Special functions. Lecture and seminar 1

Three definitions of Euler Gamma function.

1 Weierstrass infinite product.

$$\Gamma^{-1}(z) = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \tag{1}$$

Here γ is Euler constant

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right) = 0,57721...$$

Convergence of infinite product equivalent to convergence of the series of logarithms

$$\sum_{n\geq 1} \log\left(1 + \frac{z}{n}\right) - \frac{z}{n} \tag{2}$$

For z, $|z| < N/2$ and $n > N$

$$\left|\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right| = \left|\frac{z^2}{2n^2}\right| \left|1 - \frac{2z}{3n} + \ldots\right| \leq \left|\frac{N^2}{4n^2}\right| \left(1 + \frac{1}{2} + \ldots\right) \leq \frac{N^2}{2n^2}$$

which shows that the series (2) is majorated by convergent series

$$\sum_{n\geq 1} \frac{1}{n^2}$$

From (1) one can check basic functional equation

$$\Gamma(z+1) = z\Gamma(z). \tag{3}$$

For that consider the ratio of corresponding approximations of both sides of equation (3)

$$(z+1)e^{\gamma(z+1)} \prod_{n=1}^{N} \left(1 + \frac{z+1}{n}\right) e^{-\frac{z+1}{n}} \ : \ z^{-1} \cdot z e^{\gamma z} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

It is equal to

$$e^{\gamma - 1 - \ldots - \frac{1}{N}}(z + N + 1) = e^{\log N + O(\frac{1}{N}) - \log(z+N+1)} = e^{-\log(1 + \frac{z+1}{N}) + O\left(\frac{1}{N}\right)}$$

and clearly tends to 1 when N tends to infinity.

From Weierstrass presentation we see that $\Gamma(z)$ is a meromorphic functions with simple poles at $z = 0, -1, ...$ without zeros.

2. Euler product

$$\Gamma(z) = \lim_{N\to\infty} \frac{N! N^z}{z(z+1)\cdots(z+N)} \tag{4}$$

Let us derive (4) from (1). Rewrite the approximation

$$ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

as

$$\begin{aligned}\frac{z(z+1)\cdots(z+N)}{N!}e^{(\gamma-1-\ldots-\frac{1}{N})z} =& \frac{z(z+1)\cdots(z+N)}{N!}e^{(-\log N+O(\frac{1}{N}))z}\\ =& \frac{z(z+1)\cdots(z+N)}{N!}N^{-z}\left(1+O(\frac{1}{N})\right)\end{aligned}$$

and we arrive to Euler product.

3. Euler integral

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, \qquad \operatorname{Re} z>0 \tag{5}$$

The integral converges for $\operatorname{Re} z>0$. Let us prove that it coincides under this restriction with Euler product. We have

$$\int_0^\infty t^{z-1}e^{-t}dt=\int_0^N t^{z-1}e^{-t}dt+\int_N^\infty t^{z-1}e^{-t}dt$$

The second integral tends to zero when N tends to infinity. The first integral we compare with

$$\int_0^N t^{z-1}\left(1-\frac{z}{N}\right)^N dt.$$

The latter can be calculated by induction and is equal precisely to

$$\frac{N!N^z}{z(z+1)\cdots(z+N)}$$

so the only thing to show is that the integral

$$\int_0^N \left\{e^{-t}-\left(1-\frac{t}{N}\right)\right\}t^{z-1}dt \tag{6}$$

tends to zero when N tends to infinity.

Substitute into inequality $\log(1+u)>u$ for $|u|<1$, the variables $\pm t/N$ instead of u we get for $0<t<N$

$$\left(1-\frac{t}{N}\right)^N<e^{-t}<\left(1+\frac{t}{N}\right)^{-N},$$

and then due to Bernoulli inequality

$$(1-\alpha)^n>1-n\alpha$$

for $0<\alpha<1$, we get:

$$0<e^{-t}-\left(1-\frac{t}{N}\right)^N=e^{-t}\left\{1-e^t\left(1-\frac{t}{N}\right)^N\right\}<e^{-t}\left\{1-\left(1-\frac{t^2}{N^2}\right)^N\right\}<e^{-t}\frac{t^2}{N}.$$

Thus the integral (7) is less then C/N, where $C = \int_0^\infty e^{-t} t^{s+1} ds$, and tends to zero when N tends to infinity.

We want to extend the Euler integral to other values of z. It can be done as follows/

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) d\,t = \int_0^1 t^{z-1} \exp(-t) d\,t + \int_1^\infty t^{z-1} \exp(-t) d\,t =$$

$$\int_0^1 t^{z-1} d\,t + \int_0^1 t^{z-1} (\exp(-t) - 1) d\,t + \int_1^\infty t^{z-1} \exp(-t) d\,t =$$

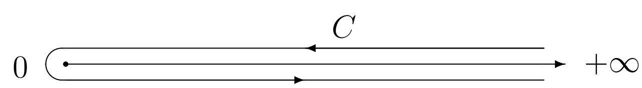
$$1/z + \int_0^1 t^{z-1} (\exp(-t) - 1) d\,t + \int_1^\infty t^{z-1} \exp(-t) d\,t$$

The first integral converges for $\mathrm{Re}(s) > -1$ since $\exp(-t) - 1 \sim -t$ for $t \to 0$. This procedure can be repeated by using Taylor decomposition of e^{-t} in a vicinity of zero and gives analitycal continuation of Γ function to the left half plane.

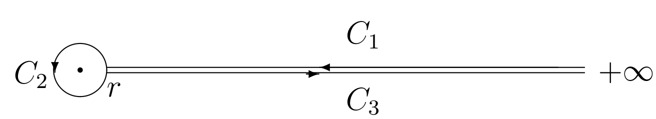
Another way to extend Euler integral is known as Hankel integral representation. Consider the contour integral

$$\oint_C (-t)^{z-1} e^{-t} dt = \oint_C e^{(z-1)(\log|t| + i \arg t - i\pi) - t} dt$$

where contour C starts in $+\infty$ goes over the real line, encircle zero and returns to $+\infty$ below the real line



Assume first that $\mathrm{Re}\, z > 0$ and contract contour close to real line so that it can be decomposed into three parts C_1, C_2 and C_3



Then itegrals over C_1 and C_3 are equal to

$$\oint_{C_1} (-t)^{z-1} e^{-t} dt = e^{-\pi i} \int_0^{+\infty} t^{z-1} e^{-t} dt, \qquad \oint_{C_3} (-t)^{z-1} e^{-t} dt = -e^{\pi i} \int_0^{+\infty} t^{z-1} e^{-t} dt$$

For the integral over C_2 we have

$$\oint_{C_2} e^{(z-1)(\log|t| + i \arg t - i\pi) - t} dt = i r^z \int_{-\pi}^{\pi} e^{iz\phi + r(\cos\phi + i \sin\phi)} d\phi.$$

When r tends to zero the integrand tends to $e^{iz\phi}$, while the integral has a finite limit $\frac{e^{\pi i z} - e^{-\pi i z}}{z}$, so that for $\mathrm{Re}\, z > 0$ the whole expression tends to zero

Finally for $\operatorname{Re} z > 0$ we get

$$(e^{-\pi iz} - e^{\pi iz})\Gamma(x) = \oint_C (-t)^{z-1} e^{-t} dt$$

and

$$\Gamma(x) = \frac{-1}{2i \sin \pi x} \oint_C (-t)^{z-1} e^{-t} dt$$

Since both sides are analytical functions the equality takes place for any $z \notin \mathbb{Z}$.

Euler beta integral

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

It converges when $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$. Let us prove that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

for $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$. Then $\Gamma(p)\Gamma(q)$ can be presented, due to absolute convergence, as a double integral

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-x} x^{p-1} dx \cdot \int_0^\infty e^{-y} y^{q-1} dy = \int_0^\infty \int_0^\infty e^{-x-y} x^{p-1} y^{q-1} dx dy$$

Change variables $u = x + y$, $v = x/(x+y)$, that is, $x = uv$, $y = u(1-v)$, $dxdy = ududv$. Then by Fubini theorem

$$\int_0^\infty \int_0^\infty e^{-x-y} x^{p-1} y^{q-1} dx dy = \int_0^\infty \int_0^1 e^{-u} u^{p+q-1} v^{p-1} (1-v)^{q-1} du dv,$$

The latter integral equals to $\Gamma(p+q)B(p,q)$.

Consider infinite product

$$\prod_{n=0}^\infty \frac{(a_1+n)\cdots(a_k+n)}{(b_1+n)\cdots(b_l+n)}$$

It is not difficult to see, looking to the corresponding series of logarithms and repeating the arguments given in considerations of Weierstrass infinite product, that it converges if and only if $k = l$ and $a_1 + \cdots + a_k = b_1 + \cdots + b_l$. Moreover substitution of Weierstrass product says that

$$\prod_{n=0}^\infty \frac{(a_1+n)\cdots(a_k+n)}{(b_1+n)\cdots(b_k+n)} = \frac{\Gamma(b_1)\cdots\Gamma(b_k)}{\Gamma(a_1)\cdots\Gamma(a_k)}.$$

This product can be also presented as an integral. For that we use Frullani integral

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) \log \frac{b}{a} \tag{7}$$

valid for any continuos function $f(x)$ for which integral

$$\int_c^\infty \frac{f(x)}{x} dx$$

converges for any $a > 0$.

The proof of (7) is elementary:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x} dx = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \frac{f(ax)-f(bx)}{x} dx = \lim_{\varepsilon\to 0} \left(\int_\varepsilon^\infty \frac{f(ax)}{x} dx - \int_\varepsilon^\infty \frac{f(bx)}{x} dx \right) =$$
$$\lim_{\varepsilon\to 0} \left(\int_{a\varepsilon}^\infty \frac{f(x)}{x} dx - \int_{b\varepsilon}^\infty \frac{f(x)}{x} dx \right) = \lim_{\varepsilon\to 0} \int_{a\varepsilon}^{b\varepsilon} \frac{f(x)}{x} dx = f(0) \lim_{\varepsilon\to 0} \int_{a\varepsilon}^{b\varepsilon} \frac{dx}{x} = f(0) \log \frac{b}{a}$$

Applying Frullani integral to the function e^{-x} we get the equality

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} \frac{1-e^{-Nx}}{1-e^{-x}} = \sum_{n=0}^{N} \int_0^\infty \frac{e^{-(a+n)x}-e^{-(b+n)x}}{x} dx = \sum_{n=0}^{N} \log \frac{b+n}{a+n} \qquad (8)$$

Then tending N to infinity we obtain the equality

$$\int_0^\infty \frac{\sum_{j=1}^{k} \left(e^{-a_j x} - e^{-b_j x}\right)}{x(1-e^{-x})} dx = \sum_{j=1}^{k} \sum_{n=0}^{\infty} \log \frac{b_j+n}{a_j+n}$$

or

$$\prod_{n=0}^{\infty} \frac{(a_1+n)\cdots(a_k+n)}{(b_1+n)\cdots(b_k+n)} = \exp \int_0^\infty \frac{\sum_{j=1}^{k} \left(e^{-a_j x} - e^{-b_j x}\right)}{x(1-e^{-x})} dx$$