$$F_{2,1}(a,b;c;z) = (1-z)^{-a} F_{21}\left(a, c-b, c, \frac{z}{z-1}\right)$$

$$\| $$

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt =$$

$$z = \frac{y}{y-1}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1 - \frac{ty}{y-1}\right)^{-a} dt =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(\frac{-y+1+ty}{1-y}\right)^{-a} dt \quad \left(\frac{1-(1-t)y}{(1-y)}\right)^{-a}$$

$$F\left(a,b;c;\frac{y}{y-1}\right) = (1-y)^{+a} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\cdot(1-\underbrace{(1-t)}y)^{-a}\,dt$$

$$1-t = s$$

$$= (1-y)^a \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{c-b-1}(1-s)^{b-1}(1-sy)^{-a}\,ds =$$

$$b \qquad c-b$$

$$= (1-y)^a F(a, c-b, c, y)$$

$$F\left(a,b;c;\frac{y}{y-1}\right) = (1-y)^a F(a, c-b; c; y)$$

$$F(a, c-b; c; y) = (1-y)^{-a} F\left(a, b; c; \frac{y}{y-1}\right)$$ 17 φрафар

Differ. equations with regular singularities.

$\bar{y}' = A(z)y$ system of linear dif. equations

$z \in \mathbb{C}$ time is complex $A(z)$ is holom. function
complex derivative

Start from $z \in \mathbb{R}$ $\longrightarrow$ solution is analytical
can extend to complex an. function

z_0 is a singular point of eq. z_0 is a pole of ess. sing. point of $A(z)$

Regular singular point.

Dfn. The solution of d. eq. around z_0 growth not faster than a power of z

$$y = \sum \left((z-z_0)^{\lambda_i} \log^{\mu_i}(z-z_0) \right) u_i(z)$$

u_i are anal. at z_0

n=1. $y' = \varphi(z)\cdot y$ $\quad z=0 \quad$ $\varphi(z) = z^{k}\cdot u(z)$ holom. at 0

$$\frac{dy}{dz} = \frac{\lambda}{z} y$$

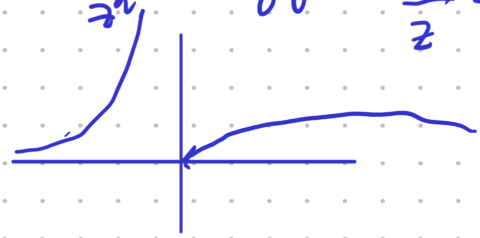
a) $\varphi(z) = \frac{\lambda}{z}\cdot u(z)$ holom

$$\frac{dy}{y} = \lambda \frac{dz}{z} \qquad \ln|y| = \lambda \log z \qquad \underline{y = Cz^{\lambda}}$$

$$\frac{dy}{dz} = \frac{\lambda y}{z^2} \qquad \frac{dy}{y} = \lambda \frac{dz}{z^2} \qquad \log y = -\frac{\lambda}{z} + C$$

$$y = C\cdot \underline{e}^{-\frac{\lambda}{z}}$$

ess. singul-ty



singular point is irregular.

$y' = \varphi(z) y$ $\quad$ 0 regul. sing. point $\iff$ $\varphi(z)$ has pole of order 1

$\bar{y}' = \frac{A(z)}{z}\bar{y}$ $\quad$ system has a reg. sing. point at 0 if $A(z)$ is holom. at 0.

Question. $y' = \varphi(z) y$. $\quad n=1.$ $\quad \bar{\mathbb{C}}^1$

$$y' = y \qquad \infty \qquad \mathbb{C}$$

$$w = \frac{1}{z} \qquad \frac{dy}{dz} = \frac{dy}{dw}\cdot\frac{dw}{dz} = \frac{dy}{dw}\cdot\left(-\frac{1}{z^2}\right) = \frac{dy}{dw}\cdot(-w^2)$$

$$-w^2\frac{dy}{dw} = y \qquad \frac{dy}{dw} = -w^{-2} dy$$

∞ is a irr. singular point

$$y = Ce^{z} \qquad y_z' = \frac{\lambda y}{z} \Rightarrow_{w=\frac{1}{z}} \frac{dy}{dw} = -\frac{\lambda}{w} y$$

0, ∞ are regular sing. points.

$(z-a)^\lambda$ $\quad F_{1,0}(\lambda; z-a)$

$y'' + a(z)y' + b(z)y = 0$ $\quad z_0.$ $\quad$ when z_0 is a sing. regular point of eq.

Statement $\quad z_0$ ~ is regular s.p. if
$a(z)$ has pole of order ≤ 1
$b(z)$ has pole of order ≤ 2

$$\begin{cases} y_1' = y_2 \\ y_2' = -a(z)y_2 - b(z)y_1 \end{cases}$$ $\quad y_2 = y' = y_1$

$$\dot{\bar{y}} = \begin{pmatrix} 0 & 1 \\ -b(z) & -a(z) \end{pmatrix} \bar{y}$$

$$y_2 = \frac{\tilde{y}_2}{z-z_0} \qquad \begin{cases} y_1' = \dfrac{\tilde{y}_2}{z-z_0} \\ \dfrac{\tilde{y}_2'}{z-z_0} = \dfrac{\tilde{y}_2}{(z-z_0)^2} - \dfrac{a(z)\tilde{y}_2}{z-z_0} - b(z)y_1 \end{cases}$$

$$y_2' = \frac{\tilde{y}_2'}{z-z_0} - \frac{\tilde{y}_2}{(z-z_0)^2}$$

$$\begin{cases} y_1' = \dfrac{\tilde{y}_2}{z-z_0} \\ \tilde{y}_2' = \tilde{y}_2\left(\dfrac{1}{z-z_0} - a(z)\right) - (z-z_0)\, b(z) y_1 \end{cases}$$

$a(z)$ should have a pole of order 1
$b(z)$ has pole of order 2.

Now let z_0 be a ~~regular~~ sing. point of eq.

$y'' + a(z)y' + b(z)y = 0$ $\qquad z_0 = 0$

we a looking for sol. of the form

$$y = z^\lambda \cdot \underbrace{(1 + u_1 z + u_2 z^2 + \dots)}_{u(z)}$$

$$a(z) = \frac{p(z)}{z} \qquad b(z) = \frac{q(z)}{z^2} \qquad \begin{aligned} p(z) &= p_0 + p_1 z \dots \\ q(z) &= q_0 + q_1 z + \dots \end{aligned}$$

$y = z^\lambda \cdot u(z)$

$$\lambda(\lambda-1) z^{\lambda-2} u(z) + 2\lambda z^{\lambda-1} u'(z) + z^\lambda u''(z) + \lambda z^{\lambda-2} \underbrace{p(z)u(z)}_{(p_0+p_1 u)} + z^{\lambda-1} p(z) u'(z)$$
$$+ q(z) z^{\lambda-2} u(z) = 0$$

$$\underset{\underline{\underline{0}}}{z^{\lambda-2}} \quad \underset{0}{z^{\lambda-1}} \quad \underset{0}{z^\lambda} \quad z^{\lambda+1} + \dots$$

$u(z) = 1 + u_1 z + \dots$
$p(z) = p_0 + p_1 z + \dots$

$z^{\lambda-2}$ | $\lambda(\lambda-1) + \lambda p_0 + q_0 = 0$ quadr. eq. on λ λ_1 λ_2

$P(\lambda)=0$

$z^{\lambda-1}$ | $\lambda(\lambda-1)\cdot u_1 + 2\lambda u_1 + (p_0\lambda u_1 + p_1) + p_0 + q_0 u_1 + q_1 = 0$

z^{λ} $\left(\lambda(\lambda+1)\cdots + (\lambda+1)p_0 + q_0\right)u_1 + p_1 + p_0 + q_1 = 0$

$\lambda = \lambda_1$

$$u_1 = \frac{p_1 + p_0 + q_1}{P(\lambda+1)}$$

$P(\lambda_1 + n) \neq 0$

$\forall n \geq 1$

$z^{\lambda+n}$ $u_n = \dfrac{\text{previous}}{P(\lambda+n)}$ Then we have unique solution.

$P(\lambda)=0$ character. eq. λ_1 and λ_2 – exponent of sing. point

if $\lambda_1 - \lambda_2 \notin \mathbb{Z}$ $\Rightarrow$ we have two indep. solutions converge around 0.

$y_1 = (z-z_1)^{\lambda_1} u(z)$ $\quad y_2 = (z-z_0)^{\lambda_2} v(z)$

If $\lambda_1 - \lambda_2 \in \mathbb{Z}$

$y_1 = (z-z_1)^{\lambda_1} u(z)$ $\quad y_2 = (z-z_1)^{\lambda_1} \cdot \log(z-z_1) \cdot v(z)$

Hypergeom. equation

$$z(1-z)y'' + (c-(a+b+1)z)y' - ab\,y = 0$$

sing. points: $0, 1, \infty$

0 $\quad y'' + \dfrac{(c-(a+b+1)z)}{z(1-z)}y' - \dfrac{ab\,y}{z(1-z)} = 0$

$y'' + \dfrac{p(z)}{z}y' + \dfrac{q(z)}{z^2}y = 0$ $\quad p(z) = \dfrac{(c-(a+b+1)z)}{1-z}$

$$q(z) = -\frac{abz}{1-z}$$

$p(0)=c$ $\quad q(0)=0$ $\quad \lambda(\lambda-1) + c\lambda = 0$

$\lambda(\lambda + c - 1) = 0$

exponents are 0, $1-c$

$\Rightarrow$ we have two disting. solutions

$$1 + u_1 z + \cdots \qquad\qquad z^{1-c}\left(1 + v_1 z + \cdots\right)$$
$$\shortparallel \qquad\qquad\qquad \shortparallel\ ?$$
$$F(a, b; c; z)$$

$z_0 = 1$
$$y'' + \frac{(c - (a+b+1)z)}{z(1-z)} y' - \frac{ab\, y}{z(1-z)} = 0$$

$$y'' + \frac{p(z)}{z-1} y' + \frac{q(z)}{(z-1)^2} y = 0 \qquad p(z) = \frac{(a+b+1)z - c}{z} \qquad p(1) = a+b+1-c$$

$$q(z) = \frac{(z-1)ab}{z} \qquad q(1) = 0$$

$$\lambda(\lambda - \cancel{1}) + (a+b+\cancel{1}-c)\lambda = 0 \qquad \lambda_1 = 0$$
$$\lambda_2 = c - a - b$$

∞ $\quad w = \frac{1}{z}$
$$\frac{dy}{dz} = -w^2 \frac{dy}{dw} \qquad \frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) =$$

$$= -\frac{w^2 d}{dw}\left(-w^2 \frac{dy}{dw}\right) = w^4 \frac{d^2y}{dw^2} + 2w^3 \frac{dy}{dw}$$

$$\frac{1}{w}\left(1 - \frac{1}{w}\right)\left(w^4 y''_{ww} + 2w^3 y'\right) + \left(c - \frac{(a+b+1)}{w}\right)\left(-w^2 y'\right) - aby = 0$$

$$y'' + \frac{(2-c)w + (a+b-1)}{w(w-1)} y' - \frac{ab}{w^2(w-1)} y = 0$$

$z = \infty \qquad w = 0$
$$p(w) = \frac{(2-c)w + (a+b-1)}{w-1} \qquad q(w) = -\frac{ab}{w-1}$$

$$p(0) = 1 - a - b \qquad q(0) = ab$$

$$\lambda(\lambda - \cancel{1}) + (\cancel{1} - a - b)\lambda + ab = 0$$
$$\lambda^2 - (a+b)\lambda + ab = 0 \qquad \lambda_1 = a, \quad \lambda_2 = b$$

$\Rightarrow$ for Big z
$$y_1 = z^{-a}\left(1 + \frac{a_1}{z} + \cdots\right) \quad \text{two solut.}$$
$$y_2 = z^{-b}\left(1 + \frac{b_1}{z} + \cdots\right)$$

$z(1-z)y'' + (c+(a+b-1)z)y' - ab\,y = 0$ 3 регул. особые точки

$0, 1, \infty$

0	1	∞
$0, 1-c$	$0, c-a-b$	a, b

$n=1$ 2 рег. особ. точки

$n=2$ 3 особые точки

<u>Theorem</u> (Papperitz) For any $a, b, c \in S^2 = \overline{\mathbb{C}}$

and exponents $\alpha_1 \alpha_2$ a; β_1, β_2 b; γ_1, γ_2 c there exists unique d.e. with sing. points a, b, c and exponents

$$y'' + \left\{ \frac{1-\alpha_1-\alpha_2}{x-a} + \frac{1-\beta_1-\beta_2}{x-b} + \frac{1-\gamma_1-\gamma_2}{x-c} \right\} y' +$$
$$+ \left\{ \frac{(a-b)(a-c)\alpha_1\alpha_2}{x-a} + \frac{(b-a)(b-c)\beta_1\beta_2}{x-b} + \frac{(c-a)(c-b)\gamma_1\gamma_2}{x-c} \right\} \cdot \frac{y}{(x-a)(x-b)(x-c)} = 0$$

Riemann equation   Hypergeom. eq. particular case of Riemann eq. $a=0$, $b=1$, $c=\infty$

Riemann scheme: $P\left\{ \begin{matrix} a & b & c \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \middle| z \right\}$ 1) equation 2) set of specific local solution $(z-a)^{\alpha_1} \cdot (1 + \dots)$

conformal maps $t = \dfrac{\lambda z + \mu}{\delta z + \nu}$

$a, b, c \to a', b', c'$

multiply solutions by $(z-a)^{\delta_1}(z-b)^{\delta_2} \cdot (z-c)^{\delta_3}$ if $\delta_1 + \delta_2 + \delta_3 = 0$

$(z-a_1)^{\alpha_1} u_1(z) \to (z-a_1)^{\alpha_1 + \delta_1},$

$(z-b)^{\beta_1} \to (z-b)^{\beta_1 + \delta_2}$

if ∞ was regular $\Rightarrow$ it still regular

$$(x-a)^{\delta_1}(x-b)^{\delta_2}(x-c)^{\delta_3} P\left(\begin{matrix} a & b & c \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \middle| z \right) = P\left(\begin{matrix} a & b & c \\ \alpha_1+\delta_1 & \beta_1+\delta_2 & \gamma_1+\delta_3 \\ \alpha_2+\delta_1 & \beta_2+\delta_2 & \gamma_2+\delta_3 \end{matrix} \right)$$

$\delta_1 + \delta_2 + \delta_3 = 0$

<u>ex.</u> Hypergeom. equation: $P\left(\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{matrix} \middle| z \right)$

$$z^{\delta_1}(z-1)^{\delta_2} P\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{pmatrix} = P\begin{pmatrix} 0 & 1 & \infty \\ \delta_1 & \delta_2 & a-\delta_1-\delta_2 \\ 1-c+\delta_1 & c-a-b-\delta_2 & b-\delta_1-\delta_2 \end{pmatrix}$$

$z^{\delta_1},\ \frac{1}{z}(z-\infty)^{\delta_3}$

Task: For hypergeom. eq-n find out precisely all 6 special solutions of h. eq.

$$z^{c-1} P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{pmatrix} = P\left(\begin{matrix} 0 & 1 & \infty \\ c-1 & 0 & a+1-c \\ 0 & c-a-b & b+1-c \end{matrix}\ \middle|\ z\right)$$

$$= P\left(\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a+1-c \\ c-1 & c-a-b & b+1-c \end{matrix}\ \middle|\ z\right)$$

$c-1 = 1-c'$ $\quad c' = 2-c$

$a+1-c = a'$ $\quad a' = a-c+1$

$b+1-c = b'$ $\quad b' = b-c+1$

$$\| $$

$$P\left(\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a' \\ 1-c' & c'-a'-a' & b' \end{matrix}\ \middle|\ z\right)$$ – hypergeom. eq-n has unique sol. regular at 0 $=1$ $\quad F(a',b',c',z)$

$z^{c-1}F(a,b;c,z)$ $\qquad z^{1-c}\cdot(1+\ldots)\cdot z^{c-1}$

The solution is $\quad z^{1-c}F(a-c+1, b-c+1, 2-c, z)$

$y_2 \cdot z^{c-1} = F(a-c+1, b-c+1, 2-c; z)$

$\Rightarrow y_2 = z^{1-c}F(a-c+1, b-c+1, 2-c; z)$

$z_0 = 1$ $\qquad x \to 1-x$ $\qquad 0\to 1$, $1 \to 0$, $\infty \to \infty$

$$P\left(\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{matrix}\ \middle|\ x\right) \to P\left(\begin{matrix} 1 & 0 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{matrix}\ \middle|\ 1-x\right) \quad // \quad P\left(\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & a \\ c-a-b & 1-c & b \end{matrix}\ \middle|\ 1-x\right)$$

$1-c = c'-a'-b'$, $c-a-b = 1-c'$ $\qquad a' = a$, $b' = b$

$c' = a+b-c+1$

1) $F(a,b,1+a+b-c; 1-x)$ is regular at 1

2) $(1-x)^{1-c'}F(a',b',c';1-x) = (1-x)^{c-a-b}F(c-b, c-a, c+1-a-b; 1-x)$

these are local solutions at $x=1$

$0 \Leftrightarrow \infty$ $\qquad x \to \frac{1}{x}$

3) ∞

$$P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{pmatrix} = P\begin{pmatrix} 0 & 1 & \infty & \\ a & 0 & 0 & \frac{1}{x} \\ b & c-a-b & 1-c & \end{pmatrix} =$$

$$= x^{-a} P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{1}{x} \\ b-a & c-a-b & 1-c+a & \end{pmatrix}$$

$a \to a$
$b \to 1-c+a$

$(-x)^{-a} F(a, a+1-c, a+1-b; \frac{1}{x})$ solut. with asympt. $\left(\frac{1}{x}\right)^{\infty}$ at ∞

$(-x)^{-b} F(b, b+1-c, b+1-a; \frac{1}{x})$

$y_1 \sim F$
$y_2 \quad z^{1-c}\ldots$ $\Big\}$ basis of sol. cor. to $z \to 0$

y_3
y_4 $\quad z \to 1$

y_5
y_6 $\quad z \to \infty$

Find out transition matrices.