Solution to the last problem

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For any $z, -k-1 < \operatorname{Re} z < -k$

$$\Gamma(z) = \int_0^\infty t^{z-1} \left(e^{-t} - 1 + t - \cdots + (-1)^{k+1} \frac{t^k}{k!} \right) \mathrm{d}t.$$

Solution. Let $I(z) = \int_0^\infty t^{z-1} \left(e^{-t} - 1 + t - \cdots + (-1)^{k+1} \frac{t^k}{k!} \right) \mathrm{d}t$, where k is the integer s.t. $-k-1 < x < -k$, $x = \operatorname{Re} z$.

For $x < -1$, note that $x + k < 0$ and $x + k + 1 > 0$, then by partial integration,

$$\begin{aligned} I(z) &= \frac{t^z}{z} \left(e^{-t} - 1 + t - \cdots + (-1)^{k+1} \frac{t^k}{k!} \right) \bigg|_0^\infty + \frac{1}{z} \int_0^\infty t^z \left(e^{-t} - 1 + t - \cdots + (-1)^k \frac{t^{k-1}}{(k-1)!} \right) \mathrm{d}t \\ &= 0 + \frac{I(z+1)}{z} = \frac{I(z+1)}{z}. \end{aligned}$$

For $-1 < x < 0$,

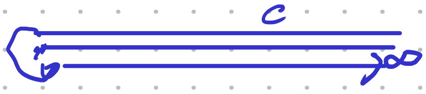
$$I(z) = \int_0^\infty t^{z-1}(e^{-t} - 1) \,\mathrm{d}t = \frac{t^z}{z}(e^{-t} - 1) \bigg|_0^\infty + \frac{1}{z} \int_0^\infty t^z e^{-t} \,\mathrm{d}t = \frac{\Gamma(z+1)}{z}.$$

It follows that $I(z) = \Gamma(z)$ for $-1 < x < 0$. The equation for $x < -1$ implies that $I(z) = \Gamma(z)$ for $x < -1$. As a result, for any $z, -k-1 < \operatorname{Re} z < -k$

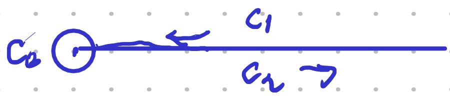
$$\Gamma(z) = \int_0^\infty t^{z-1} \left(e^{-t} - 1 + t - \cdots + (-1)^{k+1} \frac{t^k}{k!} \right) \mathrm{d}t. \qquad \blacksquare$$

2) $\Gamma(x)\Gamma(1-x) = B(x, 1-x) = \int_0^1 t^{x-1}(1-t)^{-x}\,dt =$

$$= \int_0^\infty \frac{t^{x-1}}{(1+t)}\,dt \qquad B(m,n) = \int \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$



$$\int_C \frac{(-t)^{x-1}}{(1+t)}\,dt$$



$\arg(-t) = -\pi$

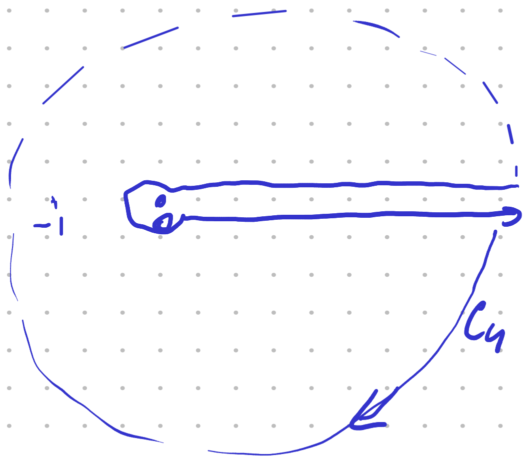
$$\int_{C_1} \frac{e^{(x-1)\log(-t)}}{1+t}\,dt = \int_\infty^0 \frac{e^{(x-1)(\log t - \pi i)}}{1+t} =$$

$$= e^{-\pi i x}\int_0^\infty \frac{t^{x-1}}{1+t}\,dt$$

$$\int_{C_2} = \int_0^\infty \frac{e^{(x-1)(\log t + \pi i)}}{1+t} = -e^{\pi i x}\int_0^\infty \frac{t^{x-1}}{1+t}\,dt$$

$$\int_{C_0} = \int_{-\pi}^{\pi} \frac{\varepsilon^{x-1} e^{i(x-1)\varphi}}{1-\varepsilon e^{i\varphi}}\, i\varepsilon\, d\varphi \sim \qquad t = \varepsilon \cdot e^{i\varphi}$$

$\sim \varepsilon^x \to 0 \quad \operatorname{Re} x > 0 \qquad \operatorname{Re} x < 1$
$\to 0$



$\int_{C_4}$ $\qquad t = Re^{i\varphi}$

$\to 0$

$$\frac{R^x}{R} \sim R^{x-1}$$

For $0 < \operatorname{Re} x < 1$ integrales over small and big contours vanish.

$$\int_{C_1+C_2+C_0+C_4} = -2\pi i \operatorname{Res}_{-1} \frac{(-t)^{x-1}}{1+t} = -2\pi i$$

$$\left(e^{-\pi i x} - e^{\pi i x}\right)\Gamma(x)\Gamma(1-x) = -2\pi i$$

$$\Updownarrow$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Homework

$$1)\ -\frac{d\log\Gamma(z)}{dz} = \frac{d\log\Gamma^{-1}(z)}{dz} = \frac{1}{z} + \gamma + \sum\left(\log'\left(1+\frac{z}{n}\right) - \frac{1}{n}\right) =$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z}\prod_{n\geq 1}\left(1+\frac{z}{n}\right)e^{-z/n} \qquad = \frac{1}{z} + \gamma + \sum_{n=1}^{N}\left(\frac{\frac{1}{n}}{1+\frac{z}{n}} - \frac{1}{n}\right)$$

$$= \frac{1}{z} + \gamma + \sum_{n=1}^{N}\left(\frac{1}{n+z} - \frac{1}{n}\right)$$

Substitute $z = 1$

$$= 1 + \gamma + \sum_{n\geq 1}\left(\frac{1}{n+1} - \frac{1}{n}\right) = \gamma$$

$$\log'\Gamma(z)\Big|_{z=1} = -\gamma$$

But $\frac{d\log\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$ and

$$\Gamma(1) = 1 \Rightarrow \Gamma'(1) = -\gamma$$

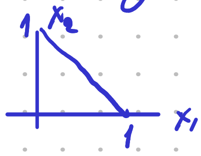
Dirichlet integral

$$1)\ \underset{\substack{x_i \geq 0 \\ \sum x_i \leq 1}}{\int\cdots\int} x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\,dx_1\ldots dx_n \overset{?}{=} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\ldots+\alpha_n+1)}$$

$n = 1$

$$\int_0^1 x_1^{\alpha_1-1}dx_1 = \frac{1}{\alpha_1} = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+1)}$$

$$\int_0^1 dx_1\int_0^{1-x_1}dx_2\, x_1^{\alpha_1-1}x_2^{\alpha_2-1} = \int_0^1 dx_1\, \frac{(1-x_1)^{\alpha_2}}{\alpha_2}x_1^{\alpha_1-1} = \frac{B(\alpha_1,\alpha_2+1)}{\alpha_2} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2+1)}{\alpha_2\Gamma(\alpha_1+\alpha_2+1)} =$$



$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2+1)}$$

$\Downarrow$

2) Vol $\sum\left(\frac{x_i}{\alpha_i}\right)^{p_i} \leq 1$

$p_i = 2$, $\alpha_i = R$ – ball with radius R

$$\| \quad \prod \alpha_i \cdot \frac{\Gamma(1+1/p_i)}{\Gamma(1+\sum 1/p_i)}$$

Euler reflection formula

1) Weierstrass $\frac{1}{x}\prod\left(1-\frac{x^2}{n^2}\right)$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

by known from Fourier series product f-la for $\sin\pi x$

Asymptotical expansions.

Taylor f-la

error terms

- Lagrange
- Cauchy
- Peano

diff.

$f^{(n+1)}(\xi)$

$\sim$

$o(x-x_0)^{n+1}$

1. $F(x) = \int_0^\infty e^{-xt} \underbrace{\cos t}_{\text{series}}\, dt = \int_0^\infty e^{-xt}\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots\right) dt \underset{xt=\tau}{=}$

$$= \int_0^\infty e^{-\tau} \cdot \sum (-1)^n \frac{\tau^{2n}}{(2n)!\, x^{2n+1}} d\tau = \sum_{n\geq 0} (-1)^n \frac{\Gamma(2n+1)}{(2n)!\, x^{2n+1}} = \sum (-1)^n \frac{1}{x^{2n+1}} = \frac{x}{1+x^2}$$

2. $G(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt = \int_0^\infty e^{-xt}(1 - t + t^2 - \dots) dt =$ $\quad xt=\tau \quad dt = \frac{d\tau}{x}$

$$= \int e^{-\tau}\left(\frac{1}{x} - \frac{\tau}{x^2} + \dots\right) d\tau = \sum \frac{\Gamma(n+1)}{x^n} (-1)^{n+1} =$$

$$= \sum \frac{(-1)^n n!}{x^{n+1}} \leftarrow \text{diverges } \forall x$$

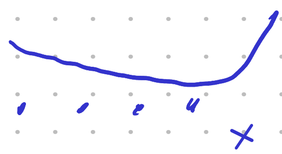
$$G_n(x) = \sum_{k=0}^{N} \frac{(-1)^k k!}{x^{k+1}}$$

$$\frac{1}{1+x} \to 1 - x + x^2 - \dots + (-1)^{n-1} x^{n-1} = \frac{1-(-1)^n x^n}{1+x}$$

difference is $\frac{(-1)^n x^n}{1+x}$

$$\text{error} = \left|\int_0^\infty \frac{(-1)^n e^{-xt} t^n}{1+t} dt\right| < \int_0^\infty e^{-xt} t^n dt = \frac{n!}{x^{n+1}}$$

$1+t > 1$



$$G(x) = \frac{1}{x} - \frac{1}{x^2} + \dots + \frac{(n-1)!}{x^n} \cdot (-1)^{n-1} + R_n \qquad n \to n+1 \qquad R_{n+1} = R_n \cdot \frac{n+1}{x}$$

$$|R_n| < \frac{n!}{x^{n+1}}$$

Fix $n = 10$ $\quad x > 10 \quad |R_n| \leq 0{,}01.$

Fix x $\quad R_n$ decreases until $n = [x]$
increases.

$$G(x) \sim \frac{1}{x} - \frac{1}{x^2} + \dots + \frac{(-1)^{n-1}}{x^n} \qquad x \to \infty$$

asymp. expansion

<u>Poincare</u>. ~ 1900

$\mathcal{D} \ni x_0$ $\quad \mathcal{D}$ - region in $\mathbb{C}$

$x_0 \in \overline{D}$

$x \to x_0 \quad , x \in D$

$f_1(x), f_2(x), \dots, f_n(x)$ $\quad x \in D$

$$f_{n+1}(x) = o(f_n(x))$$
$$x \to x_0,\ x \in D$$

asympt. sequence $\quad$ ~~ex~~ $\quad f_{n+1}(x) = \alpha(x) f_n(x)$

$\alpha(x) \to 0$

$x \to x_0$

ex. $\quad x_0 = 0$

$D = \mathbb{R}$

~~$f_0 =$~~

$f_0 = 1$ $\quad$ Taylor series

$f_1 = x$

$f_2 = x^2$

$\vdots$

$x_0 = \infty$

$D =$ $\quad \operatorname{Re} x > 0$

$f_0 = 1,\ f_1 = \frac{1}{x},\ f_2 = \frac{1}{x^2}, \dots$

$f_0 = 1,\ f_1 = \log x,\ f_2 = \frac{1}{x}, \dots$

__Dfn__ $\quad f(x) \sim a_1 f_1(x) + a_2 f_2(x) + \dots$ — asymp. ex.

$x \to x_0,\ x \in D$ $\qquad |\ | \le C \cdot |f_{n+1}(x)|$

if $\quad f(x) - a_1 f_1(x) - \dots - a_n f_n(x) = O(f_{n+1}(x))$

$o \quad O$

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ex. $\quad x_0 = a \qquad f_k(x) = (x - x_0)^k$

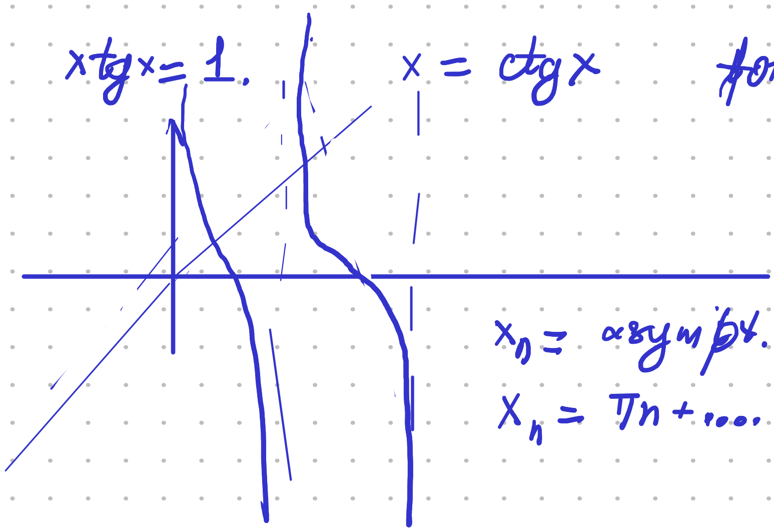
asymp. exp. $\Leftrightarrow$ Taylor in a form of Peano

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n + O(x^{n+1})$$

$$G(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\,dt \sim \sum \frac{(-1)^n n!}{x^{n+1}} \qquad x \to +\infty$$

$\operatorname{Re} x > 0$

ex. $\quad x \operatorname{tg} x = 1. \qquad x = \operatorname{ctg} x \qquad$ for each $n = 1, 2, 3, \dots$ there is a solution



$x_n =$ asympt. over n

$x_n = \pi n + \dots$

$x_n = \pi n + \alpha_n \qquad x_n = \operatorname{ctg} x_n \qquad \frac{1}{x_n} = \operatorname{tg} x_n$

arcctg

$\operatorname{arcctg} \frac{1}{x_n} + \pi n = \alpha_n \qquad < \alpha_n \leq O(1)$

$\operatorname{arcctg} y = y - \frac{y^3}{3} + \frac{y^5}{5} - \dots$

$\frac{1}{\pi n + \alpha_n} + O\left(\frac{1}{n^2}\right) + \cancel{\pi_1} = \cancel{\pi_1} - \alpha_y \qquad \alpha_n \approx \frac{1}{\pi_n + \alpha_n} + O\left(\frac{1}{n^2}\right)$

$\alpha_n \simeq \frac{1}{\pi n} + O\left(\frac{1}{\pi n}\right)$

$x_n = \pi n + \frac{1}{\pi n} + O\left(\frac{1}{n}\right) = \pi n + \frac{1}{\pi n} + O\left(\frac{1}{n^2}\right)$

proceed further

$x_n = \pi_n + \frac{1}{\pi n} - \frac{4}{3(\pi n)^3} + O\left(\frac{1}{n^4}\right)$

...

Laplace integrals

$F(x) = \int_0^\infty e^{-xt} f(t)\,dt \qquad f(t)$ integrable

finite growth

$|f(t)| < Ce^{x_0 t}$

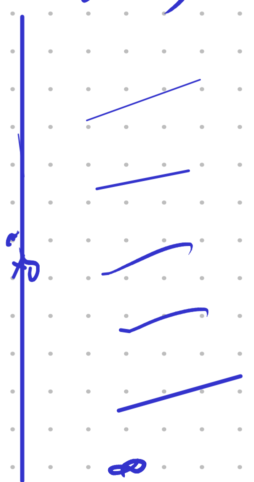
If $\operatorname{Re} x > x_0 \Rightarrow$ integral converges

<u>Problem</u> study $F(x)$ for big x

<u>Watson lemma</u> Let $f(t) = O(t^a)$ $t \to 0^+$ $a > -1$

Then $F(x) = \int_0^\infty e^{-xt} f(t)\,dt = O\left(\frac{1}{x^{a+1}}\right)$

$x \to \infty, \quad \operatorname{Re} x > x_0$



$x \rightarrow \qquad \infty$

<u>Proof</u> $\int_0^\infty e^{-xt} f(t)\,dt =$

$= \underbrace{\int_0^\varepsilon e^{-xt} f(t)\,dt}_{\mathrm{I}} + \underbrace{\int_\varepsilon^\infty e^{-xt} f(t)\,dt}_{\mathrm{II}}$

$x_1 > x_0 \qquad |f(t)| < ce^{-x_0 t}$

$\mathrm{II} \quad \left|\int_\varepsilon^\infty e^{-(x-x_1)t} f(t)\,dt \cdot e^{-x_1 t}\right| < e^{-\operatorname{Re}(x-x_1)\varepsilon} \int_\varepsilon^\infty f(t) e^{-x_1 t}\,dt$

converges $< C'$

$\downarrow 0 \qquad O\left(e^{-\operatorname{Re}(x-x_1)\varepsilon}\right)$

I is $O(e^{-\varepsilon \operatorname{Re} x})$ $\qquad f(t) = O(t^a)$ $\quad t \to 0$

$$I \ \left| \int_0^\varepsilon e^{-xt} f(t)\,dt \right| < \left| \int_0^\varepsilon e^{-xt} t^a\,dt \right| < \left| \int_0^\infty e^{-xt} t^a\,dt \right|$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{x^{a+1}}\Gamma(a+1)$$

$$\int = O\left(\frac{1}{x^{a+1}}\right) + \underset{\text{over } x}{\text{exp. decreasing}}$$

example $\qquad f(t) = a_1 t^{\alpha_1} + \dots + a_n t^{\alpha_n} + O(t^{\alpha_n})$

$$\operatorname{Re}\alpha_1 < \dots < \operatorname{Re}\alpha_n$$

$$F(x) = \int_0^\infty f(t) e^{-xt}\,dt = \int_0^\infty \left(a_1 t^{\alpha_1} + \dots + a_n t^{\alpha_n}\right) e^{-xt}\,dt + \underset{\text{Watson}}{O}\left(\frac{1}{x^{\alpha_n+1}}\right)$$

$$xt = \tau$$

$$\| \quad a_1 \frac{\Gamma(\alpha_1+1)}{x^{\alpha_1+1}} + \dots + a_n \frac{\Gamma(\alpha_n+1)}{x^{\alpha_n+1}} + O\left(\frac{1}{x^{\alpha_n+1}}\right)$$

In particular, if f has a Taylor decomp. at 0

$$f(t) = f(0) + f'(0)t + \dots + \frac{f^{(n)}(0)}{n!} t^n + \dots$$

$\Rightarrow$ $F(x)$ has asymp. expansion

$$F(x) = \frac{f(0)\Gamma(1)}{x} + \dots + \frac{f^{(n)}(0)}{\cancel{n!}} \frac{\Gamma(n+1)}{x^{n+1}} + \dots =$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{x^{k+1}} + O\left(\frac{1}{x^{n+2}}\right)$$

II. $F(x) = \int_{-\infty}^{\infty} f(t) e^{-xt^2}\,dt$ $\qquad F(x) \sim ?$ $\qquad f(t)$ groth less than some exponent

Gauss integrals $\qquad t^2 = \tau \qquad 2t\,dt = d\tau$

$$= \int_0^\infty \frac{f(\tau^{1/2})}{2\tau^{1/2}} e^{-x\tau}\,d\tau + \int_0^\infty \frac{f(-\tau^{1/2})}{2\tau^{1/2}} e^{-x\tau}\,d\tau$$

Laplace integrals.

$$f(t) = f(0) + t f'(0) + \dots$$

$$= \sum_{n=0}^{N} a_n t^n + O(t^N)$$

$$\frac{f(\tau^{1/2}) + f(-\tau^{1/2})}{2\tau^{1/2}} = a_0 \tau^{-1/2} + a_2 \tau^{1/2} + a_4 \tau^{3/2} + \dots$$

$$F(x) = a_0 \frac{\Gamma(1/2)}{x^{1/2}} + a_2 \frac{\Gamma(3/2)}{x^{3/2}} + \dots + a_{2n} \frac{\Gamma(1/2+n)}{x^{n+1/2}} + O\left(\frac{1}{x^{n+3/2}}\right)$$

Asympt. expansion of Gauss integral

$\operatorname{Re} x > 0 \quad , x \to \infty$