1. (M. Evseev)

$$\underset{\substack{\sum x_i \le 1,\\ x_1,\dots x_n \le 1}}{\int \cdots \int} dx_1 \cdots dx_n \, x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} = \frac{1}{\alpha_n} \underset{\substack{\sum x_i \le 1\\ x_1,\dots, x_{n-1} \le 1}}{\int\int} dx_1 \dots dx_{n-1} \, (1 - x_1 - \dots - x_{n-1})^{\alpha_n} \cdot x_1^{\alpha_1 - 1} \cdots x_{n-1}^{\alpha_{n-1} - 1} \qquad (A)$$

On the other hand, $\Gamma(\alpha_1) \dots \Gamma(\alpha_m) = \int_0^\infty u_1^{\alpha_1 - 1} e^{-u_1} du_1 \cdot \int_0^\infty u_m^{\alpha_m - 1} e^{-u_m} du_m$

Change of variables

$$u_1 = y_m y_1, \quad u_2 = y_m y_2, \dots \quad u_{m-1} = y_m y_{m-1}, \quad u_m = y_m (1 - y_1 - \dots - y_{m-1})$$

$$|J| = y_m^{m-1} (1 - y_1 - \dots - y_{m-1}) \qquad u_1 + u_2 + \dots + u_m = y_m$$

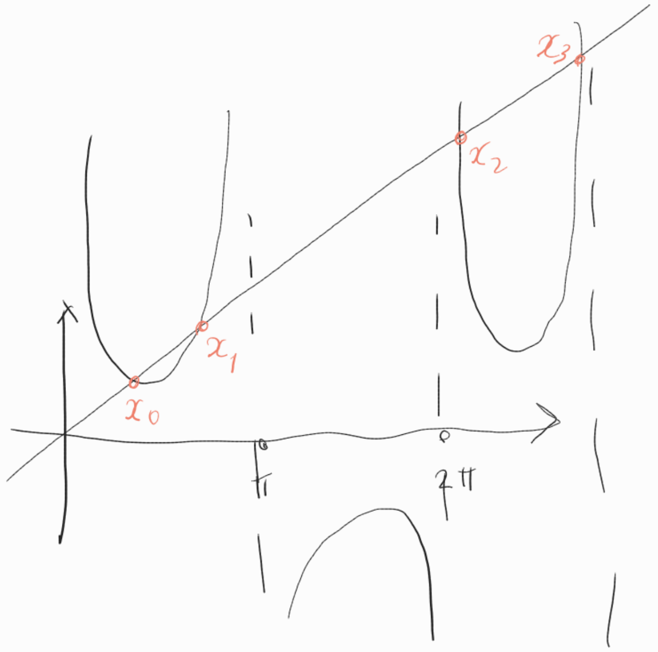
$$\Gamma(\alpha_1) \dots \Gamma(\alpha_m) = \int e^{-\sum y_m} y_m^{\sum \alpha_j - \cancel{m} + \cancel{m} - 1} dy_m \int\int dy_1 \dots dy_{m-1} \, y_1^{\alpha_1 - 1} \dots y_{m-1}^{\alpha_{m-1} - 1} \cdot (1 - y_1 \dots - y_{m-1})^{\alpha_m}$$

$$= \Gamma(\sum \alpha_i) \underset{\substack{\sum y_i \le 1\\ y_1, \cdot y_{m-1} \ge 0}}{\int \cdots \int} dy_1 \dots dy_{m-1} \, y_1^{\alpha_1 - 1} \cdots y_{m-1}^{\alpha_{m-1} - 1} \cdot (1 - y_1 - \dots - y_m)^{\alpha_m} \qquad (B)$$

Comparing (A) and (B) $\Longrightarrow$

$$\underset{\substack{\sum x_i \le 1,\\ x_1,\dots x_n \le 1}}{\int \cdots \int} dx_1 \cdots dx_n \, x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_m + 1)}{\alpha_m \Gamma(\alpha_1 + \dots + \alpha_m + 1)} = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)}{\Gamma(\alpha_1 + \dots + \alpha_m + 1)}$$

3. $x_k = \pi k + f_k, \quad -\pi \le f_k \le \pi$




$$\sin(\pi k + f_k) = \sin\left((-1)^k f_k\right) = \frac{1}{x_k}$$

$$\arcsin(y) = y + \frac{y^3}{6} + O(y^5)$$

$$y = \frac{1}{x_k}$$

$$(-1)^k f_k = \arcsin \frac{1}{x_k} = \frac{1}{x_k} + O\left(\frac{1}{x_k^3}\right) = \frac{1}{\pi k + f_k} + O\left(\frac{1}{k^3}\right) \sim \frac{1}{\pi k}$$

$x_k = \pi k + f_k$ since $f_k = O(1)$

$$(-1)^k f_k = \frac{1}{\pi k} + \dots = \frac{1}{\pi k + (-1)^k \frac{1}{\pi k}} + \frac{1}{6(\pi k)^3} + O\left(\frac{1}{k^4}\right) =$$

$$= \frac{1}{\pi k}\left(1 - (-1)^k \frac{1}{(\pi k)^2}\right) + \frac{1}{6(\pi k)^3} + O\left(\frac{1}{k^4}\right) = \frac{1}{\pi k} + \left((-1)^{k+1} + \frac{1}{6}\right)\cdot\frac{1}{(\pi k)^3} + O\left(\frac{1}{k^4}\right)$$

Hence

$$x_k = \pi k + (-1)^k \left(\frac{1}{\pi k} + \left((-1)^{k+1} + \frac{1}{6}\right)\frac{1}{(\pi k)^3} + O\left(\frac{1}{k^4}\right)\right)$$

$$k = 2n, \quad x_{2n} = 2\pi n + \frac{1}{2\pi n} - \frac{5}{6(2\pi n)^3} + O\left(\frac{1}{n^4}\right)$$

Laplace integral $F(x) = \int_0^\infty e^{-xt} f(t)\,dt$

Watson lemma $\quad |f(t)| < Ce^{x_0 t} \quad t \to \infty$

and $f(t) = O(t^a) \quad t \to 0 \quad \Rightarrow$

$F(x) = O\left(\frac{1}{x^{a+1}}\right) \quad x \to \infty \quad a \in \mathbb{C}$

$\Downarrow$

If $f(t)$ has asymp. exp. $\quad f(t) \sim a_0 t^{\alpha_0} + \dots a_n t^{\alpha_n} + O(t^{\alpha_n})$

$-1 < \operatorname{Re} \alpha_0 < \dots < \operatorname{Re} \alpha_n$

$\Rightarrow F(x) \sim \frac{a_0 \Gamma(\alpha_0+1)}{x^{\alpha_0+1}} + \dots + \frac{a_n \Gamma(\alpha_n+1)}{x^{\alpha_n+1}} + O\left(\frac{1}{x^{\alpha_n+1}}\right)$

$0 < \operatorname{Re} x \to \infty$

__Remark.__ Does not matter whether exp. of $f$ at $0$ converges or not

$\Rightarrow F(x)$ – asymptotical exp.

$F(x) \int_{-\infty}^{\infty} f(t) e^{-xt^2}\,dt \quad \Rightarrow$

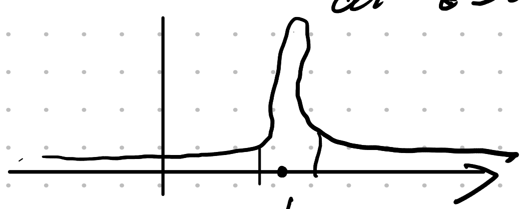
$|f(t)| < Ce^{x_0 t^2} \quad f(t) \sim \sum_{k=0}^{\infty} a_k t^k \quad t \to 0$

$F(x) \sim \sum_{k=0}^{n} a_{2k} \frac{\Gamma(k+\frac{1}{2})}{x^{k+1/2}} + O\left(\frac{1}{x^{n+1/2}}\right) \quad \operatorname{Re} x \to \infty$

## Saddle point method

$F(x) = \int_{-b}^{c} g(t) e^{+x\varphi(t)}\,dt \quad \varphi(t)$ has global uniq maximum at $t = t_0$

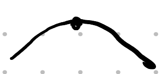
$F(\lambda)$ ~~for big~~ $x$



$t_0$

$F(x) = \int_{-\infty}^{\infty} g(t) e^{+x\varphi(t)}\,dt = e^{+x\varphi(t_0)} \int_{-\infty}^{\infty} g(t) e^{+x(\varphi(t)-\varphi(t_0))}\,dt$

$\varphi(t) - \varphi(t_0) = -\tau^2$



$\varphi(t) - \varphi(t_0) = +a_2(t-t_0)^2 + a_3(t-t_0)^3 + \dots = -\tau^2$

$(t-t_0)^2 \cdot a_2\left(1 + \frac{a_3}{a_2}(t-t_0) + \dots\right) = -\tau^2$

$\tau = c_1(t-t_0) + c_2(t-t_0)^2 + \dots \qquad \tau = \left\{\sqrt{-a_2} \cdot (t-t_0)\right) + \dots$

$dt = \frac{d\tau}{\sqrt{-a_2}} \qquad a_2 = \frac{\varphi''(t_0)}{2}$

$$F(x) = e^{x\varphi(t_0)} \cdot \int g(t(\tilde\tau)) \cdot e^{-x\tilde\tau^2} \cdot \left(\sqrt{\frac{2}{-\varphi''(t_0)}} + O(\tilde\tau)\right) d\tilde\tau \Rightarrow$$

$$F(x) = e^{x\varphi(t_0)} \cdot g(t_0) \cdot \sqrt{\frac{-2}{\varphi''(t_0)}} \cdot \frac{\Gamma(\frac{1}{2})}{\sqrt{x}} \left(1 + O\left(\frac{1}{x}\right)\right) =$$

$$= e^{x\varphi(t_0)} g(t_0) \sqrt{\frac{-2\pi}{\varphi''(t_0)}} \frac{1}{\sqrt{x}} \left(1 + O\left(\frac{1}{x}\right)\right)$$

Stirling formula Asymp. behaviour of $\Gamma(x)$

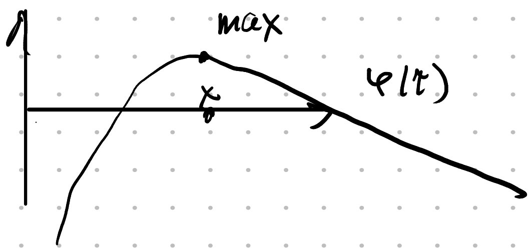
$\operatorname{Re} x \to \infty$

$\operatorname{Re} x > 0$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{1}{x} \int_0^\infty e^{-\tau} \tau^x d\tau =$$

$$= \frac{1}{x} \int_0^\infty e^{\underbrace{-\tau + x\log\tau}_{\varphi(\tau)}} d\tau \qquad \varphi(\tau) = -\tau + x\log\tau$$

$$\varphi'(\tau) = -1 + \frac{x}{\tau} \qquad \varphi' = 0 \qquad \tau = x$$




change of var.

$\tau = x(t+1)$ $\qquad$ $\tau \to 0$, $t \to -1$

$\tau = x \Leftrightarrow t = 0$ $\qquad$ $\tau \to \infty$, $t \to \infty$

$d\tau = x\,dt$

$$\frac{1}{x}\int_{-1}^\infty e^{-x(t+1) + x\log x(t+1)} dt = \frac{1}{x}\int_{-1}^\infty e^{-x + x\log x - xt + x\log(t+1)} \cancel{x} dt =$$

$$= e^{-x} x^x \int_{-1}^\infty e^{x\underbrace{(-t + \log(t+1))}_{\varphi(t)}} dt$$

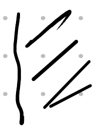
$$\varphi' = -1 + \frac{1}{t+1} \qquad \varphi' = 0, \quad t = 0$$

$$\varphi'' = -\frac{1}{(t+1)^2} = -1$$

Saddle point

$$\Gamma(x) \sim x^x e^{-x} \cdot \sqrt{\frac{2\pi}{x}} \left(1 + O\left(\frac{1}{x}\right)\right)$$

$\operatorname{Re} x \to \infty$



$\operatorname{Re} x > 0$

$x \to +i\infty$

$$|\Gamma(ix)|^2 = \Gamma(ix)\Gamma(-ix) = \frac{\Gamma(ix)\Gamma(1-ix)}{-ix} = \frac{\pi}{\sin\pi ix \cdot -ix}$$

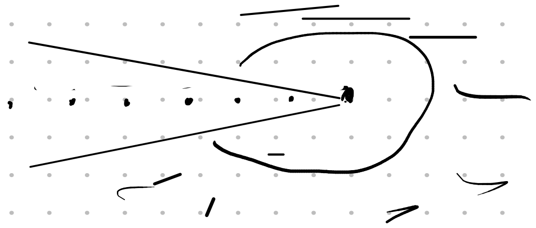
$$|\sin\pi ix| = |\operatorname{sh}\pi x| = \left|\frac{e^{\pi x} - e^{-x}}{2}\right| \sim \frac{e^{|\pi x|}}{2} \quad x \to +\infty$$

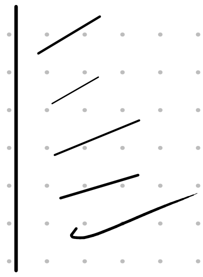
$|\Gamma(ix)|^2 \sim \frac{2\pi}{|x|\, e^{|\pi x|}}$ $\qquad |\Gamma(ix)| \sim \sqrt{2\pi}\cdot e^{-\frac{\pi|x|}{2}} \cdot |x|^{-1/2}\left(1+O(1)\right)$

More general

$$\log\Gamma(x) = \frac{1}{2}\log 2\pi + \left(x-\frac{1}{2}\right)\log x - x + \sum_{j>0}^{m} \frac{B_{2j}}{(2j-1)2j}\cdot\frac{1}{x^{2j-1}} + O\left(\frac{1}{x^{2m+1}}\right)$$

$|x| \to \infty \qquad |\arg x| < \pi - \varepsilon$ $\qquad x \to \infty$





$f_0, f_1, f_2, \dots \qquad f_{n+1} = o(f_n) \qquad x \to x_0$

$x \in D$

$f_0 = x\log x \qquad f_1 = x \qquad f_2 = \log x \quad f_3 = 1 \qquad f_4 = \frac{1}{x}, \dots$

$B_n(x)$: $\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$ $\qquad B_n(x)$ – Bernoulli polyn

$B_n(0) = B_n$ numbers

$B_n \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ $\qquad B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2n+1} = 0$ $\quad n>0$

$B_{2n}$ – rational numbers

$$\log\Gamma(x) = \frac{1}{2}\log 2\pi + \left(x-\frac{1}{2}\right)\log x - x + \sum_{j>0}^{m} \frac{B_{2j}}{(2j-1)2j}\cdot\frac{1}{x^{2j-1}} + O\left(\frac{1}{x^{2m+1}}\right)$$

a) Stirling $\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\, x^x e^{-x}\left(1+O\left(\frac{1}{x}\right)\right)$

б) $|\Gamma(ix)|$ $\qquad x \to \pm i\infty$ $\qquad \log x = \log|x| + i\arg x$

## Riemann $\zeta$-function.

$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ $\quad \operatorname{Re} s > 1$ converges $\qquad \zeta(s)$ analyt at $\operatorname{Re} s > 1$

$$\int_0^\infty e^{-xt} t^s \frac{dt}{t} \underset{xt=\tau}{=} \int_0^\infty e^{-(xt)} \frac{(xt)^s}{x^s} \frac{d(xt)}{xt} = \frac{1}{x^s}\cdot\Gamma(s)$$

$x = 1, 2, 3, \dots$ sum up

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-nt}\, t^s \frac{dt}{t} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^s}}_{=\zeta(s)}\Gamma(s) \Rightarrow \zeta(s)\cdot\Gamma(s) = \int_0^{\infty}\frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt \qquad \operatorname{Re} s > 1$$

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt \qquad \text{analyt. contin.}$$

converges at $\infty$     sing. point $0$

$$\zeta(s)\Gamma(s) = \int_1^{\infty}\frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt + \underbrace{\int_0^1 t^{s-1}\left(\frac{e^{-t}}{1-e^{-t}} - \sum_{k=0}^{n}\frac{B_k t^{k-1}}{k!}\right)dt}_{\text{conv. for } \operatorname{Re} s > -n} + \int_0^1 t^{s-1}\sum_{k=0}^{n}\frac{B_k}{k!}t^k\,dt =$$

$$= \int_1^{\infty}\frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt + \int_0^1 t^{s-1}\left(\frac{e^{-t}}{1-e^{-t}} - \sum_{k=0}^{n}\frac{B_k t^{k-1}}{k!}\right)dt + \sum_{k=0}^{n}\frac{B_k}{k!}\,\frac{t^{s+k-1}}{s+k-1}\Bigg|_0^1$$

$$\sum_{k=0}^{n}\frac{B_k}{k!}\,\frac{1}{s+k-1}$$

$\Rightarrow$ $\zeta(s)\Gamma(s)$ is a merom. function in $\mathbb{C}$ with poles at $s = 1, 0, -1, -2, \ldots$

$$\operatorname{Res} = \frac{B_k}{k!}$$

$\Gamma(s)$ has poles at $s = 0, -1, -2, \ldots$ first order

$\Rightarrow$ $\zeta(s)$ has the only first order pole at $s = 1$

$$\operatorname{Res} = \frac{B_0}{\Gamma(1)} = 1$$

Can compute $\zeta(-n)$    $n = 0, -1, \ldots$

$$\operatorname{Res}\, \zeta(s)\cdot\Gamma(s)\Big|_{s=-n} = \frac{B_{n+1}}{(n+1)!}$$

$$\operatorname{Res}\, \Gamma(s)\Big|_{s=-n} = \frac{(-1)^n}{n!} \qquad \Rightarrow \zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}$$

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2) = \zeta(-2n) = \ldots = 0$$

$$\zeta(1-2n) = -\frac{1}{2k}B_{2k}$$

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = 1 + \int_1^\infty \frac{d[t]}{t^s} =$$

$\mathrm{Re}\, s > 1$ $\qquad$ Re $s > 1$

Stieltjes integral

$$\int f\,dF = \sum_\Delta (F(x_{i+1}) - F(x_i)) f(\xi_i)$$

int. by parts

$$= 1 + \underbrace{\frac{[t]}{t^s}\Big|_1^\infty}_{=0} + s\int_1^\infty \frac{[t]\,dt}{t^{s+1}} = 1 + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\,dt + s\int_1^\infty \frac{dt}{t^s} =$$

$$= 1 - \frac{1}{s-1} + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\,dt = \frac{s}{s-1} + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\,dt$$

converges for Re $s > 0$

$$\zeta(s) = \frac{s}{s-1} + \text{holom. for Re}\, s > 0$$

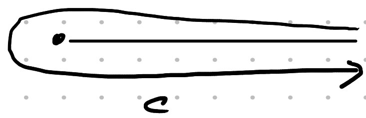
$s = 1$ pole $\quad$ Res $= 1$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \left(1 - \frac{1}{2^s}\right) + 2\left(\frac{1}{2^s} - \frac{1}{3^s}\right) + 3\left(\frac{1}{3^s} - \frac{1}{4^s}\right) + \ldots$$

$$= s\int_1^2 \frac{dt}{t^{s+1}} + 2s\int_2^3 \frac{dt}{t^{s-1}} + \ldots = s\int_1^\infty \frac{[t]}{t^{s+1}}\,dt = 1 + \frac{1}{s-1} + \int_1^\infty \frac{[t]-t}{t^{s+1}}\,dt$$

Hankel integral presentation

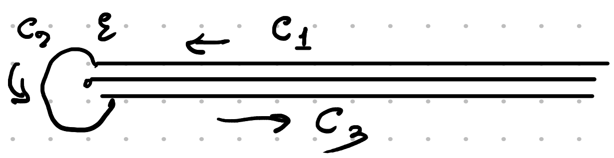
$$\zeta(s)\Gamma(s) = -\frac{1}{2i\sin\pi s}\int_C (-t)^{s-1}\frac{e^{-t}}{1-e^{-t}}\,dt$$




$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$

$$\zeta(s) = -\frac{1}{2\pi i\,\Gamma(1-s)}\int_C (t)^{s-1}\frac{e^{-t}}{1-e^{-t}}\,dt$$

Re $s > 1$ $\qquad$ $\varepsilon \to 0$




$$\int_{C_1} (-t)^{s-1}\frac{e^{-t}}{1-e^{-t}}\,dt \qquad C_2 \quad C_3$$

$t \to$ real

$$(-t)^{s-1} = e^{(s-1)(\log t - \pi i)} = -t^{s-1}\cdot e^{-\pi i s}$$

$$\int_{C_1} \to -e^{-\pi i s}\int_\infty^0 \frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt = e^{-\pi i s}\int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt$$

$$\int_{C_3} \to -e^{\pi i s}\int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt$$

$$\int_{C_2} \to 0 \qquad \varepsilon \to 0 \qquad \int_{C_2} \underbrace{t^{s-1}}_{\varepsilon^{s-1}} \frac{e^{-t}}{1-e^{-t}}\,dt \to \varepsilon^{s-1}\cdot \text{finite} \to 0$$

$dt \to \varepsilon\, d\varphi$, $\frac{e^{-t}}{1-e^{-t}} \to \varepsilon^{-1}$

$\operatorname{Re} s > 1$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} \to -2i \sin \pi s \int_0^\infty t^{s-1}\frac{e^{-t}}{1-e^{-t}}\,dt$$