Special functions. Problems for seminar 4

1. Using asymptotics of logarithm of Γ function, find the asymptotics of

$$|\Gamma(a+ix)|, \qquad a, x \in \mathbb{R}$$

for fixed a and x tending to $\pm\infty$.

2. *

   a) Using saddle point method, show that $\int_0^{\pi/2} \sin^n t dt = \left(\frac{\pi}{2n}\right)^{1/2} (1 + O(n^{-1}))$ when $n \to \infty$.

   b) derive from that Vallis formula $\pi = \lim\limits_{n\to\infty} \frac{2}{n} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2$.

3. By definition of Euler - Masceroni constant γ,

$$1 + \frac{1}{2} + \ldots + \frac{1}{n} = \log n + \gamma + o(1)$$

Make this statement more precise: find the constant a such that

$$1 + \frac{1}{2} + \ldots + \frac{1}{n} = \log n + \gamma + \frac{a}{n} + o\left(\frac{1}{n}\right)$$

It can be done by purely geometric considerations of areas under corresponding plots