 & 0 & w_1 & w_2 & w_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b \end{pmatrix}$$

$$N = N_1 \cdot N_2 \cdot N_1 - N_2 \cdot N_1 \cdot N_2 = 0$$

$$5)\sum_3 = 7, (i_1 i_2 i_3) \in \{(1,3,3),(2,2,3),(2,3,2),(3,1,3),(3,2,2),(3,3,1)\}$$

$$O_1 = \begin{pmatrix} u_1 & u_2 & 0 & u_3 & 0 & 0 \\ v_1 & v_2 & 0 & v_3 & 0 & 0 \\ 0 & 0 & a' & 0 & b' & 0 \\ w_1 & w_2 & 0 & w_3 & 0 & 0 \\ 0 & 0 & c' & 0 & d' & 0 \\ 0 & 0 & 0 & 0 & 0 & q' \end{pmatrix}, O_2 = \begin{pmatrix} q' & 0 & 0 & 0 & 0 & 0 \\ 0 & a' & b' & 0 & 0 & 0 \\ 0 & c' & d' & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & u_2 & u_3 \\ 0 & 0 & 0 & v_1 & v_2 & v_3 \\ 0 & 0 & 0 & w_1 & w_2 & w_3 \end{pmatrix}$$

$$O = O_1 \cdot O_2 \cdot O_1 - O_2 \cdot O_1 \cdot O_2 = 0$$

$$6)\sum_3 = 8, (i_1 i_2) \in \{(2,3,3),(3,2,3),(3,3,2)\}$$

$$P_1 = \begin{pmatrix} a' & b' & 0 \\ c' & d' & 0 \\ 0 & 0 & q' \end{pmatrix}, P_2 = \begin{pmatrix} q' & 0 & 0 \\ 0 & a' & b' \\ 0 & c' & d' \end{pmatrix}$$

$$P = P_1 \cdot P_2 \cdot P_1 - P_2 \cdot P_1 \cdot P_2 = 0$$

$$7)\ \sum_3 = 9, (i_1 i_2 i_3) = (3,3,3),\ S_1 = S_2 = (q')$$

The braid relations for blocks 1 and 7 are trivial. We will be solving the system of relations with the help of Wolfram Mathematica in the remaining blocks $2,\ldots,6$ sequentially, starting with block 2.

Symmetry Transformations

Consider the symmetry transformations of R-matrix, compatible with our ansatz.

a) Transposition $R <\!\!-\!\!> R^T$.
Under this transformation, all blocks of the ansatz are transposed

$$B <\!\!-\!\!> B^T, C <\!\!-\!\!> C^T, D <\!\!-\!\!> D^T. \qquad (a)$$

b) Rearrangement of the spaces $V_1 \otimes V_2 <\!\!-\!\!> V_2 \otimes V_1$, in which R operates

$$R_{12} <\!\!-\!\!> R_{21} = P_{12} R_{12} P_{12}.$$

To understand the effect of this transformation on R, let's define the *reflection transformation* of a matrix M relative to its center.

$$M <\!\!-\!\!> M^\circ : M^\circ_{ij} = M_{n+1-i, n+1-j},\ \text{where } n \text{ is the size of M.}$$

On the blocks of the R-matrix, this transformation acts as:

$$R_{12} <\!\!-\!\!> R_{21} : B <\!\!-\!\!> B^\circ, C <\!\!-\!\!> C^\circ, D <\!\!-\!\!> D^\circ. \qquad (b)$$

c) Inversion of the basis in V: $\{v_1, v_2, v_3\} <\!\!-\!\!> \{v_3, v_2, v_1\}$ (v_i - basis in V).We are redenoting the indices of the matrix R $\{1,2,3\} <\!\!-\!\!> \{3,2,1\}$. Under inversion $R <\!\!-\!\!> R^*$, the blocks of R^* A^*, B^*, C^*, D^*, E^*, F^* are related to the blocks of R in the following way:

$$A^* = F, F^* = A, B^* = D^\circ, D^* = B^\circ, C^* = C^\circ. \qquad (c)$$

d) Additionally, if in block C $v_1 = u_2 = v_3 = w_2 = 0$,

$$C = \begin{pmatrix} u_1 & 0 & u_3 \\ 0 & v_2 & 0 \\ w_1 & 0 & w_3 \end{pmatrix},$$

then it splits into two blocks: $\begin{pmatrix} u_1 & u_3 \\ w_1 & w_3 \end{pmatrix}$, и $\begin{pmatrix} v_2 \end{pmatrix}$ .In this case, the group of symmetries expands: any permutation of vectors in the basis becomes possible. It leads to permutations and reflections of the triples of 2×2 blocks and 1×1 blocks of R-matrix.

For all these symmetry transformations, the transformed R-matrix satisfies both the ansatz conditions and the Yang-Baxter equation. Therefore, when listing R-matrices, we will not separately consider R-matrices connected by the symmetry transformations. In addition, we assume $detR \neq 0$.

Classification

We obtained a complete classification, which is consistent with (and slightly simplifies) the results established in [1, 2]. We organize our results in 5 families of the R-matrices. In the formulas below symbols q, r, s, t, m, p are reserved for the eigenvalues of the R-matrices (excepting cases B_6 and C); symbols ε, η, ω are reserved for different roots of 1:

$$\varepsilon = \pm 1, \quad \eta^2 = -1, \quad \omega^3 = -1, \ \omega \neq -1.$$

- **Family A: multivalued R-matrices**

Case $A_{1,1}$

$$A = (q), \ B = \begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 0 & ty \\ 0 & q & 0 \\ \frac{t}{y} & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & tz \\ \frac{t}{z} & 0 \end{pmatrix}, \ F = (s).$$

Eigenvalues: $(q, r, s, t, -t)$ with multiplicities (3, 1, 1, 2, 2).

Case $A_{1,2}$

$$A = (q), \ B = \begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 0 & ty \\ 0 & r & 0 \\ \frac{t}{y} & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & tz \\ \frac{t}{z} & 0 \end{pmatrix}, \ F = (s).$$

Eigenvalues: $(q, r, s, t, -t)$ with multiplicities (2, 2, 1, 2, 2).

Case A_2

$$A = (q), \ B = \begin{pmatrix} 0 & tx \\ \frac{t}{x} & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & 0 & my \\ 0 & r & 0 \\ \frac{m}{y} & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & pz \\ \frac{p}{z} & 0 \end{pmatrix}, \ F = (s).$$

Eigenvalues: $(q, r, s, t, -t, m, -m, p, -p)$ all with multiplicity 1.

Case A_3

$$A=(q),\ B=\begin{pmatrix}0 & tx\\ \frac{t}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}0 & 0 & ty\\ 0 & r & 0\\ \frac{t}{y} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}r & 0\\ 0 & r\end{pmatrix},\ F=(r).$$

Eigenvalues: $(q, r, t, -t)$ with multiplicities (1, 4, 2, 2).

- **Family B: Hecke type R-matrices (two eigenvalues q and r: $q \neq \pm r$)**

Case $B_{1,1}$ ($GL(3)$-type 'Drinfe'ld-Jimbo' R-matrix)

$$A=(q),\ B=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}q+r & 0 & qy\\ 0 & q & 0\\ -\frac{r}{y} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}q+r & qz\\ -\frac{r}{z} & 0\end{pmatrix},\ F=(q).$$

Eigenvalues: (q, r) with multiplicities (6, 3).

Case $B_{1,2}$ ($GL(2|1)$-type 'Kulish-Sklyanin' R-matrix)

$$A=(q),\ B=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}q+r & 0 & qy\\ 0 & q & 0\\ -\frac{r}{y} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}q+r & qz\\ -\frac{r}{z} & 0\end{pmatrix},\ F=(r).$$

Eigenvalues: (q, r) with multiplicities (5, 4).

Case $B_{2,1}$ ($GL(3)$-type 'Cremmer-Gervais' R-matrix)

$$A=(q),\ B=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}q+r & 0 & -\frac{q^2x^2}{r}\\ -\frac{r^2\,u}{q^2x^2} & q & u\\ \frac{r^2}{q\,x^2} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ F=(q).$$

Eigenvalues: (q, r) with multiplicities (6, 3).

Case $B_{2,2}$ ($GL(2|1)$-type R-matrix)

$$A=(q),\ B=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}q+r & 0 & -\frac{q^2x^2}{r}\\ -\frac{r\,u}{q\,x^2} & r & u\\ \frac{r^2}{q\,x^2} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ F=(q).$$

Eigenvalues: (q, r) with multiplicities (5, 4).

Case $B_{3,1}$

$$A=(q),\ B=\begin{pmatrix}q+r & qx\\ -\frac{r}{x} & 0\end{pmatrix},\ C=\begin{pmatrix}q+r & 0 & qy\\ 0 & q & 0\\ -\frac{r}{y} & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}q & 0\\ 0 & q\end{pmatrix},\ F=(q).$$

Eigenvalues: (q, r) with multiplicities (7, 2).

Case $B_{3,2}$

$$A=(q),\ B=\begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix},\ C=\begin{pmatrix} q+r & 0 & qy \\ 0 & r & 0 \\ -\frac{r}{y} & 0 & 0 \end{pmatrix},\ D=\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},\ F=(r).$$

Eigenvalues: (q, r) with multiplicities (3, 6).

Case $B_{4,1}$

$$A=(q),\ B=\begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix},\ C=\begin{pmatrix} q+r & 0 & -\frac{q^2x^2}{r} \\ -\frac{r^2u}{q^2x^2} & q & u \\ \frac{r^2}{q\,x^2} & 0 & 0 \end{pmatrix},\ D=\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix},\ F=(q).$$

Eigenvalues: (q, r) with multiplicities (7, 2).

Case $B_{4,2}$

$$A=(q),\ B=\begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix},\ C=\begin{pmatrix} q+r & 0 & -\frac{q^2x^2}{r} \\ -\frac{r\,u}{q\,x^2} & r & u \\ \frac{r^2}{q\,x^2} & 0 & 0 \end{pmatrix},\ D=\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},\ F=(r).$$

Eigenvalues: (q, r) with multiplicities (3, 6).

Cases $B_{5,1},\ B_{5,2}$

$$A=(q),\ B=\begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix},\ C=\begin{pmatrix} 0 & 0 & -\frac{q^2x^2}{r} \\ u & q & \frac{qx^2u}{r} \\ \frac{r^2}{q\,x^2} & 0 & q+r \end{pmatrix},\ D=\begin{pmatrix} 0 & -rx \\ \frac{q}{x} & q+r \end{pmatrix},\ F=(-\frac{r^3}{q^2}).$$

with an additional relation among eigenvalues q and r

- in case $B_{5,1}$: $r=\eta q$, $\eta^2=-1$. Eigenvalues: $(q, \eta q)$ with multiplicities (5, 4);
- in case $B_{5,2}$: $r=\omega q$, $\omega^3=-1, \omega\neq -1$. Eigenvalues: $(q, \omega q)$ with multiplicities (6, 3).

Case B_6

$$A=(q),\ B=\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix},\ C=\begin{pmatrix} r & \frac{rs(q-r)}{q\,y} & \frac{(r-q)x}{y} \\ y & q-\frac{r\,s}{q} & x \\ \frac{y(s-q)}{x} & \frac{rs(q-s)}{q\,x} & s \end{pmatrix},\ D=\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix},\ F=(q).$$

with an additional restriction on the parameters: $r^2s^2=q^2(r-q)(s-q)$.
Eigenvalues: $(q, -\frac{r^2s^2}{q^3})$ with multiplicities (8, 1).

- **Family C: Birman-Murakami-Wenzl type R-matrices**

Case C ($O(3)$-type R-matrix)

$$A=F=(q),\ B=D=\begin{pmatrix} q(1-t^2) & qx \\ \frac{qt^2}{x} & 0 \end{pmatrix},\ C=\begin{pmatrix} q(1-t)(1-t^2) & q(1-t)u & qx^2 \\ \frac{qt(1+t)(1-t^2)}{u} & qt & 0 \\ \frac{qt^4}{x^2} & 0 & 0 \end{pmatrix}.$$

Eigenvalues: $(q, -qt^2, qt^3)$ with multiplicities (5, 3, 1).

- **Family D: R-matrices related to cyclic representations of $U_q gl(2)$ at $q = \sqrt[3]{1}$**

Case D

$$A = (q),\ B = \begin{pmatrix} q+r & qx \\ -\frac{r}{x} & 0 \end{pmatrix},\ C = \begin{pmatrix} \frac{(q+r)(q-\omega r)}{q} & r(\omega-1)u & qx^2 \\ \frac{(q+r)(q-\omega r)}{qu} & \omega r & 0 \\ \frac{r^2}{qx^2} & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} \frac{r(q-\omega r)}{q} & \frac{rx}{\omega} \\ \frac{r^2\omega^2}{qx} & 0 \end{pmatrix},\ F = (\frac{r^2}{\omega^2 q}).$$

Eigenvalues: $(q, r, \frac{r^2}{\omega^2 q})$ with multiplicities (3, 3, 3), $\omega^3 = -1, \omega \neq -1$.

- **Family F: Permutation type R-matrices (two eigenvalues: q and $-q$)**

Cases $F_{1,1},\ F_{1,2}$

$$A = (q), B = \begin{pmatrix} 0 & qx \\ \frac{q}{x} & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & qx^2 \\ u & \varepsilon q & v \\ \frac{q}{x^2} & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & qx \\ \frac{q}{x} & 0 \end{pmatrix}, F = (q).$$

- In case $F_{1,1}$ $\varepsilon = 1$. Eigenvalues: $(q, -q)$ with multiplicities (6, 3).
- In case $F_{1,2}$ $\varepsilon = -1$. Eigenvalues: $(q, -q)$ with multiplicities (5, 4).

Cases $F_{2,1},\ F_{2,2}$

$$A = (q),\ B = \begin{pmatrix} 0 & qx \\ \frac{q}{x} & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & qx^2 \\ u & \varepsilon q & -\varepsilon u x^2 \\ \frac{q}{x^2} & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} \varepsilon q & 0 \\ 0 & \varepsilon q \end{pmatrix},\ F = (\varepsilon q).$$

- In case $F_{2,1}$ $\varepsilon = 1$. Eigenvalues: $(q, -q)$ with multiplicities (7, 2).
- In case $F_{2,2}$ $\varepsilon = -1$. Eigenvalues: $(q, -q)$ with multiplicities (3, 6).

Remark Let us comment on the relation of our list of R-matrices and the lists from [1, 2]. In section 5 of [1] authors classified degree conserving R-matrices for which all the components u_2, v_1, v_3 and w_2 of the 3×3 matrix C vanish. Such R-matrices are named there *'Strict Charge Conserving'*. The correspondence between their and our lists is as follows: our case A_2 was considered in Lemma 5.4; cases $A_{1,2}$ and $A_{1,1}$ are given in Lemma 5.5; cases $B_{1,1}$ and $B_{1,2}$ — in Lemma 5.8; cases A_3, $B_{3,2}$ and $B_{3,1}$ — in Lemma 5.12.

The rest of R-matrices from the classification list are derived in [2]. Our cases $B_{2,1/2}$ correspond to (32)/(33) in [2]; cases $B_{4,1/2}$ correspond to (30)/(31) in [2]; cases $B_{5,1/2}$ correspond to (35)/(34) in [2]; case B_6 corresponds to (27a,b) in [2].

Case C corresponds to (28) in [2].

Case D corresponds to (29) in [2]. With particular choices $r = -q$, and $\omega r = q$ case D also gives cases (38), and (39) in [2].

Cases $F_{1,1}$ and $F_{1,2}$ correspond to (36) in [2] of which (37) is a particular subcase.

Note also that for the family of the Hecke type R-matrixes one can consider the limiting point $q = -r$ of the cases $B_{1,*}, \ldots, B_{4,*}$. In this way one obtains subamilies of the cases A_2, $F_{1,1/2}$ and $F_{2,1/2}$ from the cases $B_{1,1/2}$, $B_{2,1/2}$ and $B_{3,1/2}$, respectively. By contrast, letting $q = -r$ in cases $B_{4,1/2}$ one reproduces cases $F_{2,1/2}$ in their full generality. That is the reason why cases $F_{2,1/2}$ does not show up in the classification scheme of [2]: these cases are included in solutions (30)/(31) there.

We got four distinct multivalued (at least four different eigenvalues) R-matrices to consider: $A_{1,1}$, $A_{1,2}$, A_2 and A_3. We are particularly interested in them, since R-matrices with 2 or 3 different eigenvalues produce a version of already known invariants.

Connection between R-matrices and link/knot invariants

We define a skew-invertible R-matrix as an R-matrix, for which there exists a matrix Ψ satisfying the following condition:

$$Tr_2(R_{12}\Psi_{23}) = P_{13} = Tr_2(\Psi_{12}R_{23}),$$

where Tr_i denotes a trace over the space i, and P_{13} is a permutation matrix. For a skew-invertible R-matrix, we define $D = Tr_2(\Psi_{12}.)$

Every braid corresponds to an oriented link or to a knot. To obtain a link/knot from the given braid diagram, we should “close” all strands, i.e. connect upper and lower ends of all strands. So using R-matrix representation of braids, we can also obtain a polynomial link/knot invariant associated with the given R-matrix, if that R-matrix is skew-invertible, and if it satisfies the following condition

$$Tr_2 D_2 R_{12}^{-1} = Id_1.$$

To do so, we have to compute the following expression for a braid with n strings:

$$\mathrm{Inv} = Tr_1 Tr_2 \dots Tr_n(D_1 D_2 \dots D_n f(B)),$$

where $f(B)$ denotes an R-matrix representation of the particular braid.

The R-matrix A_3 is not skew-invertible, and therefore doesn't produce link/knot invariants. Case A_2 gives trivial (numerical instead of polynomial) answers. Case $A_{1,1}$ produces potentially meaningful results. Case $A_{1,2}$ produces the same results as $A_{1,1}$.

Detailed overview of case $\mathbf{A}_{1,1}$

$$R = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q+r & 0 & qx & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ty & 0 & 0 \\ 0 & -\frac{r}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & zt & 0 \\ 0 & 0 & \frac{t}{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{t}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{pmatrix}$$

From $Tr_2(R_{12}\Psi_{23}) = P_{13}$ and $D = Tr_2(\Psi_{12})$ we get

$$\mathrm{D} = \begin{pmatrix} -\frac{r}{q^2} & 0 & 0 \\ 0 & \frac{1}{q} & 0 \\ 0 & 0 & \frac{1}{s} \end{pmatrix}.$$

From $Tr_2 D_2 R_{12}^{-1} = Id_1$, we get $r = -q^3, s^2 = 1$, that leads to

$$\mathrm{D} = \begin{pmatrix} q & 0 & 0 \\ 0 & \frac{1}{q} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix},$$

$$R=\left(\begin{array}{ccccccccc} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q-q^3 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & q^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 \end{array}\right).$$

Here we also set x = y = z =1 for simplicity, since these parameters do not participate in the resulting invariant.

Remarkable feature of $A_{1,1}$

An embedding $f : R \to R^3$ is called a long knot if there exist $a, b \in R$ such that $f(t) = (0, 0, t)$ for any $t < a$ or $t > b$.

A long knot R-matrix invariant is derived similarly to a regular R-matrix invariant, with the exception that we do not take the trace over the first space,

$$\text{Inv1} = Tr_2 \dots Tr_n(D_2 \dots D_n f(B)).$$

For previously considered R-matrices Inv1 was a scalar matrix [3], $c \times Id$, where c is the long knot invariant. However, in our case, we get a diagonal, but not a scalar matrix,

$$\text{X} = \left(\begin{array}{ccc} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{array}\right).$$

Presumably, $d_1 = d_2$, but so far it is an observation (checked for all prime knots with up to 9 crossings), not a proven fact. Under this assumption, we can rewrite our invariant as

$$I = qd_1 + \frac{1}{q}d_1 + sd_3 = (q + \frac{1}{q})d_1 + sd_3, (s^2 = 1).$$

Now, we can focus on d_1 and d_3. Let's split them into two components, $d_1 = (A) + (B)$, $d_3 = (C) + (D)$. Here, A, C are parts of the trace with $i, j \neq 3$ for all matrix components a_{ij}. Parts of the trace with $i = 3$ or $j = 3$ form B and D.

Invariant calculation examples

1) Knot 7_2, braid notation {1,1,1,2,-1,2}:

$$d_1 = (q^2 - q^4 + 2q^6 - 2q^8 + 2q^10 - q^12 + q^14 - q^16) + (0),$$

$$d_3 = (0) + (1).$$

2) Link $L7a7\{0, 0\}$, braid notation {-1, 2, 2, -1, -3, 2, -3}:

$$d_1 = (\frac{1}{q^8} - q^6 - \frac{1}{q^6} + 3q^4 + \frac{4}{q^4} - 3q^2 - \frac{3}{q^2} + 4) + (\frac{sq^5}{t^4} + \frac{sq}{t^4} + 1 + \frac{s}{q^5} + \frac{s}{q}),$$

$$d_3 = (\frac{1}{q^6} + \frac{1}{q^4} + \frac{1}{q^2} + 1) + (\frac{sq}{t^4} + \frac{s}{qt^4} + 1 + sq + \frac{s}{q}).$$

Further analysis

The first component of d_1, A, represents Jones invariant, so we get rid of it to focus on the new parts:

$$\tilde{I} = I - (q + \frac{1}{q})A = I_1(q,t) + I_2(q,t).$$

Key observations:

- For knots, $I_1 = 0$, $I_2 = 1$, so our invariant does not provide any new information.
- For links, I_1, I_2 are usually (but not always) non-trivial polynomials.
- Our invariant can differentiate some of the links that are not distinguishable by Jones, HOMFLY, and Kauffman Polynomials, e.g. $L10a38\{1\}$ and $L10a108\{1\}$.
- Our invariant can differentiate some of the links that are not distinguishable by Multivariable Alexander Polynomial, e.g. $L10a136\{0,1\}$ and $L10a136\{1,0\}$.

It follows from the last two points that our invariant is different from all the main polynomial link invariants, mentioned above.

References

[1] Paul Martin and Eric C. Rowell. Classification of spin-chain braid representations. *arXiv: 2112.04533*, 2021.

[2] Jarmo Hietarinta, Paul Martin, Eric C. Rowell. Solutions to the constant Yang-Baxter equation: additive charge conservation in three dimensions. *arXiv: 2310.03816*, 2023.

[3] A. P. Isaev, Quantum Groups And Yang-Baxter Equations, preprint MPIM (Bonn), MPI 2004-132, page 32, proposition 4.