Problems for seminar 6

1. Generalize Riemann functional equation to Hurwitz ζ function:

$$\zeta(x,s)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi n x}{n^{1-s}}+\cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi n x}{n^{1-s}}\right\},\quad \operatorname{Re} s<0.$$

Solution. Recall that Hurwitz ζ function has integral representations

$$\zeta(x,s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{z^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z=-\frac{1}{2i\Gamma(s)\sin(\pi s)}\int_{C_H}\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z=-\frac{\Gamma(1-s)}{2\pi i}\int_{C_H}\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z.$$

where C_H is the Hankel contour that starts at infinity on the positive real axis, encircles the origin once in the positive direction, excluding the points $\pm 2n\pi i$, $n>1$ an integer, and returns to positive infinity.
Let C be a contour consisting of a circle of radius $(2n+1)\pi$, n is an integer, starting at the point $(2n+1)\pi$ on the real axis and encircling the origin in the positive direction, $\arg(-z)=0$ at $z=-(2n+1)\pi$.
Let C_n be a contour that starts from $(2n+1)\pi$, then encircles the origin once in the positive direction, and returns to $(2n+1)\pi$.

Claim. *The following equality holds:*

$$\frac{1}{2\pi i}\int_C\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z=\frac{1}{2\pi i}\int_{C_n}\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z+\sum_{m=1}^{n}(R_m+R_m'),\tag{$*$}$$

where R_m, R_m' are the residues of the integrand at $2m\pi i$, $-2m\pi i$ respectively.

Proof. Note that in the region between C and the contour C_n, the function $\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}$ is holomorphic and one-valued except at the simple poles $\pm 2\pi i,\dots,\pm 2n\pi i$. Hence ($*$) holds. ■

At the point at which $-z=2m\pi e^{-\pi i/2}$, the residue is

$$\begin{aligned}\lim_{z\to 2m\pi i}(z-2m\pi i)\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}&=(2m\pi e^{-\pi i/2})^{s-1}e^{-2xm\pi i}\lim_{z\to 2m\pi i}\frac{z-2m\pi i}{1-e^{-z}}\\&=(2m\pi e^{-\pi i/2})^{s-1}e^{-2xm\pi i}e^{2m\pi i}\\&=(2m\pi)^{s-1}e^{-\frac{1}{2}\pi i(s-1)}e^{-2xm\pi i}.\end{aligned}$$

Similarly, at the point at which $-z=2m\pi e^{\pi i/2}$, the residue is

$$\begin{aligned}\lim_{z\to -2m\pi i}(z+2m\pi i)\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}&=(2m\pi e^{\pi i/2})^{s-1}e^{2xm\pi i}\lim_{z\to -2m\pi i}\frac{z+2m\pi i}{1-e^{-z}}\\&=(2m\pi e^{\pi i/2})^{s-1}e^{2xm\pi i}e^{-2m\pi i}\\&=(2m\pi)^{s-1}e^{\frac{1}{2}\pi i(s-1)}e^{2xm\pi i}.\end{aligned}$$

Hence

$$R_m+R_m'=(2m\pi)^{s-1}\cdot 2\cos\left(\frac{\pi}{2}(s-1)+2xm\pi\right)=2(2m\pi)^{s-1}\sin\left(\frac{\pi s}{2}+2\pi m x\right).$$

Therefore,

$$-\frac{1}{2\pi i}\int_{C_n}\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z=\frac{2\sin\frac{\pi s}{2}}{(2\pi)^{1-s}}\sum_{m=1}^{n}\frac{\cos(2\pi x m)}{m^{1-s}}+\frac{2\cos\frac{\pi s}{2}}{(2\pi)^{1-s}}\sum_{m=1}^{n}\frac{\sin(2\pi x m)}{m^{1-s}}-\frac{1}{2\pi i}\int_C\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z$$

Now, note that $x\in(0,1]$, there is a number K independent of n s.t. $\left|\frac{e^{-xz}}{1-e^{-z}}\right|<K$ for $z\in C$. It follows that

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z\right|&<\frac{K}{2\pi}\int_{-\pi}^{\pi}|[(2n+1)\pi]^s e^{\mathrm{i}s\theta}|\,\mathrm{d}\theta\\&<K[(2n+1)\pi]^{\sigma}e^{\pi|s|},\quad \sigma=\operatorname{Re} s.\end{aligned}$$

If $\sigma < 0$, then $K[(2n+1)\pi]^\sigma e^{\pi|s|} \to 0$ as $n \to \infty$. We concolude that

$$\begin{aligned} -\frac{1}{2\pi i}\int_{C_H} \frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z &= -\frac{1}{2\pi i}\lim_{n\to\infty}\int_{C_n} \frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z \\ &= \frac{2\sin\frac{\pi s}{2}}{(2\pi)^{1-s}}\sum_{m=1}^{\infty}\frac{\cos(2\pi xm)}{m^{1-s}} + \frac{2\cos\frac{\pi s}{2}}{(2\pi)^{1-s}}\sum_{m=1}^{\infty}\frac{\sin(2\pi xm)}{m^{1-s}}, \end{aligned}$$

and hence

$$\zeta(x,s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{C_H}\frac{(-z)^{s-1}e^{-xz}}{1-e^{-z}}\,\mathrm{d}z = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^{1-s}} + \cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n^{1-s}}\right\}. \quad \blacksquare$$

№2

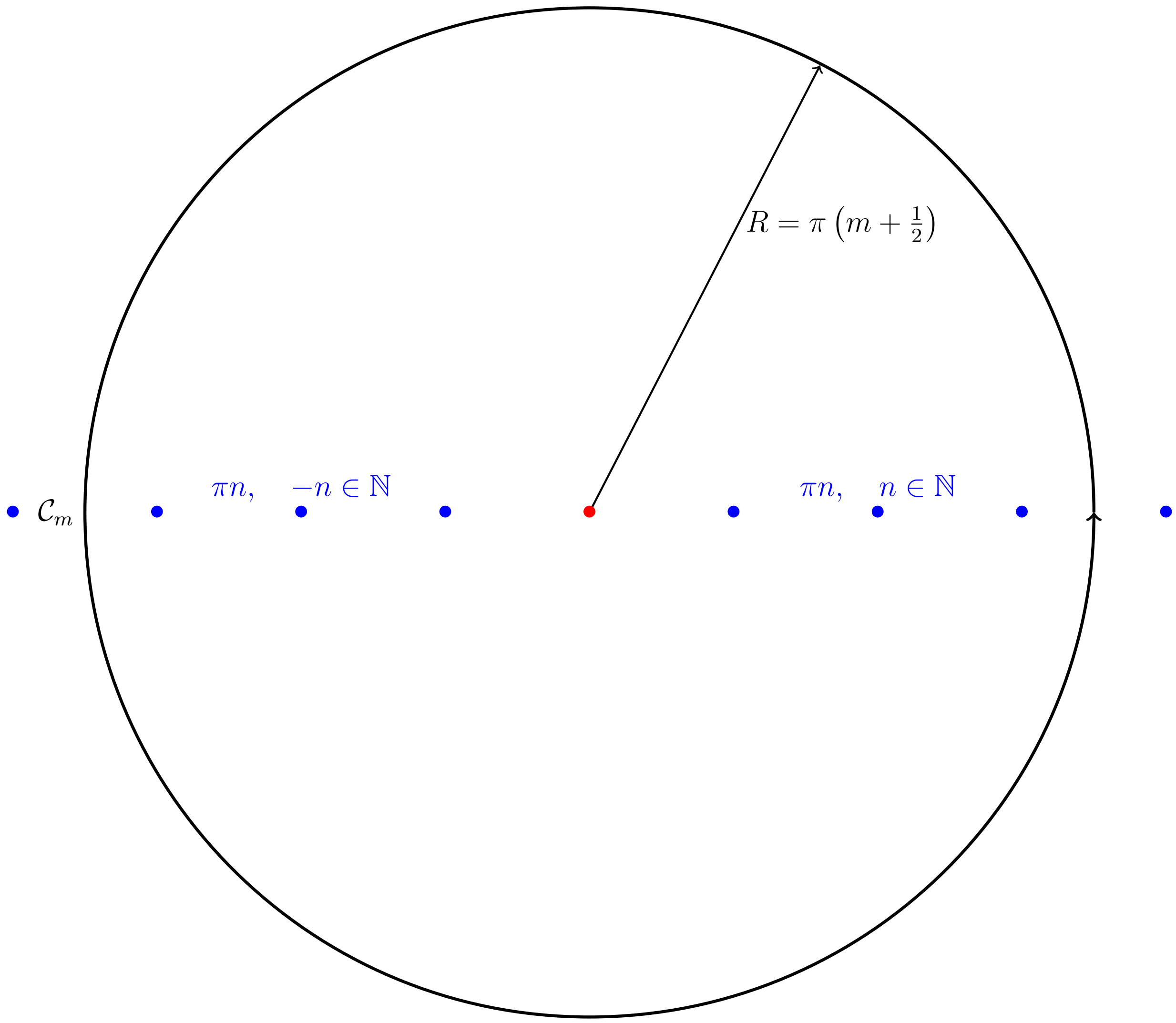


Figure 1: Contours for Mittag-Leffler theorem. Blue dots represent poles of the function $\cot z - \frac{1}{z}$.

We first show that function $f(z) = \cot z = \frac{\cos z}{\sin z}$ is bounded on a system of contours shown on a figure 1. On this contour we have:

$$
\begin{aligned}
|f(z)| &= \left|\frac{\cos z}{\sin z}\right| = \left\langle \begin{array}{l} z = Re^{i\varphi}, \\ R = \pi\left(m+\frac{1}{2}\right) \end{array} \right\rangle = \left|\frac{\cos(Re^{i\varphi})}{\sin(Re^{i\varphi})}\right| = \left|\frac{\cos(R\cos\varphi + iR\sin\varphi)}{\sin(R\cos\varphi + iR\sin\varphi)}\right| = \\
&= \left|\frac{\cos(R\cos\varphi)\cos(iR\sin\varphi) - \sin(R\cos\varphi)\sin(iR\sin\varphi)}{\sin(R\cos\varphi)\cos(iR\sin\varphi) + \sin(iR\sin\varphi)\cos(R\cos\varphi)}\right| = \left\langle \begin{array}{l} \sin(iz) = i\sinh(z), \\ \cos(iz) = \cosh(z) \end{array} \right\rangle = \\
&= \left|\frac{\cos(R\cos\varphi)\cosh(R\sin\varphi) - i\sin(R\cos\varphi)\sinh(R\sin\varphi)}{\sin(R\cos\varphi)\cosh(R\sin\varphi) + i\sinh(R\sin\varphi)\cos(R\cos\varphi)}\right| = \\
&= \sqrt{\frac{\cos^2(R\cos\varphi)\cosh^2(R\sin\varphi) + \sin^2(R\cos\varphi)\sinh^2(R\sin\varphi)}{\sin^2(R\cos\varphi)\cosh^2(R\sin\varphi) + \cos^2(R\cos\varphi)\sinh^2(R\sin\varphi)}} = \left\langle \cosh^2 z = \sinh^2 z + 1 \right\rangle = \\
&= \sqrt{\frac{\sinh^2(R\sin\varphi) + \cos^2(R\cos\varphi)}{\sinh^2(R\sin\varphi) + \sin^2(R\cos\varphi)}}
\end{aligned}
$$

We can't have both $\sinh(R\sin\varphi)$ and $\sin(R\cos\varphi)$ to be equal to zero, because the only real zero of $\sinh z$ is 0, and if $R\sin\varphi$ equal to zero, then $R\cos\varphi = \pm R = \pm\pi\left(m+\frac{1}{2}\right)$. This means that our denominator is always nonzero. Now we will estimate from the top this expression.

$$|\cot z|_{\mathcal{C}_m} = \sqrt{\frac{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\sin\varphi\right)+\cos^2\left(\pi\left(m+\frac{1}{2}\right)\cos\varphi\right)}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\sin\varphi\right)+\sin^2\left(\pi\left(m+\frac{1}{2}\right)\cos\varphi\right)}} \leqslant$$
$$\leqslant \sqrt{\frac{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\sin\varphi\right)+1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\sin\varphi\right)+\sin^2\left(\pi\left(m+\frac{1}{2}\right)\cos\varphi\right)}} \leqslant \sqrt{\frac{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\right)+1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)\right)}} \leqslant 2.$$

Now we can apply Mittag-Leffler theorem. By the theorem our function can be decomposed into series

$$f(z) = h(z) + \sum_{k=1}^{+\infty}\Big(g_k(z) - h_k(z)\Big), \tag{1}$$

where $h(z)$ and $h_n(z)$ are polynomial of power no more than m, $|f(z)|_{\mathcal{C}_m} \leqslant A_m|z|^m$. As we saw previously, in our case $m = 0$. To decompose into elementary fractions our function $\cot z$ we must work with the function $\cot z - \frac{1}{z}$, because to compute coefficients

$$h(z) = \sum_{n=0}^{m}\frac{f^{(n)}(0)}{n!}\,z^n,$$
$$h_k(z) = \sum_{n=0}^{m}\frac{g_k^{(n)}(0)}{n!}\,z^n,$$

the function $f(z)$ should be holomorphic at the origin. Now let us obtain the decomposition. First we will evaluate principal parts $g_k(z)$. We have:

$$g_k(z) = \frac{a}{z-\pi k} + \frac{b}{z+\pi k}.$$

Coefficients a and b can be extracted from the residue theorem:

$$g_k(z) = \mathrm{Res}_{z=\pi k}\left[\cot z - \frac{1}{z}\right]\frac{1}{z-\pi k} + \mathrm{Res}_{z=-\pi k}\left[\cot z - \frac{1}{z}\right]\frac{1}{z+\pi k},$$

so

$$\mathrm{Res}_{z=\pi k}\left[\frac{z\cos z - \sin z}{z\sin z}\right] = \lim_{z\to\pi k}\frac{z\cos z - \sin z}{z\cos z} = 1,$$
$$\mathrm{Res}_{z=-\pi k}\left[\frac{z\cos z - 1}{z\sin z}\right] = \lim_{z\to-\pi k}\frac{z\cos z - \sin z}{z\cos z} = 1.$$

After this we can write our principal parts:

$$g_k(z) = \frac{1}{z-\pi k} + \frac{1}{z+\pi k}. \tag{2}$$

Next we evaluate corrections:

$$h(z) = \left(\cot z - \frac{1}{z}\right)\Bigg|_{z=0} = \left(\frac{z\cos z - \sin z}{z}\right)\Bigg|_{z=0} = \lim_{z\to 0}\frac{z - z + \mathcal{O}(z^3)}{z} = 0, \tag{3}$$

$$h_k(z) = \left(\frac{1}{z-\pi k} + \frac{1}{z+\pi k}\right)\Bigg|_{z=0} = \frac{1}{\pi k} + \frac{1}{\pi k}. \tag{4}$$

Combining together (1), (2), (3) and (4) we obtain:

$$\cot z - \frac{1}{z} = \sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[\frac{1}{z-\pi n} - \frac{1}{\pi n}\right].$$

Bound of ctg z

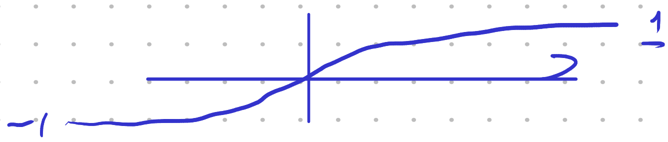
orh. ±1

$z = \pm(N+\frac{1}{2})\pi + iy$

$z = \pm iN + x$

$$ctg(\pm(N+\tfrac{1}{2})\pi + iy) = tg\, iy = th\, y$$

$$\frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$



$$ctg(iN\pi + x) = cth(N\pi + ix) = \frac{e^{N\pi + ix} + e^{-N\pi - ix}}{e^{N\pi + ix} - e^{-N\pi - ix}} \to 1$$

for $N \to \infty$ ...

Infinite products

$f(z)$ entire function (analyt. in $\mathbb{C}$)

$a_1, a_2, \dots a_n \dots$ – zeros

ex. $f(z)$ – polynom. $\Rightarrow f(z) = A(z-a_1)(z-a_2)\dots(z-a_N)$

cruc. idea log – derivative

$$G(z) = \frac{d \log f(z)}{dz} = \frac{f'(z)}{f(z)}$$

$G(z)$ has only simple poles corresp. to zeros or poles of $f(z)$

$f(z) = (z-a)^n \cdot g(z)$ $\quad g(a) \neq 0$

$$\frac{d \log f(z)}{f(z)} = \frac{((z-a)^n)'}{(z-a)^n} \cdot g(z) + \dots \cdot \frac{g'(z)}{f(z)} = \frac{n}{z-a} \cdot g(z) + \underset{\text{at } a}{\text{regular}}$$

$$\frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}$$

$\frac{f'}{f}$ has simple pole with Res $= n =$ order of zero

Th. f is entire with zeros $a_1 \dots a_n \dots$
and $\exists$ seq of contours $C_n \to \infty$ $\quad G(z) = \left|\frac{f'(z)}{f(z)}\right| < A|z|^p$

$$\Rightarrow f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\cdot e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots\frac{1}{p+1}\left(\frac{z}{a_n}\right)^{p+1}} \qquad |f| \not< e^{z^p}$$

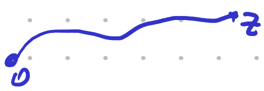
$g(z)$ polyn. of degree $p+1$

Proof. Mittag-L. for $G(z) = \frac{f'(z)}{f(z)}$

$$G(z) = h(z) + \sum_{n=1}^{\infty}\big(\underbrace{g_n(z)}_{\text{singular part}} - h_n(z)\big)$$

$$G(z)_{z\to a_n} = \frac{m_n}{z-a_n} = g_n(z) \qquad h_n(z) = -\frac{m_n}{a_n\left(1-\frac{z}{a_n}\right)} = \underbrace{-\frac{m_n}{a_n}\left(1+\frac{z}{a_n}+\frac{z^2}{a_n^2}+\dots\right)}_{\text{Taylor } g_n(z) \text{ in } z}$$

$$G(z) = h(z) + \sum_{n=1}^{\infty}\left(\frac{m_n}{z-a_n} + \frac{m_n}{a_n}\left(1+\frac{z}{a_n}+\dots\frac{z^p}{a_n^p}\right)\right) = (\log f(z))'$$



$$\log f(z) = \int_0^z G(z)\,dz = \underbrace{\int_0^z h(z)}_{= g(z)} + \sum_{n=1}^{\infty} m_n\Big(\underbrace{\log\left(1-\frac{z}{a_n}\right)}_{= \frac{\log(z-a_n)}{\log(-a_n)}} +$$

$$+ m_n\left(\frac{z}{a_n}+\frac{z^2}{2a_n^2}+\dots\frac{z^{p+1}}{(p+1)a_n^{p+1}}\right)\Big)$$

$$f(z) = z^m e^{g(z)} \cdot \prod_n \left(1-\frac{z}{a_n}\right)\cdot \exp\left(\frac{z}{a_n}+\dots+\frac{z^{p+1}}{(p+1)a_n^{p+1}}\right)$$

Weierstraß $\quad p=0 \quad \frac{\Gamma'(z)}{\Gamma(z)}$ is bounded on contours

Mellin transform and asymp. expansion of $\Gamma(z)$

<u>Mellin</u>: $\check{\varphi}(s) = \int_0^{\infty} \varphi(t)\, t^{s-1}\, dt$ $\qquad$ Mell. of exponent $\varphi(t) = e^{-t} \Rightarrow \check{\varphi}(s) = \Gamma(s)$

Properties: $|\varphi(t)| < t^{b}$ $\quad t \to +\infty$
$|\varphi(t)| < t^{a}$ $\quad t \to 0$ $\quad$ and integrable

$\Rightarrow$ $\check{\varphi}(s)$ is analytical $-a < \mathrm{Re}\, s < -b$

because $-a < \mathrm{Re}\, s \Rightarrow$ converges at 0

$\mathrm{Re}\, s < -b \Rightarrow$ converges at ∞

$$\varphi(t) = \sum_{n=1}^{\infty} \psi(nt) \Rightarrow \check{\varphi}(s) = \check{\psi}(s)\cdot \zeta(s)$$

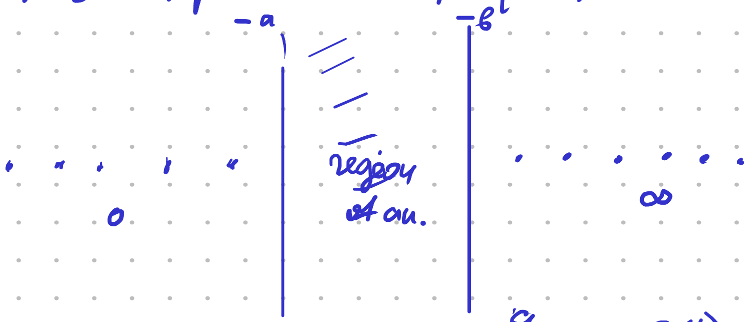
$$\varphi(t) = \sum_{n=1}^{\infty} \psi\left(\frac{t}{n}\right) \Rightarrow \check{\varphi}(s) = \check{\psi}(s)\cdot \zeta(-s)$$

$$\check{\varphi}(s) = \sum_{n=1}^{\infty} \int_0^{\infty} \psi(nt)\, t^s \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^{\infty} \psi(\tau)\left(\frac{\tau}{n}\right)^s \frac{d\tau}{\tau} = \sum_{n=1}^{\infty} \frac{1}{n^s}\cdot \int_0^{\infty} \psi(\tau)\tau^s \frac{d\tau}{\tau}$$

$nt = \tau$

Asymp. expansion of $\varphi(t)$ at 0 $\xrightarrow{\text{translates}}$ the poles of $\check{\varphi}(s)$ in left half plane

As. expansion of $\varphi(t)$ at ∞ $\to$ poles of $\check{\varphi}(s)$ in right half plane



example $\varphi(t) = c\, t^a + O(t^{a+1}) \quad t \to 0$

$$\check{\varphi}(s) = \int_0^1 \varphi(t) t^{s-1} dt + \int_1^{\infty} \varphi(t) t^{s-1} dt = I_1 + I_2$$

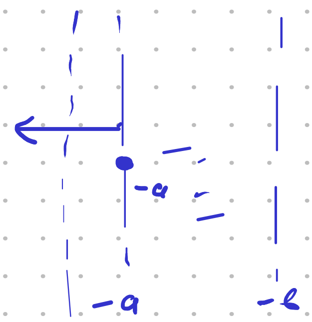
converges

t^{a+s-1}

$s > -a$

$$I_1 = \int_0^1 (\varphi(t) - ct^a)\, t^{s-1} dt + c\int_0^1 t^{a+s-1} dt$$

$s > -a-1$ $\qquad \int_0^1 t^{a+s-1}dt = \frac{c}{a+s}$



$\mathrm{Res} = c$

Resume If $\check{\varphi}(s)$ is analyt. in the Region $-a < \mathrm{Re}\, s < -b$ and has simple poles $-a, -a-1, -a-2, \ldots$
$c_1 \quad c_2 \quad c_3 \quad \ldots$

has simple poles $\overset{d_1}{-b} \quad \overset{d_2}{-b+1} \quad \ldots$

$$\check{\varphi}(s) \simeq \text{entire} + \left(\sum \frac{c_0}{s+a} + \frac{c_1}{s+a+1} + \cdots\right) + \sum\left(\frac{d_0}{s+b} + \frac{d_1}{s+b-1} + \cdots\right) \Longrightarrow$$

$\varphi(t)$ has as. exp. at 0 $\sim -c_k t^{a+k}$

as. exp. at ∞ $\sim -d_k t^{b+k}$