Special functions. Problems for seminar 7

Let $\varphi(t)$, $t > 0$ be a locally integrable function such that

$$|\varphi(t)| < C_1 t \qquad \text{for } 0 < t < 1, \qquad |\varphi(t)| < C_1 t^{-1} \qquad \text{for } t > 1,$$

for some positive constants C_1 and C_2.

1. Show that its Mellin transform

$$\check{\varphi}(s) = \int_0^\infty \varphi(t) t^{s-1} dt$$

   is a function, analytical in the strip

$$-1 < \operatorname{Re} s < 1$$

2. Assume that the function $\check{\varphi}(s)$ extends to meromorphic function with simple poles at the points $s = n$, $n = \pm 1, \pm 2, \ldots$ with residues c_n. What can you say about asymptotical expansion of $\varphi(t)$ at zero and at infinity?

3. Assume now that the points $s = \pm 1$ are poles of $\check{\varphi}(s)$ of the second order with singular parts

$$\frac{\alpha_{\pm 1}}{(s \mp 1)^2} + \frac{\beta_{\pm 1}}{s \mp 1}.$$

   What can you say about asymptotics of $\varphi(t)$ in this case?