Special functions. Problems for seminar 2

1. Compute $\Gamma(1)$ and $\Gamma'(1)$.

2. Show that first and second logarithmic derivatives of $\Gamma(z)$ are given by the following seria, absolutely convergent for $z \neq 0, -1, \ldots$:

$$\frac{d\log\Gamma(z)}{dz} = -\gamma - \frac{1}{z} + z\sum_{n=1}^{\infty}\frac{1}{n(z+n)}, \qquad \frac{d^2\log\Gamma(z)}{dz^2} = \sum_{n=1}^{\infty}\frac{1}{(z+n)^2}.$$

3. Compute integrals

   a) $\int_0^{\pi/2}\cos^{m-1}x\,\sin^{n-1}x\,dx$

   b) $\int_0^1 \frac{dx}{\sqrt[n]{1-x^m}}$, where $m > 0$.

   c) $\int_0^\infty x^n e^{-x^2}dx$

4. Show that

   a) show that $B(m,n) = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{n+m}}$, where $\mathrm{Re}\, n, m > 0$.

   b) For any z, $-k-1 < \mathrm{Re}\, z < -k$

$$\Gamma(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \ldots + (-1)^{k+1}\frac{t^k}{k!}\right)dt$$