Special functions seminar work

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Contents

Problems for seminar 2

1. Compute $\Gamma(1)$ and $\Gamma'(1)$.

Solution.

$$\begin{aligned}
\frac{1}{\Gamma(1)} &= e^{\gamma}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} \\
&= e^{\gamma}\lim_{N\to\infty} e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)}(N+1) \\
&= e^{\gamma}\lim_{N\to\infty} e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)+\log N}\left(1+\frac{1}{N}\right) \\
&= e^{\gamma}e^{-\gamma} \\
&= 1.
\end{aligned}$$

Hence $\Gamma(1)=1$.
By the result of problem 2,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma-\frac{1}{z}+z\sum_{n=1}^{\infty}\frac{1}{n(z+n)}.$$

Therefore,

$$\Gamma'(1)=\Gamma(1)\left[-\gamma-1+\sum_{n=1}^{\infty}\frac{1}{n(1+n)}\right]=-\gamma-1+1=-\gamma. \qquad \blacksquare$$

2. Show that first and second logarithmic derivatives of $\Gamma(z)$ are given by the following seria, absolutely convergent for $z\neq 0,-1,\ldots$:

$$\frac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z}=-\gamma-\frac{1}{z}+z\sum_{n=1}^{\infty}\frac{1}{n(z+n)}, \qquad \frac{\mathrm{d}^2\log\Gamma(z)}{\mathrm{d}z^2}=\sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

Solution. Recall: $\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}$ or $\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$.

$$\begin{aligned}
\log\Gamma(z) &= \lim_{n\to\infty}\left(\log(n!)+z\log n-(\log z+\log(z+1)+\cdots+\log(z+n))\right) \\
\frac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z} &= \lim_{n\to\infty}\left(\log n-\left(\frac{1}{z}+\frac{1}{z+1}+\cdots+\frac{1}{z+n}\right)\right) \\
&= \lim_{n\to\infty}\left[\left(-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\log n\right)-\frac{1}{z}+\sum_{m=1}^{n}\left(\frac{1}{m}-\frac{1}{m+z}\right)\right] \\
&= -\gamma-\frac{1}{z}+z\sum_{n=1}^{\infty}\frac{1}{n(z+n)}. \\
\frac{\mathrm{d}^2\log\Gamma(z)}{\mathrm{d}z^2} &= \frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{1}{(z+n)^2} \\
&= \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}. \qquad \blacksquare
\end{aligned}$$

3. Compute integrals

(a) $\int_0^{\frac{\pi}{2}}\cos^{m-1}x\sin^{n-1}x\,\mathrm{d}x$;

(b) $\int_0^1\frac{\mathrm{d}x}{\sqrt[n]{1-x^m}}$, where $m>0$;

(c) $\int_0^{\infty}x^ne^{-x^2}\,\mathrm{d}x$.

Solution.

(a) Recall: $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x$.

Let $I_{m,n}=\int_0^{\frac{\pi}{2}}\cos^{m-1}x\sin^{n-1}x\,\mathrm{d}x$.

$$\begin{aligned}B(p,q)&\xlongequal{x=\sin^2\theta}\int_0^{\frac{\pi}{2}}\sin^{2p-2}\theta\cos^{2q-2}\theta(2\sin\theta\cos\theta)\,\mathrm{d}\theta\\&=2\int_0^{\frac{\pi}{2}}\sin^{2p-1}\theta\cos^{2q-1}\theta\,\mathrm{d}\theta.\end{aligned}$$

Then

$$I_{m,n}=\frac{B\left(\frac{n}{2},\frac{m}{2}\right)}{2}=\frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n+m}{2}\right)}.$$

For even n,

$$\Gamma\left(\frac{n}{2}\right)=\left(\frac{n}{2}-1\right)!.$$

For odd $n=2k+1$, $\Gamma(z+1)=z\Gamma(z)$ gives

$$\Gamma\left(\frac{n}{2}\right)=\Gamma\left(k+\frac{1}{2}\right)=\frac{(2k)!}{4^k k!}\sqrt{\pi}=\frac{(2k-1)!!}{2^k}\sqrt{\pi}=\frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi}.$$

If $m=2l,n=2k$ are even integers, then

$$I_{m,n}=\frac{1}{2}\frac{\left(\frac{n}{2}-1\right)!\left(\frac{m}{2}-1\right)!}{\left(\frac{m+n}{2}-1\right)!}=\frac{1}{2}\frac{\frac{(n-2)!!}{2^{k-1}}\frac{(m-2)!!}{2^{l-1}}}{\frac{(m+n-2)!!}{2^{k+l-1}}}=\frac{(n-2)!!(m-2)!!}{(m+n-2)!!}.$$

If $m=2l+1,n=2k$, then

$$I_{m,n}=\frac{1}{2}\frac{(k-1)!\frac{(m-2)!!}{2^{\frac{m-1}{2}}}\sqrt{\pi}}{\frac{(m+n-2)!!}{2^{\frac{m+n-1}{2}}}\sqrt{\pi}}$$

Note that

$$\frac{(k-1)!}{2}=\frac{(n-2)!!}{2^{k-1}\cdot 2}=\frac{(n-2)!!}{2^{\frac{n}{2}}},$$

thus

$$I_{m,n}=\frac{(n-2)!!(m-2)!!}{(m+n-2)!!}.$$

If $m=2l+1,n=2k+1$, then

$$I_{m,n}=\frac{1}{2}\frac{\frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi}\cdot\frac{(m-2)!!}{2^{\frac{m-1}{2}}}\sqrt{\pi}}{(k+l)!}=\frac{\pi}{2}\frac{(n-2)!!(m-2)!!\cdot\frac{1}{2^{\frac{m+n}{2}-1}}}{(m+n-2)!!\cdot\frac{1}{2^{\frac{m+n}{2}-1}}}=\frac{\pi}{2}\frac{(n-2)!!(m-2)!!}{(m+n-2)!!}.$$

In summary, we have the following conclusion:

$$I_{m,n}=\begin{cases}\frac{\pi}{2}\frac{(n-2)!!(m-2)!!}{(m+n-2)!!}, & m,n\text{ odd integers}\\ \frac{(n-2)!!(m-2)!!}{(m+n-2)!!}, & \text{else.}\end{cases}$$

(b) Let $J_{m,n}=\int_0^1\frac{\mathrm{d}x}{\sqrt[n]{1-x^m}}$.

$$\begin{aligned}J_{m,n}&\xlongequal{x^m=t}\int_0^1\frac{1}{(1-t)^{\frac{1}{n}}}\frac{\mathrm{d}t}{mt^{1-\frac{1}{m}}}\quad(mx^{m-1}\,\mathrm{d}x=\mathrm{d}t,\ x=t^{\frac{1}{m}})\\&=\frac{1}{m}\int_0^1 t^{\frac{1}{m}-1}(1-t)^{\left(1-\frac{1}{n}\right)-1}\,\mathrm{d}t\\&=\frac{1}{m}B\left(\frac{1}{m},1-\frac{1}{n}\right)\\&=\frac{1}{m}\frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(1+\frac{1}{m}-\frac{1}{n}\right)}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^\infty x^n e^{-x^2}\,\mathrm{d}x &\xlongequal{x^2=t} \int_0^\infty t^{\frac{n}{2}} e^{-t}\frac{1}{2t^{\frac{1}{2}}}\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^\infty t^{\left(\frac{n+1}{2}\right)-1}e^{-t}\,\mathrm{d}t\\ &= \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\\ &= \begin{cases}\frac{\left(\frac{n-1}{2}\right)!}{2}, & n \text{ odd}\\ \sqrt{\pi}\frac{(n-1)!!}{2^{\frac{n}{2}+1}}, & n \text{ even.}\end{cases}\end{aligned}$$

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4. Show that

(a) $B(m,n) = \int_0^\infty \frac{x^{m-1}\,\mathrm{d}x}{(1+x)^{n+m}}$, where $\operatorname{Re} m, \operatorname{Re} n > 0$

(b) For any $z, -k-1 < \operatorname{Re} z < -k$

$$\Gamma(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \cdots + (-1)^{k+1}\frac{t^k}{k!}\right)\mathrm{d}t.$$

Solution.

(a) Recall: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$.

$$\begin{aligned}B(m,n) &\xlongequal{y=\frac{x}{1-x}} \int_0^\infty x^{m-1}(1-x)^{n-1}(1-x)^2\,\mathrm{d}y \quad (\mathrm{d}x = (1-x)^2\,\mathrm{d}y)\\ &= \int_0^\infty x^{m-1}x^{n+1}y^{-n-1}\,\mathrm{d}y \quad \left(1-x = \frac{x}{y}\right)\\ &= \int_0^\infty \frac{y^{m+n}}{(1+y)^{m+n}}y^{-n-1}\,\mathrm{d}y \quad \left(x = \frac{y}{1+y}\right)\\ &= \int_0^\infty \frac{y^{m-1}\,\mathrm{d}y}{(1+y)^{m+n}}.\end{aligned}$$

(b) Let $I(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \cdots + (-1)^{k+1}\frac{t^k}{k!}\right)\mathrm{d}t$, where k is the integer s.t. $-k-1 < x < -k$, $x = \operatorname{Re} z$.

For $x < -1$, note that $x + k < 0$ and $x + k + 1 > 0$, then by partial integration,

$$\begin{aligned}I(z) &= \frac{t^z}{z}\left(e^{-t} - 1 + t - \cdots + (-1)^{k+1}\frac{t^k}{k!}\right)\Bigg|_0^\infty + \frac{1}{z}\int_0^\infty t^z\left(e^{-t} - 1 + t - \cdots + (-1)^k\frac{t^{k-1}}{(k-1)!}\right)\mathrm{d}t\\ &= 0 + \frac{I(z+1)}{z} = \frac{I(z+1)}{z}.\end{aligned}$$

For $-1 < x < 0$,

$$I(z) = \int_0^\infty t^{z-1}(e^{-t}-1)\,\mathrm{d}t = \frac{t^z}{z}(e^{-t}-1)\Big|_0^\infty + \frac{1}{z}\int_0^\infty t^z e^{-t}\,\mathrm{d}t = \frac{\Gamma(z+1)}{z}.$$

It follows that $I(z) = \Gamma(z)$ for $-1 < x < 0$. The equation for $x < -1$ implies that $I(z) = \Gamma(z)$ for $x < -1$. As a result, for any $z, -k-1 < \operatorname{Re} z < -k$

$$\Gamma(z) = \int_0^\infty t^{z-1}\left(e^{-t} - 1 + t - \cdots + (-1)^{k+1}\frac{t^k}{k!}\right)\mathrm{d}t.$$

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Problems for seminar 3

1.

(a) Compute Dirichlet integral

$$\int\limits_{x_1>0,\ldots,x_n>0,\sum x_i\leqslant 1} x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,\quad \operatorname{Re}\alpha_i>0.$$

(b) Compute the volume of ellipsoid

$$E=\left\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n:\ \sum_{i=1}^n\left|\frac{x_i}{a_i}\right|^{p_i}\leqslant 1\right\},$$

where $a_1,\ldots,a_n,p_1,\ldots,p_n$ are positive.

Solution.

(a) Let

$$J_n(f,\alpha_1,\ldots,\alpha_n)=\int\limits_{x_1>0,\ldots,x_n>0,\sum x_i\leqslant 1} f(x_1+\cdots+x_n)x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where f is continuous, and $\operatorname{Re}\alpha_i>0$, $i=1,\ldots,n$. Let $I_n(\alpha_1,\ldots,\alpha_n)=J_n(1,\alpha_1,\ldots,\alpha_n)$.

First calculate the double integral of x_1 and x_2: let $\lambda=x_3+x_4+\cdots+x_n$, $\tau_2=x_1+x_2$, then

$$\begin{aligned}&\int_0^{1-\lambda}\mathrm{d}x_2\int_0^{1-\lambda-x_2} f(x_1+x_2+\lambda)x_1^{\alpha_1-1}x_2^{\alpha_2-1}\,\mathrm{d}x_1\\ &=\int_0^{1-\lambda}\mathrm{d}x_2\int_{x_2}^{1-\lambda} f(\tau_2+\lambda)(\tau_2-x_2)^{\alpha_1-1}x_2^{\alpha_2-1}\,\mathrm{d}\tau_2.\end{aligned}$$

Change the order of integration, and let $x_2=\tau_2 x$, we obtain

$$\begin{aligned}&\int_0^{1-\lambda}\mathrm{d}\tau_2\int_0^{\tau_2} f(\tau_2+\lambda)(\tau_2-x_2)^{\alpha_1-1}x_2^{\alpha_2-1}\,\mathrm{d}x_2\\ &=\int_0^{1-\lambda} f(\tau_2+\lambda)\tau_2^{\alpha_1+\alpha_2-1}\,\mathrm{d}\tau_2\int_0^1(1-x)^{\alpha_1-1}x^{\alpha_2-1}\,\mathrm{d}x\\ &=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\int_0^{1-\lambda} f(\tau_2+\lambda)\tau_2^{\alpha_1+\alpha_2-1}\,\mathrm{d}\tau_2.\end{aligned}$$

This reduces one integral, while the integral form remains unchanged. Applying the above method to τ_2 and x_3 can reduce another integral, and the factor before the integral is

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_3)}.$$

By continuously applying the above method, we finally get:

$$J_n(f,\alpha_1,\ldots,\alpha_n)=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}\int_0^1 f(\tau)\tau^{\left(\sum\limits_{i=1}^n\alpha_i\right)-1}\,\mathrm{d}\tau.$$

Take $f\equiv 1$, then

$$\begin{aligned}\int\limits_{x_1>0,\ldots,x_n>0,\sum x_i\leqslant 1} x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_n&=I_n(\alpha_1,\ldots,\alpha_n)=J_n(1,\alpha_1,\ldots,\alpha_n)\\ &=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}\int_0^1\tau^{\left(\sum\limits_{i=1}^n\alpha_i\right)-1}\,\mathrm{d}\tau\end{aligned}$$

$$= \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)} \frac{1}{\sum\limits_{i=1}^{n}\alpha_i}$$

$$= \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n+1)}$$

(b) Consider the change of variables

$$X_i = \left(\frac{x_i}{a_i}\right)^{p_i}, i = 1, \ldots, n, x_i > 0$$

then $x_i = a_i X_i^{1/p_i}$ and

$$\mathrm{d}x_i = a_i \frac{1}{p_i} X_i^{\frac{1}{p_i}-1}\,\mathrm{d}X_i.$$

Now, the volume of ellipsoid is

$$\begin{aligned} V &= \int_E \mathrm{d}x_1\cdots\mathrm{d}x_n = 2^n \int_{\{x\in E:\, x_i>0,\forall i\}} \mathrm{d}x_1\cdots\mathrm{d}x_n \\ &= 2^n \frac{a_1\cdots a_n}{p_1\cdots p_n} \int\limits_{X_1,\ldots,X_n>0,\sum X_i\leqslant 1} X_1^{\frac{1}{p_1}-1}\cdots X_n^{\frac{1}{p_n}-1}\,\mathrm{d}X_1\cdots\mathrm{d}X_n \\ &= 2^n \frac{a_1\cdots a_n}{p_1\cdots p_n} I_n\left(\frac{1}{p_1},\ldots,\frac{1}{p_n}\right) \\ &= 2^n \frac{a_1\cdots a_n}{p_1\cdots p_n} \frac{\Gamma\left(\frac{1}{p_1}\right)\cdots\Gamma\left(\frac{1}{p_n}\right)}{\Gamma\left(\frac{1}{p_1}+\cdots+\frac{1}{p_n}+1\right)} \end{aligned}$$ ■

2. Show that for $\operatorname{Re} z > 0$

$$\frac{\mathrm{d}^2 \log\Gamma(z)}{\mathrm{d}z^2} = \int_0^\infty \frac{te^{-tz}}{1-e^{-t}}\,\mathrm{d}t.$$

Solution. Let $I = \int_0^\infty \frac{te^{-tz}}{1-e^{-t}}\,\mathrm{d}t$, then

$$\begin{aligned} I &= \int_0^\infty te^{-tz} \sum_{n=0}^\infty (e^{-t})^n\,\mathrm{d}t \\ &= \sum_{n=0}^\infty \int_0^\infty te^{-t(z+n)}\,\mathrm{d}t \\ &= \sum_{n=0}^\infty \left[-\frac{t}{z+n}e^{-t(z+n)}\Big|_0^\infty + \frac{1}{z+n}\int_0^\infty e^{-t(z+n)}\,\mathrm{d}t\right] \\ &= \sum_{n=0}^\infty \left(0 - \frac{1}{(z+n)^2}e^{-t(z+n)}\Big|_0^\infty\right) \\ &= \sum_{n=0}^\infty \frac{1}{(z+n)^2} = \frac{\mathrm{d}^2\log\Gamma(z)}{\mathrm{d}z^2}. \end{aligned}$$ ■

3. Find first three terms of asymptotical expansion over big n of n-th positive root of the equation

$$x\sin x = 1.$$

Solution. Suppose $x_n \sin x_n = 1$, then we write

$$x_n = \pi n + \alpha_n.$$

We get

$$(-1)^n \sin\alpha_n = \frac{1}{\pi n + \alpha_n}.$$

Let

$$z = \frac{1}{\pi n} = \frac{1}{(-1)^n \frac{1}{\sin \alpha_n} - \alpha_n} = f(\alpha_n),$$

where $f(x) = \frac{1}{(-1)^n \frac{1}{\sin x} - x}$.
By Lagrange Inversion Theorem, the inverse of f will be expressed to a power series of z:

$$\begin{aligned}\alpha_n(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k!} \lim_{x\to 0} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \left(\frac{x}{f(x)}\right)^k \\ &= \sum_{k=1}^{\infty} \frac{z^k}{k!} \lim_{x\to 0} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \left((-1)^n \frac{x}{\sin x} - x^2\right)^k.\end{aligned}$$

- Case $k = 1$.

$$\lim_{x\to 0} \left((-1)^n \frac{x}{\sin x} - x^2\right) = (-1)^n.$$

- Case $k = 2$.

$$\lim_{x\to 0} \frac{\mathrm{d}}{\mathrm{d}x} \left((-1)^n \frac{x}{\sin x} - x^2\right)^2 = \lim_{x\to 0} 2\left((-1)^n \frac{x}{\sin x} - x^2\right)\left((-1)^n \frac{\sin x - x\cos x}{\sin^2 x} - 2x\right) = 0.$$

- Case $k = 3$. Wolframalpha told me the following result:

$$\lim_{x\to 0} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left((-1)^n \frac{x}{\sin x} - x^2\right)^3 = -6 + (-1)^n.$$

Therefore,

$$\alpha_n = (-1)^n \frac{1}{\pi n} + \frac{-6 + (-1)^n}{3!} \frac{1}{(\pi n)^3} + O\left(\frac{1}{n^4}\right).$$

It follows that

$$x_n = \pi n + (-1)^n \frac{1}{\pi n} + \frac{-6 + (-1)^n}{3!} \frac{1}{(\pi n)^3} + O\left(\frac{1}{n^4}\right).$$

■

Problems for seminar 4

1. Using asymptotics of logarithm of Γ function, find the asymptotics of

$$|\Gamma(a+\mathrm{i}x)|, \quad a, x \in \mathbb{R}$$

for fixed a and x tending to $\pm\infty$.

Solution. Recall:

$$\log\Gamma(z) = \frac{1}{2}\log 2\pi + \left(z - \frac{1}{2}\right)\log z - z + \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)2j}\frac{1}{z^{2j+1}} + O\left(\frac{1}{z^{2m+1}}\right).$$

Suppose $a > 0$ and $|x| > 1$. Take $m = 1$, we get

$$\log\Gamma(a+\mathrm{i}x) = \frac{1}{2}\log 2\pi + \left(a + \mathrm{i}x - \frac{1}{2}\right)\log(a+\mathrm{i}x) - (a+\mathrm{i}x) + O\left(\frac{1}{x^3}\right).$$

A simple calculation gives

$$\mathrm{Re}\left[\left(a + \mathrm{i}x - \frac{1}{2}\right)\log(a+\mathrm{i}x)\right] = \left(a - \frac{1}{2}\right)\log\sqrt{a^2+x^2} - x\arctan\frac{x}{a}.$$

Note that

$$\log\sqrt{a^2+x^2} = \frac{1}{2}\log x^2 + \frac{1}{2}\log\left(1 + \frac{a^2}{x^2}\right) = \log|x| + O\left(\frac{1}{x^2}\right),$$

and

$$\arctan\frac{x}{a} + \arctan\frac{a}{x} = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0 \\ -\frac{\pi}{2}, & \text{if } x < 0. \end{cases}$$

It follows that

$$\begin{aligned} -x\arctan\frac{x}{a} &= -x\left[\pm\frac{\pi}{2} - \frac{a}{x} + O\left(\frac{1}{x^2}\right)\right] \\ &= -\frac{\pi}{2}|x| + a + O\left(\frac{1}{x^2}\right). \end{aligned}$$

Therefore

$$\log|\Gamma(a+\mathrm{i}x)| = \frac{1}{2}\log 2\pi + \left(a - \frac{1}{2}\right)\log|x| - \frac{\pi}{2}|x| + O\left(\frac{1}{|x|}\right).$$

Note that by the formula $\Gamma(z+1) = z\Gamma(z)$, the assumption $a > 0$ can be removed. Hence

$$|\Gamma(a+\mathrm{i}x)| = \sqrt{2\pi}|x|^{a-\frac{1}{2}}e^{-\frac{\pi}{2}|x|}\left[1 + O\left(\frac{1}{|x|}\right)\right].$$ ■

2.

(a) Using saddle point method, show that $\int_0^{\frac{\pi}{2}} \sin^n t\,\mathrm{d}t = \left(\frac{\pi}{2n}\right)^{1/2}(1 + O(n^{-1}))$.

(b) Derive Wallis formula $\pi = \lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$ from (a).

Solution.

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n t\,\mathrm{d}t = \frac{1}{2}\int_0^{\pi} \sin^n t\,\mathrm{d}t$. Then

$$I_n = \frac{1}{2}\int_0^{\pi} e^{n\log\sin t}\,\mathrm{d}t.$$

Put

$$g(t) \equiv 1 \quad \varphi(t) = \log\sin t.$$

We have

$$\varphi'(t) = \frac{\cos t}{\sin t} = \cot t \quad \varphi''(t) = -\frac{1}{\sin^2 t}.$$

Note that $\varphi'(t) > 0$ for $t \in (0, \frac{\pi}{2})$, $\varphi'(t) < 0$ for $t \in (\frac{\pi}{2}, \pi)$, and $\varphi'(t) = 0 \iff t = \frac{\pi}{2}$. So φ attains maximum at $t = \frac{\pi}{2}$. Now $\varphi''(\frac{\pi}{2}) = -1$. Therefore,

$$I_n = \frac{1}{2} e^{n\varphi(\frac{\pi}{2})} \sqrt{\frac{-2\pi}{\varphi''(\frac{\pi}{2})}} \frac{1}{\sqrt{n}} (1 + O(n^{-1})) = \sqrt{\frac{\pi}{2n}} (1 + O(n^{-1})).$$

(b) Replace n by $2n$, then on the one hand we have

$$I_{2n} = \frac{1}{2}\sqrt{\frac{\pi}{n}}(1 + O(n^{-1})).$$

On the other hand, it is well known that

$$I_{2n} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}.$$

Thus

$$\sqrt{\pi} = \frac{1}{\sqrt{n}} \frac{(2n)!!}{(2n-1)!!}(1 + O(n^{-1})),$$

let $n \to \infty$,

$$\pi = \lim_{n\to\infty} \frac{1}{n} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2.$$

■

3. By definition of Euler–Mascheroni constant γ,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + o(1).$$

Make this statement more precise: find the constant a such that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + \frac{a}{n} + o\left(\frac{1}{n}\right).$$

It can be done by purely geometric considerations of areas under corresponding plots.

Solution. Recall the Euler–Maclaurin formula:

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\,\mathrm{d}x + \frac{f(m) + f(n)}{2} + \int_m^n f'(x)P_1(x)\,\mathrm{d}x,$$

where $f \in C[m,n]$, m, n are natural numbers, $P_k(x) = B_k(x - [x])$. In particular, $P_1(x) = x - [x] - \frac{1}{2}$. Put $m = 1$, $f(x) = \frac{1}{x}$, then

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{i} - \log n &= \frac{f(1) + f(n)}{2} - \int_1^n \frac{1}{x^2} P_1(x)\,\mathrm{d}x \\ &= \frac{1}{2n} + \frac{1}{2} - \int_1^n \frac{1}{x^2} P_1(x)\,\mathrm{d}x. \end{aligned}$$

Let $n \to \infty$,

$$\gamma = \frac{1}{2} - \int_1^\infty \frac{1}{x^2} P_1(x)\,\mathrm{d}x = \frac{1}{2} - \int_1^n \frac{1}{x^2} P_1(x)\,\mathrm{d}x - \int_n^\infty \frac{1}{x^2} P_1(x)\,\mathrm{d}x.$$

Thus

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \frac{1}{2n} + \gamma + \int_n^\infty \frac{1}{x^2} P_1(x)\,\mathrm{d}x.$$

For the remainder term,

$$\begin{aligned}\int_n^\infty \frac{1}{x^2} P_1(x)\,\mathrm{d}x &= \sum_{j=n}^\infty \int_j^{j+1} \frac{P_1(x)}{x^2}\,\mathrm{d}x \\ &\xlongequal{x=j+t} \sum_{j=n}^\infty \int_0^1 \frac{P_1(j+t)}{(j+t)^2}\,\mathrm{d}t \\ &= \sum_{j=n}^\infty \int_0^1 \frac{P_1(t)}{(j+t)^2}\,\mathrm{d}t \quad (P_1 \text{ has period } 1).\end{aligned}$$

Note that $\int_0^1 P_1(t)\,\mathrm{d}t = 0$, then

$$\begin{aligned}\left|\sum_{j=n}^\infty \int_0^1 \frac{P_1(t)}{(j+t)^2}\,\mathrm{d}t\right| &= \left|\sum_{j=n}^\infty \int_0^1 P_1(t)\left(\frac{1}{(j+t)^2} - \frac{1}{(j+1)^2}\right)\mathrm{d}t\right| \\ &\leqslant \sum_{j=n}^\infty \int_0^1 |P_1(t)|\left(\frac{1}{j^2} - \frac{1}{(j+1)^2}\right)\mathrm{d}t \\ &= \frac{1}{n^2}\int_0^1 |P_1(t)|\,\mathrm{d}t \\ &= \frac{1}{n^2}\int_0^1 \left|t - \frac{1}{2}\right|\mathrm{d}t \\ &= \frac{1}{4n^2}.\end{aligned}$$

Therefore

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

The constant we are looking for is $a = \frac{1}{2}$. ■

Problems for seminar 5

1. Using Riemann functional equation to compute the sums

$$1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\cdots+\frac{1}{n^{2m}}+\cdots.$$

Solution. Riemann zeta function $\zeta(s)$ satisfies the functional equation

$$\zeta(s)\Gamma(s)=\frac{2^{s-1}\pi^s}{\cos\left(\frac{\pi s}{2}\right)}\zeta(1-s).$$

Recall that

$$\zeta(1-2m)=-\frac{B_m}{2m}.$$

Hence

$$\begin{aligned}\zeta(2m)\Gamma(2m)&=\frac{2^{2m-1}\pi^{2m}}{\cos(m\pi)}\zeta(1-2m)\\&=(-1)^m2^{2m-1}\pi^{2m}\left(-\frac{B_{2m}}{2m}\right),\\ \zeta(2m)&=\frac{(-1)^{m+1}2^{2m-1}\pi^{2m}\frac{B_{2m}}{2m}}{\Gamma(2m)}\\&=\frac{(-1)^{m+1}\frac{1}{2}(2\pi)^{2m}B_{2m}}{2m\cdot(2m-1)!}\\&=\frac{1}{2}(-1)^{m+1}\frac{B_{2m}}{(2m)!}(2\pi)^{2m}.\end{aligned}$$

■

2. One can define multiparameter Hurwitz ζ function

$$\zeta(x,s|\omega_1,\omega_2,\ldots,\omega_m)=\sideset{}{'}\sum_{n_1,\ldots,n_m\geqslant 0}\frac{1}{(x+n_1\omega_1+\cdots+n_m\omega_m)^s}.$$

Here $'$ means that there is no summand with $(n_1,\ldots,n_m)=(0,\ldots,0)$. What can you say (functional equation, identifying with something known, etc.) about the functions

$$\begin{aligned}G(x|\omega_1)&=\exp\left(\left.\frac{\partial\zeta(x,s|\omega_1)}{\partial s}\right|_{s=0}\right),\\ G(x|\omega_1,\omega_2)&=\exp\left(\left.\frac{\partial\zeta(x,s|\omega_1,\omega_2)}{\partial s}\right|_{s=0}\right).\end{aligned}$$

Solution. For $m=1$,

$$\zeta(x,s|\omega_1)=\sum_{n_1\geqslant 0}\frac{1}{(x+n_1\omega_1)^s}=\frac{1}{\omega_1^s}\sum_{n_1\geqslant 0}\frac{1}{\left(\frac{x}{\omega_1}+n_1\right)^s}=\frac{1}{\omega_1^s}\zeta\left(\frac{x}{\omega_1},s\right).$$

Taking derivative w.r.t. s,

$$\frac{\partial\zeta(x,s|\omega_1)}{\partial s}=-\frac{\log\omega_1}{\omega_1^s}\zeta\left(\frac{x}{\omega_1},s\right)+\frac{1}{\omega_1^s}\frac{\partial\zeta\left(\frac{x}{\omega_1},s\right)}{\partial s}$$

Let $s=0$, note that $\zeta(a,0)=\frac{1}{2}-a$ and $\left.\frac{\partial\zeta(a,s)}{\partial s}\right|_{s=0}=\log\frac{\Gamma(a)}{\sqrt{2\pi}}$, then

$$\left.\frac{\partial\zeta(x,s|\omega_1)}{\partial s}\right|_{s=0}=\left(\frac{x}{\omega_1}-\frac{1}{2}\right)\log\omega_1+\log\frac{\Gamma\left(\frac{x}{\omega_1}\right)}{\sqrt{2\pi}},$$

and hence

$$G(x|\omega_1) = \frac{\omega_1^{\frac{x}{\omega_1}-\frac{1}{2}}}{\sqrt{2\pi}}\Gamma\left(\frac{x}{\omega_1}\right).$$

Let $\zeta_m(x, s|\omega_1, \omega_2, \ldots, \omega_m) = \zeta(x, s|\omega_1, \omega_2, \ldots, \omega_m)$.
It can be check that

$$\zeta_m(x + \omega_j, s|\omega_1, \ldots, \omega_m) - \zeta_m(x, s|\omega_1, \ldots, \omega_m) = -\zeta_{m-1}(x, s|\omega(j)), \quad j = 1, \ldots, m, \tag{$*$}$$

where $\omega(j) = (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m)$ and $\zeta_0(x, s) = x^{-s}$.
For $m = 1$,

$$\begin{aligned}
\zeta_1(x + \omega_1, s|\omega_1) - \zeta_1(x, s|\omega_1) &= \sum_{n_1 \geqslant 0} \frac{1}{(x + \omega_1 + n_1\omega_1)^s} - \sum_{n_1 \geqslant 0} \frac{1}{(x + n_1\omega_1)^s} \\
&= \sum_{n_1 \geqslant 1} \frac{1}{(x + n_1\omega_1)^s} - \sum_{n_1 \geqslant 0} \frac{1}{(x + n_1\omega_1)^s} \\
&= -\frac{1}{x^s}.
\end{aligned}$$

Noe, assume $m \geqslant 2$, then

$$\begin{aligned}
&\zeta_m(x + \omega_j, s|\omega_1, \ldots, \omega_m) - \zeta_m(x, s|\omega_1, \ldots, \omega_m) \\
&= \sideset{}{'}\sum_{n_1,\ldots,n_m \geqslant 0} \frac{1}{(x + \omega_j + n_1\omega_1 + \cdots + n_m\omega_m)^s} - \sideset{}{'}\sum_{n_1,\ldots,n_m \geqslant 0} \frac{1}{(x + n_1\omega_1 + \cdots + n_m\omega_m)^s} \\
&= \sideset{}{'}\sum_{n_1,\ldots,n_m \geqslant 0} \frac{1}{(x + n_1\omega_1 + \cdots + (n_j + 1)\omega_j + \cdots + n_m\omega_m)^s} - \sideset{}{'}\sum_{n_1,\ldots,n_m \geqslant 0} \frac{1}{(x + n_1\omega_1 + \cdots + n_m\omega_m)^s} \\
&= -\sideset{}{'}\sum_{n_1 \geqslant 0,\ldots,n_j = 0, n_m \geqslant 0} \frac{1}{(x + n_1\omega_1 + \cdots + 0 \cdot \omega_j + \cdots + n_m\omega_m)^s} \\
&= -\zeta_{m-1}(x, s|\omega(j)).
\end{aligned}$$

Let

$$\Gamma_m(x|\omega) = \exp\left(\left.\frac{\partial\zeta(x, s|\omega_1, \ldots, \omega_m)}{\partial s}\right|_{s=0}\right).$$

After differentiating above difference equations ($*$) w.r.t. s, let $s = 0$, and take exp, we get

$$\Gamma_m(x + \omega_j|\omega_1, \ldots, \omega_m) = \frac{1}{\Gamma_{m-1}(x|\omega(j))}\Gamma_m(x|\omega_1, \ldots, \omega_m),$$

where $\Gamma_0(x|\omega) = x^{-1}$.
In particular, take $j = m$, then

$$\Gamma_m(x|\omega_1, \ldots, \omega_m) = \Gamma_{m-1}(x|\omega_1, \ldots, \omega_{m-1})\Gamma_m(x + \omega_m|\omega_1, \ldots, \omega_m).$$

Now, let $m = 1$, we then have

$$\begin{aligned}
\Gamma_1(x|\omega_1) &= \frac{1}{x}\Gamma_1(x + \omega_1|\omega_1) \\
&= \frac{1}{x}\left(\frac{1}{x + \omega_1}\Gamma_1(x + 2\omega_1|\omega_1)\right) \\
&\cdots \\
&= \Gamma_1(x + l\omega_1|\omega_1)\prod_{k=0}^{l-1}\frac{1}{x + k\omega_1}, \quad l \in \mathbb{N}.
\end{aligned}$$

Similarly, let $m = 2$, we then have

$$\begin{aligned}\Gamma_2(x|\omega_1,\omega_2) &= \Gamma_1(x|\omega_1)\Gamma_2(x+\omega_2|\omega_1,\omega_2)\\ &= \Gamma_1(x|\omega_1)\Gamma(x+\omega_2)\Gamma_2(x+2\omega_2|\omega_1,\omega_2)\\ &\cdots\\ &= \Gamma_2(x+l\omega_2|\omega_1,\omega_2)\prod_{k=0}^{l-1}\Gamma_1(x+k\omega_2|\omega_1), \quad l\in\mathbb{N}.\end{aligned}$$

Next, let's derive the integral representation of Γ_m.
Recall that for $l \in \mathbb{N}\cup\{0\}$,

$$\int_0^\infty \frac{\mathrm{d}t}{t} t^{s+l} e^{-zt} = z^{-s-l}\int_0^\infty \frac{\mathrm{d}u}{u} u^{s+l} e^{-u} = z^{-s-l}\Gamma(s+l).$$

Let $l = 0$, then

$$\int_0^\infty t^{s-1} e^{-zt}\,\mathrm{d}t = \frac{1}{z^s}\Gamma(s) \tag{1}$$

The following integral representations for the Euler gamma function are well known

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t = \frac{\mathrm{i}}{2\sin(\pi s)}\int_{C_H}(-t)^{s-1}e^{-t}\,\mathrm{d}t,$$

where in the first case $\operatorname{Re} s > 0$, and in the second expression $|\arg(-t)| < \pi$ and the Hankel contour C_H starts and finishes near the $+\infty$ point, turning around the half-axis $[0,\infty)$ counterclockwise. Note that

$$\prod_{k=1}^{m}\frac{1}{1-e^{-\omega_k t}} = \sum_{n_1,\ldots,n_m=0}^{\infty} e^{-t(n_1\omega_1+\cdots+n_m\omega_m)}.$$

On the one hand,

$$\begin{aligned}\int_0^\infty \frac{t^{s-1}e^{-xt}}{\prod_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t &= \int_0^\infty t^{s-1}e^{-xt}\sum_{n_1,\ldots,n_m=0}^{\infty} e^{-t(n_1\omega_1+\cdots+n_m\omega_m)}\,\mathrm{d}t\\ &= \sum_{n_1,\ldots,n_m=0}^{\infty}\int_0^\infty t^{s-1}e^{-(x+n_1\omega_1+\cdots+n_m\omega_m)t}\,\mathrm{d}t\\ &= \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{1}{(x+n_1\omega_1+\cdots+n_m\omega_m)^s}\Gamma(s).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{C_H}\frac{(-t)^{s-1}e^{-xt}}{\prod_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t &= \int_{C_H}(-t)^{s-1}e^{-xt}\sum_{n_1,\ldots,n_m=0}^{\infty} e^{-t(n_1\omega_1+\cdots+n_m\omega_m)}\,\mathrm{d}t\\ &= \sum_{n_1,\ldots,n_m=0}^{\infty}\int_{C_H}(-t)^{s-1}e^{-(x+n_1\omega_1+\cdots+n_m\omega_m)t}\,\mathrm{d}t\\ &= \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{1}{(x+n_1\omega_1+\cdots+n_m\omega_m)^s}\cdot\frac{2\Gamma(s)\sin(\pi s)}{i}\\ &= \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{1}{(x+n_1\omega_1+\cdots+n_m\omega_m)^s}\cdot\frac{2\pi}{\mathrm{i}\Gamma(1-s)}.\end{aligned}$$

Thus

$$\zeta_m(x,s|\omega) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-xt}}{\prod_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t$$

$$= \frac{\mathrm{i}\Gamma(1-s)}{2\pi}\int_{C_H}\frac{(-t)^{s-1}e^{-xt}}{\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t. \tag{$\star$}$$

After differentiating above difference equations ($\star$) w.r.t. s,let $s=0$, and take exp, we get

$$\begin{aligned}
\Gamma_m(x|\omega) &= \exp\left\{\frac{\mathrm{i}}{2\pi}\left[(-\Gamma'(1-s))|_{s=0}\int_{C_H}\frac{(-t)^{s-1}e^{-xt}}{\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t\bigg|_{s=0} + \Gamma(1-s)|_{s=0}\int_{C_H}\frac{\log(-t)(-t)^{s-1}e^{-xt}}{\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t\bigg|_{s=0}\right]\right\}\\
&= \exp\left\{\frac{\mathrm{i}}{2\pi}\left[(-\Gamma'(1))\int_{C_H}\frac{(-t)^{-1}e^{-xt}}{\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t + \Gamma(1)\int_{C_H}\frac{\log(-t)(-t)^{-1}e^{-xt}}{\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t\right]\right\}\\
&= \exp\left\{\frac{-\mathrm{i}}{2\pi}\left[\gamma\int_{C_H}\frac{e^{-xt}}{t\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t + \int_{C_H}\frac{\log(-t)e^{-xt}}{t\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t\right]\right\}\\
&= \exp\left(\frac{1}{2\pi i}\int_{C_H}\frac{e^{-xt}(\log(-t)+\gamma)}{t\prod\limits_{k=1}^{m}(1-e^{-\omega_k t})}\,\mathrm{d}t\right) \qquad \blacksquare
\end{aligned}$$