1) Decomposition of merom. function into elementary fractions (Mittag-Leffler)

2) Decomp. of entire function into inf. product.

1. Mittag-Leffler theorem.

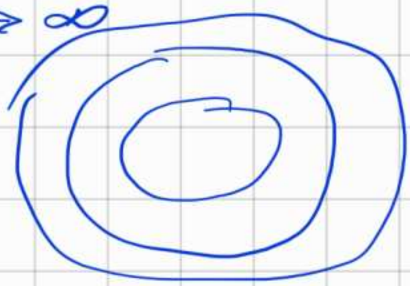
f be a merom. function on $\mathbb{C}$
poles are $a_1, a_2, \ldots a_n \ldots$
0 is not a pole.

$$g_i(z) = g_{i,0} + \frac{g_{i,1}}{z-a_i} + \dots + \frac{g_{i,n}}{(z-a_i)^n}$$

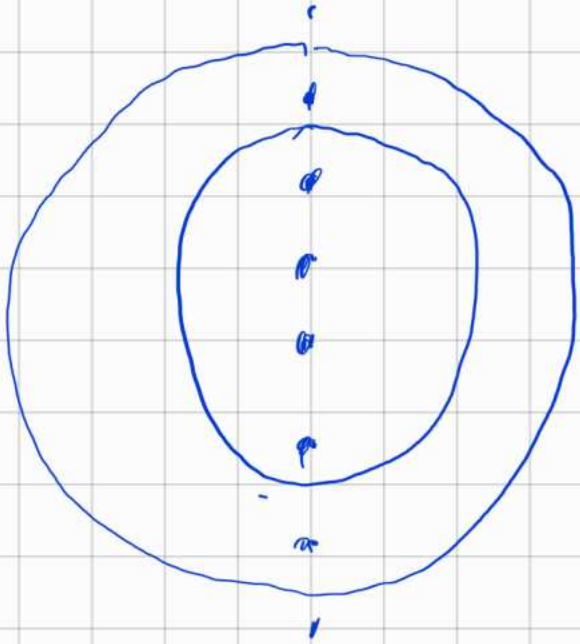
singular parts of f in a_i

Assume that there is a sequence of contours $C_n \to \infty$ such that

$$|f(z)|_{C_n} < C|z|^p$$



$$\frac{1}{\sin x} \qquad x = \pi n$$



$$\frac{1}{|\sin x|} \le \frac{C}{|x|}$$

Then $f(z) = \sum_{n \geq 0} \left(g_n(z) - h_n(z)\right) + h(z)$

$$h(z) = \sum_{k=0}^{p} \frac{f^{(k)}(0)}{k!} z^k \qquad h_n(z) = \sum_{k=0}^{p} \frac{g_n^{(k)}(0)}{k!} z^k$$

1. Assume first that f has finite number of poles

$a_0 \quad a_1 \quad \ldots \quad a_i.$

$f(z)$ $\quad g_k(z)$ is a singular part of $f(z)$ at a_k

$f(z) - \sum^{N} g_k(z)$ is regular functi

$k=0$

if it is growing like a polynom.

$\Rightarrow \quad f(z) = \sum g_n(z) + \text{polynom}$

general Mittag-Leffler theorem:

$\forall$ sequence a_n of poles $\exists$ a sequence p_n of positive numbers

a ~~sequence~~ of polynomials $h_n(z)$ of degree p_n such that

$$f(z) = h_0(z) + \sum_n g_n(z) - h_n(z),$$

$\deg h_n(z) = p_n, \quad g_n(z) =$ singular part of $f(z)$ at a_n.

<u>Products.</u> $f(z)$ entire function.
with zeros $a_1, a_2 \ldots \; a_n \ldots$ of order $m_1, m_2 \ldots \; m_n \ldots$

$\log' f(z) = \dfrac{d\log f(z)}{f(z)}$ growth not faster then z^{p-1} on some sequence of contours

Then $f(z) = z^m \underset{\approx}{e^{g(z)}} \cdot \prod\limits_{n \geq 1} \left(1 - \dfrac{z}{a_n}\right) \underset{=}{e^{\frac{z}{a_n} + \frac{z^2}{2a_n^2} + \ldots + \frac{1}{p}\frac{z^p}{a_n^p}}}$

$g(z)$ – some entire function.

$e^{z^2} \quad e^{z^4}$

$\Gamma^{-1}(z)$ – Weierstraß

$$\Gamma^{-1}(z) = z e^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

$\Gamma^{-1}(z)$ – entire funct

zeros $z = 0, -1, -2, \ldots$

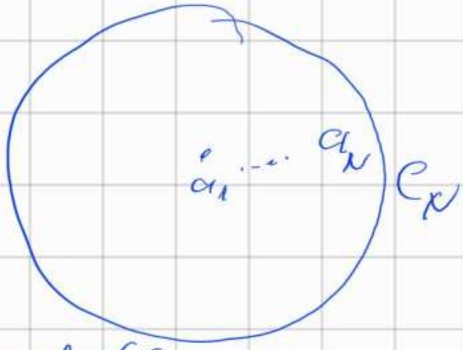
$\dfrac{d \log \Gamma^{-1}(z)}{dz} = -\dfrac{d \log \Gamma(z)}{dz} <$ cont

$f(z) \sim e^{e^z}$ ...

e^{z} - $\qquad\qquad p=1$

Proof of 1^{st} Mittag-Leffler th.

$$C_N \qquad \frac{1}{2\pi i}\int_{C_N}\frac{f(\xi)\,d\xi}{\xi-z} \neq f(z)$$



$\varphi(z)=\sum_{n\le N} g_n(z)$ the sum of singular parts of $f(z$

$f(z)-\varphi(z)$ has no poles inside C_N

$$\int_{C_N}\frac{f(\xi)-\varphi(\xi)}{\xi-z}\,d\xi = f(z) - \varphi(z)$$

$\varphi(z)$ has poles at $a_1 \dots a_N$

$\frac{\varphi(\xi)}{\xi-z}$ at ∞ has no poles. $\varphi(z)=\sum g_n(z) \Rightarrow$

$$\int_{C_N}\frac{\varphi(\xi)}{\xi-z}\,d\xi = 0, \text{ since } \frac{\varphi(\xi)}{\xi-z} = \frac{a_2(z)}{\xi^2}+\frac{a_3(z)}{\xi^3}+\dots$$

$$\int_{C_N}\frac{f(\xi)}{\xi-z}\,d\xi = f(z)-\sum_{n=1\dots N} g_n(z)$$

0

$N\to\infty$

Differentiate this equality at $z=0$

$$\frac{d}{dz}\int_{C_N}\frac{f(\xi)}{\xi-z}\,d\xi = \int_{C_N}\frac{f(\xi)}{(\xi-z)^2}\,d\xi$$

$$\frac{1}{2\pi i}\int_{C_N}\frac{f(\xi)\,d\xi}{\xi^{k+1}} = \frac{f^{(k)}(0)}{k!} - \sum \frac{g_n^{(k)}(0)}{k!}\Bigg\} z^k \quad k=0,1,\dots p$$

$$\frac{1}{2\pi i}\sum_{k=0}^{p}\int_{C_N}\frac{f(\xi)z^k}{\xi^{k+1}}\,d\xi = \sum\frac{f^{(k)}(0)}{k!}z^k - \sum\frac{g_n^{(k)}(0)}{k!}z^k$$

$$-\int_{C_N}\frac{f(\xi)}{\xi-z}\,d\xi = f(z) - \sum g_n(z)$$

$$h(z) = \sum_{k=0}^{p}\frac{f^{(k)}(0)z^k}{k!}$$

$$\frac{1}{2\pi i}\int_{C_N}\left\{\frac{1}{\xi-z} - \sum\frac{z^k}{\xi^{k+1}}\right\} f(\xi)\,d\xi = f(z) - h(z) - \sum_{n\le N}\{g_n(z) - h_n(z)\}$$

$$\| \quad \frac{z^{p+1}}{\xi(\xi-z)\cdot\xi^{p}}$$

$$h_n(z) = \sum_{k=0}^{p}\frac{g_n^{(k)}(0)z^k}{k!}$$

$$f(z) - h(z) - \sum_{n\le N}\{g_n(z) - h_n(z)\} = R_n \to 0$$

$$R_n(z) = \frac{1}{2\pi i}\int_{C_N}\frac{z^{p+1}f(\xi)}{\xi(\xi-z)\xi^{p}}\,d\xi \qquad |f(\xi)| < |\xi|^p$$

z is fixed

$$|\quad| < \frac{1}{|\xi||\xi-z|}\int_{2\pi R}\frac{dr}{R^2} \to 0$$

$$f(z) = (z-a_1)^{n_1}(z-a_2)^{n_2}\dots(z-a_k)^{n_k}$$

$$\frac{d\log f(z)}{dz} \qquad a_1 \text{ — } \text{pole.}$$

order of the pole?

Integral transforms

Integral transforms

1. Fourier $\quad f(x) \to [Ff](\lambda) = \hat{f}(\lambda) =$

$$= \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

Corr. defined for $f(x) \in L_1(\mathbb{R})$

If $f(x) \in L_1(\mathbb{R}) \cap C^1(\mathbb{R}) \Rightarrow$

inversion formula

$$f(x) = \frac{1}{2\pi} \text{ v.p.} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(\lambda) e^{i\lambda x} d\lambda$$

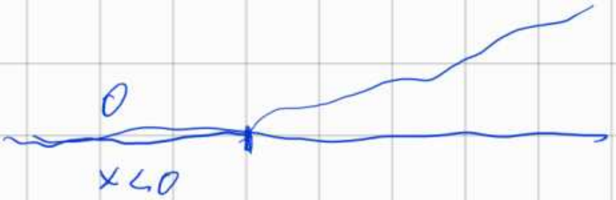
$f(x) \subset L_2(\mathbb{R})$ Fourier extends by v.p. to isomorphism $L_2(x, dx)$ and $L_2(\lambda, d\lambda)$

2. Laplace transform

$$Lf(p) = \int_0^{\infty} f(x) e^{-px} dx \qquad \text{assumption}$$

$f(x)$ abs. locally integrable and of finite groth

$$|f(x)| < Ce^{ax} \qquad x > 0$$
$$a > 0$$



Statement $[Lf]$ is an anal. function on a half plan $\operatorname{Re} p > a$

Because in $\operatorname{Re} p > a \quad \operatorname{Re} p = a + \varepsilon$

$$\Rightarrow \int_0^{\infty} \underbrace{f(x) e^{-px}}_{|\cdot| < e^{-\varepsilon x}} dx \qquad \text{it converges absolutely.}$$

$f \in C^1[0, \infty)$ and $|f(x)| < Ce^{ax}$

$\Rightarrow$ inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (Lf)(p) e^{px} dp$$

3. Two-sided Laplace transform

$$[Lf](p) = \int_{-\infty}^{\infty} f(x) e^{-px} dx$$

f is locally integrable

$$|f(x)| < Ce^{ax} \quad x \to +\infty$$

$$|f(x)| < C'e^{bx} \quad x \to -\infty$$

$$a < b$$

$[Lf](p)$ is analytical on the strip

$$a < \mathrm{Re}\, p < b$$

$$\int_{-\infty}^{\infty} f(x)e^{-px}dx = \int_{-\infty}^{0} f(x)e^{-px}dx \qquad \mathrm{Re}\, p < b$$

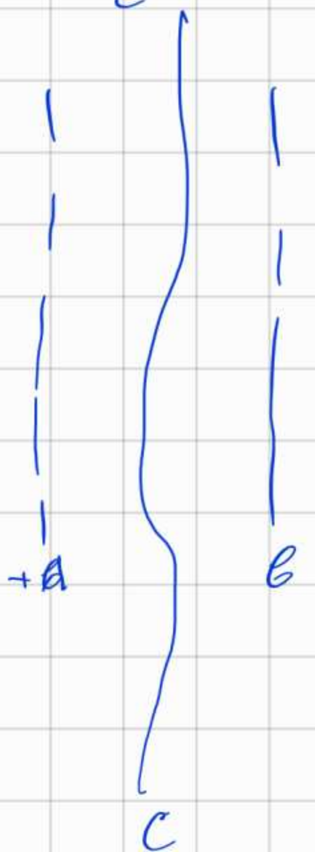
$$+ \int_{0}^{\infty} f(x)e^{-px}dx \qquad \mathrm{Re}\, p > a$$

Inversion formula

$$f(x) = \int_C [Lf](p)e^{px}dp$$

$f(x) \in C^1[R]$

$C = c + i\lambda \quad -\infty < \lambda < \infty$



3. Mellin transform

$\varphi(t)$, $t>0$

$$\check{\varphi}(s) = [M\varphi](s) = \int_0^\infty \varphi(t)\cdot t^{s-1}\,dt$$

$t = e^{-x}$

$f(x) = \varphi(e^{-x})$

$$\check{\varphi}(s) = Lf(s) = \int_{-\infty}^{\infty} \varphi(e^{-x})\, e^{-s}\,dx$$

Mellin transform is two-sided Laplace of $f(x) = \varphi(e^{-x})$

Assume that $|\varphi(t)| < C_1 t^{b}$ $t\to+\infty$

$|\varphi(t)| < C_2 t^{a}$ $t\to 0$

and locally integrable $\Rightarrow$

$\check{\varphi}(s)$ is analytical in the strip

$a < \operatorname{Re} s < b$

and there is an inversion formula

$$\varphi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \check{\varphi}(s)\, t^{-s}\,ds$$

Properties of Mellin transform:

1) $(t\varphi(t))^{\vee} = \check{\varphi}(s+1)$ $\quad (e^{2\pi i a} f(x))^{\wedge} = \hat{f}(\lambda - a)$

$$(t\varphi(t))^{\vee} = \int_0^\infty t\varphi(t)\, t^{s-1}\,dt = \int_0^\infty \varphi(t)\, t^{(s+1)-1}\,dt$$

2) $\left(t\frac{d}{dt}\varphi\right)(s) = -s\check{\varphi}(s)$ if $t^s\varphi(t)$ vanishes at 0 and ∞ in the strip $a < \mathrm{Re}\, s < b$

$$\int_0^\infty \varphi'(t)\, t^{s-1+1}\, dt = \int_0^\infty t^s\, d\varphi(t) =$$

$$= \underbrace{t^s\varphi(t)\Big|_0^\infty}_{0} - s\int_0^\infty t^{s-1}\varphi(t)\,dt$$

3) $\check{\varphi}\left(\frac{1}{t}\right)$ $\quad \psi(t) = \varphi\left(\frac{1}{t}\right) \Rightarrow \check{\psi}(s) = \check{\varphi}(-s)$

$$\check{\psi}(s) = \int_0^\infty \varphi\left(\frac{1}{t}\right) t^s \frac{dt}{t} \underset{\tau = \frac{1}{t}}{=} -\int_\infty^0 \varphi(\tau)\,\tau^{-s}\frac{d\tau}{\tau} =$$

$$= \int_0^\infty \varphi(\tau)\,\tau^{-s}\frac{d\tau}{\tau} = \check{\varphi}(-s)$$

4) $\psi(t) = \varphi(x\cdot t)$ $\quad \check{\psi}(s) = \int_0^\infty \varphi(xt)\frac{(xt)^s}{x^s}\frac{d\,xt}{xt} =$

$$= \frac{1}{x^s}\int_0^\infty \varphi(\tau)\,\tau^s\frac{d\tau}{\tau} = \frac{1}{x^s}\check{\varphi}(s)$$

Corollary

1) $\psi(t) = \sum_{n\geq 1}\varphi(nt)$

$$\check{\psi}(s) = \zeta(s)\check{\varphi}(s)$$

$$\psi(t) = \sum_{n\geq}\varphi\left(\frac{t}{n}\right)$$

$$\check{\psi}(s) = \zeta(-s)\check{\varphi}(s)$$

$\psi(t) = \varphi\left(\frac{1}{\cdot}\right)$

$$\eta(t) = \varphi(\tfrac{1}{t})$$

$$w(t) = \sum \eta(nt) = \sum \varphi(\tfrac{1}{nt})$$

For study of $\Gamma(x)$ for big x

$$\cancel{B}\ Lf(x) = \int_0^\infty f(t)e^{-xt}\,dt$$

from behaviour of $f(t)$ $t \to 0$

$$f(t) \underset{t \to 0}{\sim} \sum a_k t^k$$

$\Rightarrow$

$$F(x) \sim \sum \frac{a_k k!}{x^{k+1}} \quad x \to \infty$$

$$\int_0^\infty \varphi(t) t^{s-1}\,dt$$

I know how $\varphi(t)$ behaves at 0 and at ∞

Question: what does it mean for $\check{\varphi}(s)$?

$$\int_0^\infty \varphi(t)\,t^{s-1}\,dt = \underbrace{\int_0^1 \varphi(t)\,t^{s-1}\,dt}_{I_1} + \underbrace{\int_1^\infty \varphi(t)\,t^{s-1}\,dt}_{I_2} =$$

$$= \left(\int_0^1 \Big(\varphi(t) - \sum_{k=m}^{n} a_k t^k\Big) t^{s-1}\,dt + \int_0^1 \sum_m^n a_k t^{k+s-1}\,dt\right)$$

$$+ I_2 \qquad \varphi(t) = a_m t^m + \dots a_n t^n + O(t^{n+\varepsilon})$$

$\operatorname{Re} m < \operatorname{Re} n$

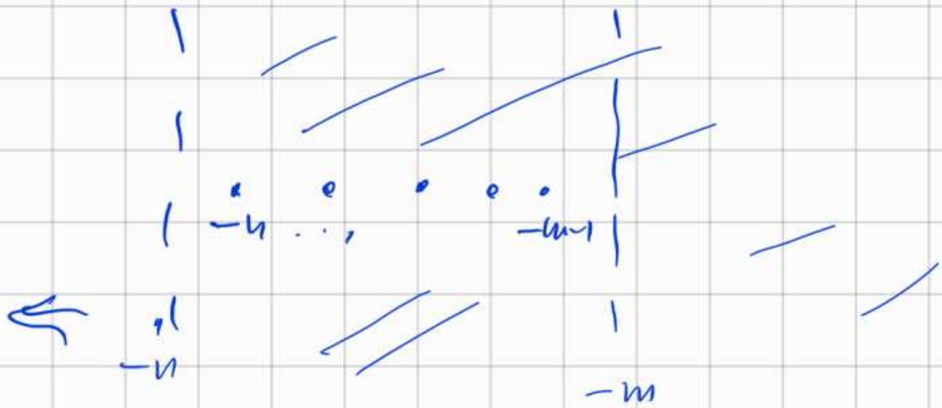
$$= I_2 + \int_0^1 O(t^{m+s-1+\varepsilon})\,dt + \sum a_k \int_0^1 t^{k+s-1}\,dt$$

converges for $\operatorname{Re}(s+n) > 0$ $\qquad \int_0^1 t^{k+s-1}\,dt = \frac{1}{k+s}$

$$= I_2 + \sum \frac{a_k}{k+s} + \text{converging in } \operatorname{Re} s > -n$$

Originally: I_1 was anal. for $\operatorname{Re} s > -m$. Continue it to $\operatorname{Re} s > -n$

$\Downarrow$



$\Rightarrow$ asymptotics of $\varphi(t)$ for $t \to 0$ and ∞ give simple pole singularities of $\check{\varphi}(s)$