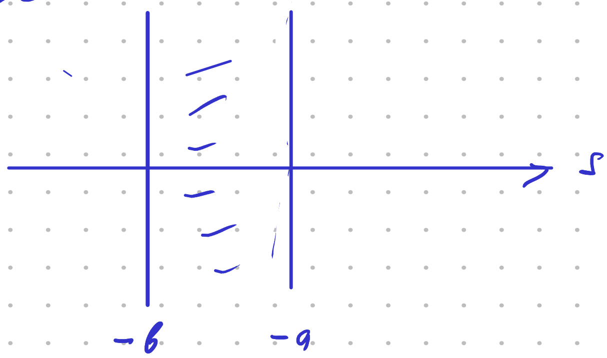
Resume: $\varphi(t) \to \check{\varphi}(s) = \int_0^\infty \varphi(t)\, t^{s-1}\, dt$

1) $\varphi(t) < c_1 t^a \quad t \to \infty$
$\varphi(t) < c_2 t^b \quad t \to 0$ $\Rightarrow$ $\check{\varphi}(s)$ is an. in the strip $-b < \operatorname{Re} s < -a$



2) $\varphi(t) \underset{t\to\infty}{\sim} \sum A_n t^n \Rightarrow \check{\varphi}(s)$ has poles $s=-n$

$\check{\varphi}(s) \sim -\sum \frac{A_n}{s+n}$ in right half plane

$\varphi(t) \underset{t\to 0}{\sim} \sum B_n t^n \Rightarrow \check{\varphi}(s)$ has poles $s=-n$

$\check{\varphi}(s) \sim \sum \frac{B_n}{s+n}$ LH plane

$\varphi(t) \underset{t\to\infty}{\sim} \dots\ t^n \log t \Rightarrow \check{\varphi}(s) \sim \frac{1}{(s+n)^2} + \dots$

$\varphi(t) \sim t^n \log t \Rightarrow \check{\varphi}(s) \sim -\frac{1}{(s+n)^2} + \dots$

4). $\varphi(t) = \sum_{n=1}^{\infty} \psi\left(\frac{t}{n}\right) \Rightarrow \check{\varphi}(s) = \check{\psi}(s) \cdot \zeta(-s)$

Study asymp. behaviour of $\log \Gamma(t)$ $t \to \infty$

Weirstr. $\Gamma^{-1}(t) = t e^{t\gamma} \prod_{n\geq 1} \left(1+\frac{t}{n}\right) \cdot e^{-\frac{t}{n}}$

$$\log \Gamma(t) = -t\gamma - \log t + \sum_{n\geq 1} \left(\frac{t}{n} - \log\left(1+\frac{t}{n}\right)\right)$$

$$\log \Gamma(t) = -\gamma t - \log t + \sum_{n\geq 1} \varphi\left(\frac{t}{n}\right)$$ where $\varphi(t) = t - \log(t+1)$

$$\underbrace{\log \Gamma(t) + \gamma t + \log t}_{u(t)} = \sum_{n\geq 1} \varphi\left(\frac{t}{n}\right)$$

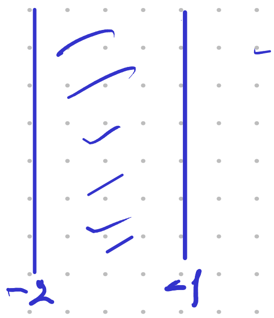
$$\check{u}(s) = \check{\varphi}(s) \cdot \zeta(-s)$$

Compute $\check{\varphi}(s) = \int_0^\infty \left(t - \log(1+t)\right) t^{s-1}\,dt$ $\qquad \log(1+t) = t - \frac{t^2}{2} + \ldots$

Region of anal. of $\check{\varphi}(s)$

$t \to 0$
integrand $\approx \frac{t^{s+1}}{2}$
converges $\operatorname{Re}(s+1) > -1$
$\operatorname{Re} s > -2$

$t \to \infty$ integrand $\approx t^s$ converges $\operatorname{Re} s < -1$



$-2 < \operatorname{Re} s < -1$

$$\int_0^\infty \underbrace{\left(t - \log(1+t)\right)}_{u} \underbrace{t^{s-1}\,dt}_{v'} = \underbrace{\left(t - \log(1+t)\right)}_{\sim t^2} \cdot \frac{t^s}{s} \Big|_0^\infty - \frac{1}{s} \int_0^\infty \left(1 - \frac{1}{t+1}\right) t^s\,dt =$$

$\downarrow 0$

$$= -\frac{1}{s} \int_0^\infty \frac{t^{s+1}}{1+t}\,dt = -\frac{1}{s} B(s+2, -1-s) = -\frac{1}{s} \frac{\Gamma(s+2)\Gamma(-1-s)}{\Gamma(1)} =$$

$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt$ $\qquad = -\frac{\pi}{s \sin \pi(s+2)} = -\frac{\pi}{s \sin \pi s}$

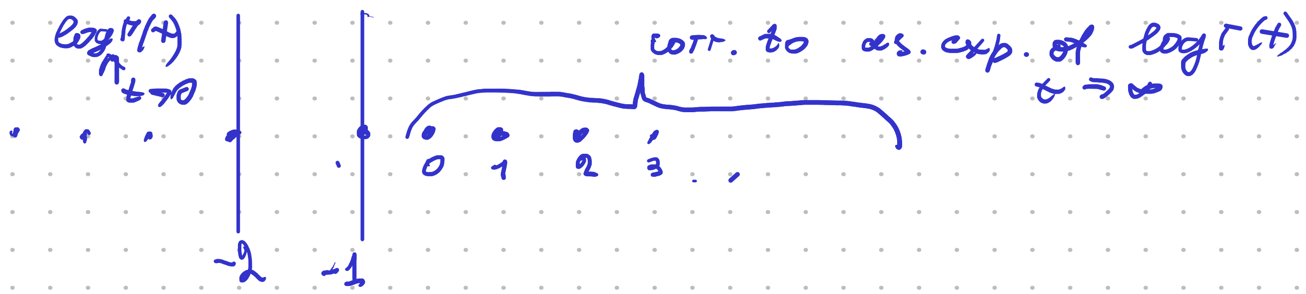
$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$

$\check{\varphi}(s) = -\frac{\pi}{s \sin \pi s}$ $\qquad \check{\varphi}(s) \cdot \zeta(-s)$
compute poles

$$\frac{\pi}{\sin \pi s} = \frac{1}{s} + \sum_{n=1,\ldots} \frac{(-1)^k 2s}{n^2 - s^2} \Downarrow$$

$$\check{\varphi}(s) = -\frac{1}{s^2} + \sum_{k \neq 0} \frac{(-1)^{k+1}}{k(s+k)}$$

$\check{\varphi}(s) \cdot \zeta(-s)$ what are the poles?



$\check{\varphi}(s)$ has poles at $s=-1,0,1,\dots$

$\zeta(-s)$ has pole at $s=-1$

$\Rightarrow$ poles $s=1,2,3,\dots$ are simple

$s\to k\geq 1$ $\quad \check{\varphi}(s)\sim \dfrac{(-1)^{k+1}}{k(s-k)}$

$\check{\varphi}(s)\cdot\zeta(-s)\sim \dfrac{(-1)^k}{k(s-k)}\cdot\zeta(-k)$

$$\mathrm{Res} = \frac{(-1)^k}{k}\zeta(-k) = \frac{B_{k+1}}{(k+1)\cdot k}$$

~~there~~ there are terms of as. expansion of $(\Gamma+\dots)$ at ∞

$$\frac{B_{k+1}}{k(k+1)}\cdot\frac{1}{t^k} \qquad \frac{B_{2m}}{2m(2m-1)}\frac{1}{x^{2m-1}} \qquad k+1=2m$$

$s=0,\ -1$

1) $s=0$ $\quad \check{\varphi}(s) = -\dfrac{1}{s^2}+O(1)$

$\zeta(-s)$ $\qquad \zeta'(0) = -\frac{1}{2}\log 2\pi \qquad \zeta(0)=-\frac{1}{2}$

$\zeta(-s) = -\frac{1}{2}+\frac{1}{2}\log 2\pi\cdot s$

$$\check{\varphi}(s)\cdot\zeta(-s) = \left(-\frac{1}{s^2}+O(1)\right)\cdot\left(-\frac{1}{2}+\frac{1}{2}\log 2\pi\cdot s\right) =$$

$$= \frac{1}{2s^2} - \frac{\log 2\pi}{2}\cdot\frac{1}{s}$$

$$\left(\log\Gamma(t)+\gamma t+\log t\right)\sim \frac{\log 2\pi}{2}\cdot t^0+\frac{1}{2}\log t$$

$s=-1$ $\quad \zeta(s)\approx \dfrac{1}{s-1}+\gamma+O(s-1) \qquad \zeta(-s)\approx\dfrac{1}{-s-1}+\gamma+O(s+1)$

$\check{\varphi}(s)\approx -\dfrac{1}{s+1}-1+O(s+1)$

$$\check{\varphi}(s) = -\frac{1}{s^2}+\sum_{k\geq 1}\frac{(-1)^{k+1}}{k(s+k)} = -\frac{1}{s+1}+\left(-\frac{1}{s^2}+\frac{1}{s-1}+\sum_{k\geq 2}\frac{(-1)^{k+1}}{k(s+k)}\right)\Bigg|_{s=-1}$$

$k=-1$

$-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}$

$$\check{\varphi}(s)\zeta(-s) \underset{s\to -1}{\simeq} -\left(\frac{1}{s+1}+1\right)\left(-\frac{1}{s+1}+\gamma\right)+\dots = \frac{1}{(s+1)^2}+\frac{(1-\gamma)}{s+1}$$

we have terms $t\log t + (\gamma-1)t$ in as. expansion

Collect:

$$\log\Gamma(t) + \cancel{\gamma t} + \log t \approx t\log t + (\cancel{\gamma}-1)t + \frac{1}{2}\log t + \frac{1}{2}\log 2\pi + \sum_{m\geq 1}\frac{B_{2m}}{2m(2m-1)}\frac{1}{x^{2m-1}} \Rightarrow$$

$$\log\Gamma(t) \simeq \left(t-\frac{1}{2}\right)\log t - t + \frac{\log 2\pi}{2} + \sum_{m\geq 1}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}$$

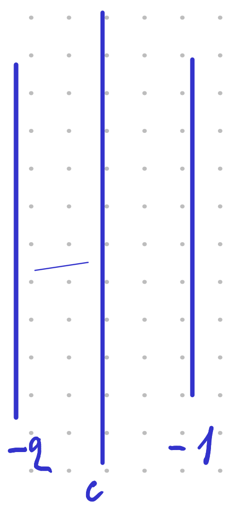
$$g(z) = \log\Gamma(z) - \gamma z - \log z$$

$$\check{g}(s) = -\frac{\pi}{s\sin\pi s}\cdot\zeta(-s)$$

$\check{g}(s)$ an. in the strip $-2<\mathrm{Re}\,s<-1$

Inversion formula

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{-\pi}{s\sin\pi s}\zeta(-s)t^{-s}\,ds$$



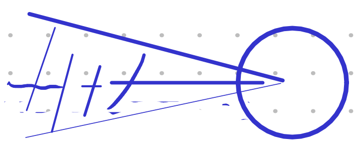
$$t^{-s} = e^{(\log|t| + i\arg t)\underset{c+i\lambda}{s}} \simeq e^{\lambda|\arg t|}$$

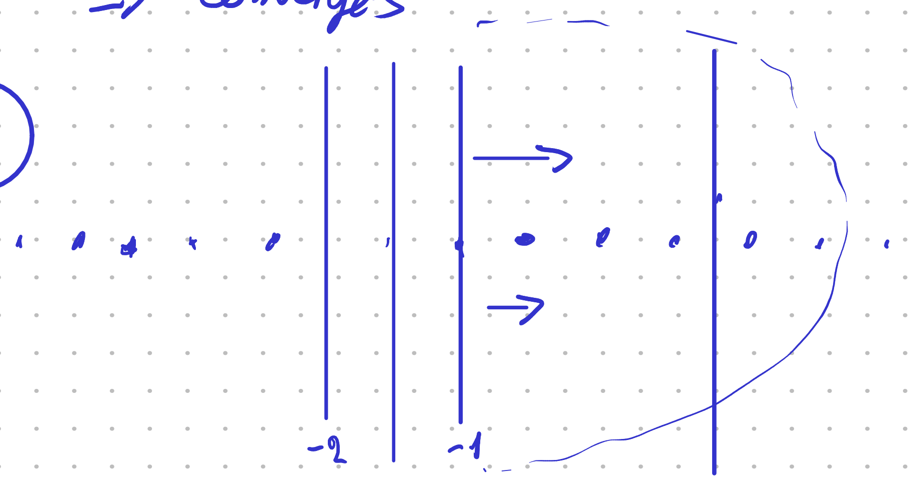
$$\frac{1}{\sin\pi s} \sim \frac{1}{\sinh\pi\lambda} \sim e^{-\pi\lambda}$$

$\zeta(-s)$ is bounded

$$e^{\lambda(|\arg t|-\pi)} \quad \text{if } |\arg t|<\pi-\delta$$

$\Rightarrow$ converges





$s = \underset{n}{\underset{\shortparallel}{R}} + i\lambda$ $\qquad e^{\overbrace{(\log|t| + i\arg t)\, s}^{}} \quad \underset{(n+i\lambda)}{\shortparallel} = t^n \cdot e^{\lambda \arg t}$

$s = n + i\lambda$ $\qquad \Rightarrow$ integral over nitted line has order t^n

For the proof we need bound for $\zeta(-z)$

$\zeta(\sigma + i\lambda) = O(|\lambda|^{1/2+\delta})$ $\qquad \operatorname{Re} s > 0$, $\quad 2 < 0$

$\delta > 0 \qquad \sum \frac{1}{n^s}$

Hypergeometric functions.

Hypergeometr. series.

$\sum_{n\geq 0} a_n z^n \qquad \frac{a_{n+1}}{a_n} = \frac{P_p(n)}{Q_q(n)}$ – rational function on n

$(a)_n = a\cdot(a+1)\dots(a+n-1)$ – Pochhammer symbol

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (a)_0 = 1$

$$F_{p,q}(a_1 \dots a_p, b_1 \dots b_q; z) = 1 + \sum_{n\geq 1} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

$\frac{(a)_{n+1}}{(a)_n} = a + n \qquad e^x = 1 + \sum \frac{x^n}{n!} =$

$= F_{0,0}(x)$

$\log(1-x) = x + \frac{x^2}{2} + \dots + \frac{x^n}{n} = x\left(1 + \frac{1}{2}x + \dots\right) = x F_{1,0}(0; x)$

$\frac{1}{n} = \frac{(0)_n}{(1)_n} = \frac{(0)_n}{n!}$

Convergence D'alamber crit.

$$\lim \frac{a_{n+1}}{a_n} = \frac{(a_1)_{n+1} \dots (a_p)_{n+1}}{(b_1)_{n+1} \dots (b_q)_{n+1}} \Big/ \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \cdot \frac{z}{n+1} =$$

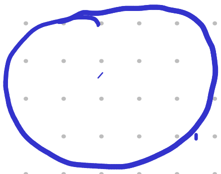
$$= \frac{(n+1+a_1)\dots(n+1+a_p)}{(n+1+b_1)\dots(n+1+b_q)\,(n+1)} \cdot z =$$

$$= \frac{\left(1+\frac{a_1}{n+1}\right)\cdots\left(1+\frac{a_p}{n+1}\right)}{\left(1+\frac{b_1}{n+1}\right)\cdots\left(1+\frac{b_q}{n+1}\right)}\cdot (n+1)^{p-q-1}\cdot z$$

$$\Rightarrow \begin{cases} p > q+1 & \infty \quad \text{diverges} \quad \forall z \\ p < q+1 & 0 \quad \text{converges} \quad \forall z \\ p = q+1 & |z|<1 \text{ conv.} \quad |z|>1 \text{ div} \end{cases}$$

let $p = q+1$

conv. on $|z|=1$?



$|z|<1$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}$$

use asymp. $\frac{\Gamma(z+a)}{\Gamma(z+b)}$ $z\to\infty$ $\overset{?}{\sim} z^{a-b}$

$$\log\frac{\Gamma(z+a)}{\Gamma(z+b)} = \log\Gamma(z+a) - \log\Gamma(z+b) =$$

$$= \left(z+a-\tfrac{1}{2}\right)\log(z+a) - (z+a) + \cancel{\tfrac{1}{2}\log 2\pi} + O\left(\tfrac{1}{z}\right)$$

$$- \left[\left(z+b-\tfrac{1}{2}\right)\log(z+b) - (z+b) + \cancel{\tfrac{1}{2}\log 2\pi}\right]$$

$$= \left(z+a-\tfrac{1}{2}\right)\Big(\log z + \underbrace{\log\left(1+\tfrac{a}{z}\right)}_{\sim \frac{a}{z}}\Big) - a -$$

$$- \left(z+b-\tfrac{1}{2}\right)\left[\log z + \tfrac{b}{z}\right] + b$$

$$= (a-b)\log z + a - b - a + b + O\left(\tfrac{1}{z}\right) =$$

$$(a-b)\log z$$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}$$

$$\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n\cdot(1)_n} = \frac{\Gamma(b_1)\cdots\Gamma(b_q)\,\Gamma(1)}{\Gamma(a_1)\cdots\Gamma(a_p)}\cdot\frac{\Gamma(a_1+n)}{\Gamma(b_1+n)}\cdots\frac{\Gamma(a_p+n)}{\Gamma(1+n)} =$$

$$= n^{\sum a_k - \sum b_k - 1}\left(1+O\left(\tfrac{1}{n}\right)\right)$$

Result: $\begin{cases} \operatorname{Re}\left(\sum a_k - \sum b_j\right) > 0 & \text{diverges for } |z|=1 \\ \operatorname{Re}\sum a_k - \operatorname{Re}\sum b_j < 0 & \text{converges for } |z|=1 \end{cases}$

Gauss hyperg. function $F_{2,1}(a,b;c;z) =$

$$= 1 + \sum_{n\geq 1} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

1. Integral presentation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$\operatorname{Re} c > \operatorname{Re} b > 0$

$t^{b-1} = \exp((b-1)\log t)$ $\quad (1-t)^{c-b-1} = \exp((c-b-1)\log(1-t))$

$1-tz \quad |z|<1$

Proof $\displaystyle\int_0^1 t^{b-1}(1-t)^{c-b-1}\left[\sum \underbrace{\frac{(a)_n}{n!}}_{\parallel} t^n z^n\right] dt =$

$$\binom{-a}{n} = \frac{(-a)(-a-1)\ldots(-a-n+1)}{n!} = (-1)^n \frac{a\cdots(a+n-1)}{n!}$$

$$= \sum_{n\geq 0} \frac{(a)_n z^n}{n!} \int t^{n+b-1}(1-t)^{c-b-1}\,dt = \sum_{n\geq 0} \frac{(a)_n}{n!} z^n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} =$$

$$= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c)} \sum \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Corollary $\displaystyle F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$ $\quad \operatorname{Re}(c-a-b) > 0$

$$F(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} \underbrace{(1-t)^{c-b-1}(1-t)^{-a}}_{\parallel\; (1-t)^{c-a-b-1}} dt =$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\,\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)}\cdot\Gamma(c-a-b)}{\Gamma(c-a)}$$

If $b = -n$ $\quad n \in \mathbb{N} \Rightarrow F$ is a polynomial of degree n

$N > n \quad (b)_N = b(b+1)\dots(b+N) = -n(-n+1)\dots(-n+n)\dots(-n+N)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$

$$F(a,-n,c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

is equality a the sum of Binom. coef ts

$b = -n$

$$\frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{\Gamma(c-a+n)}{\Gamma(c-a)} = \frac{(c-a)_n}{(c)_n}$$

Dif. equation $\quad F_{p,q}(a_1\dots a_p; b_1\dots b_q; z)$

$$\delta = z\frac{d}{dz}$$

$$\delta\cdot F = \sum_{n\geq 0} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n} \frac{\cancel{n}\, z^n}{\cancel{n!}\,(n-1)!}$$

$(\delta + a_k)\cdot F =$ $\qquad \begin{matrix}\delta \\ a_k\end{matrix} \to$ mult. by $\begin{matrix} n \\ a_n \end{matrix} \to (n+a_k)$

$$= \sum_{n\geq 0} \frac{(a_1)_n\dots(a_k)_{n+1}\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{z^n}{n!}$$

$$(\delta + b_k - 1)F = \sum_{n\geq 0} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_k)_{n-1}\dots(b_q)_n}\frac{z^n}{n!}$$

$n + b_k - 1$

$(b_k)_n \to (b_k)_{n-1}$

$$(\delta+a_1)\dots(\delta+a_p)\cdot F = \sum_{n\geq 0} \frac{(a_1)_{n+1}\dots(a_p)_{n+1}}{(b_1)_n\dots(b_q)_n}\frac{z^n}{n!}$$

$$\delta\cdot(\delta+b_1-1)\dots(\delta+b_q-1)F =$$

$$= \sum_{n\geq 0}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_{n-1}\dots(b_q)_{n-1}}\cdot\frac{z^n}{(n-1)!}$$

$n \to n+1$

$\shortparallel$

$$\frac{d}{dz}\prod(\delta+b_k-1) = \sum_{n\geq 0}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_{n-1}\dots(b_q)_{n-1}}\frac{z^{n-1}}{(n-1)!}$$

$$\left[\left(z\frac{d}{dz}+a_1\right)\dots\left(z\frac{d}{dz}+a_p\right) - \frac{d}{dz}\cdot\left(z\frac{d}{dz}+b_1-1\right)\dots\left(z\frac{d}{dz}+b_q-1\right)\right]F = 0$$

ex. $\quad F = F_{2,1}(a,b;c;z)$

$$\left[\left(z\frac{d}{dz}+a\right)\left(z\frac{d}{dz}+b\right)-\frac{d}{dz}\left(z\frac{d}{dz}+c-1\right)\right]F=0$$

$$\left(\left(z\frac{d}{dz}\right)^2+(b+a)z\frac{d}{dz}+ab-\frac{d}{dz}z\frac{d}{dz}+(1-c)\frac{d}{dz}\right)F=0$$

$$z\frac{d}{dz}z\frac{d}{dz} = z^2\frac{d^2}{dz^2}+z\frac{d}{dz}$$

$$= -z\frac{d^2}{dz^2}-\frac{d}{dz}$$

$$\frac{d}{dz}z = z\frac{d}{dz}+1$$

$$(zf)' = zf'+f$$

$$\left[z(z-1)\frac{d^2}{dz^2}+\left((b+a+1)z-c\right)\frac{d}{dz}+ab\right]F=0$$

$$z(1-z)y''+\left(c-(a+b+1)z\right)y'-aby=0$$

hypergeom equation

order $\max(p, q+1)$