$P_{\alpha,\beta}(x)$ Jacoby polynomials

orth. on $[-1,1]$ $\mu=(1-x)^{\alpha}(1+x)^{\beta}$

$$P_{\alpha,\beta}(x)=\frac{(\alpha+1)_n}{n!}F\left(-n,\, n+\alpha+\beta+1,\, \alpha+1,\, \frac{1-x}{2}\right) \qquad [0\ 1] \to [-1,1]$$

They are eigenvectors of $L=(1-x^2)\frac{\partial^2}{\partial x^2}+\left[(\beta-\alpha)-(\alpha+\beta+2)x\right]\frac{d}{dx}$

with eigenvalue $\lambda_n=-n(n+\alpha+\beta+1)$

$L P_n^{\alpha,\beta}=\lambda_n P_n^{\alpha,\beta}$ $\Longleftrightarrow$ hypergeom. equation

$\alpha=\beta=\frac{1}{2}$ Chebyshev pol. I $\cos(n.\arccos x)$

$\alpha=\beta=-\frac{1}{2}$ Chebyshev II kind

$\alpha=\beta$ Gegenbauer

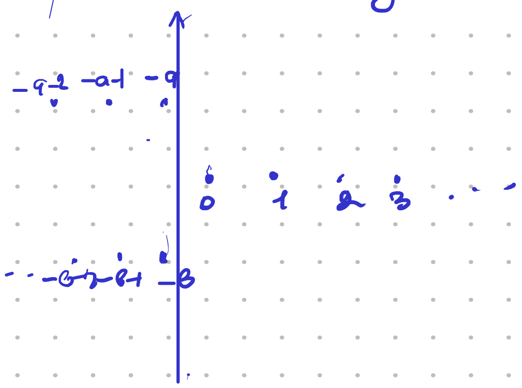
Mellin - Barnes presentation of hypergeom. function

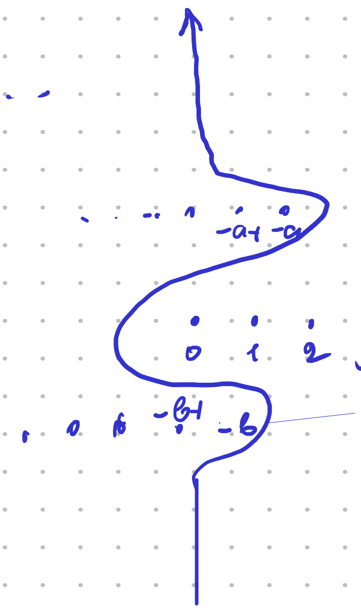
Barnes 1874-1953

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty\ C}^{+i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\,ds \qquad |\arg z|<\pi-\varepsilon$$

poles of integrand

$a+s=-n_1$
$b+s=-n_2$
$-s=-n_3$





Mellin transform

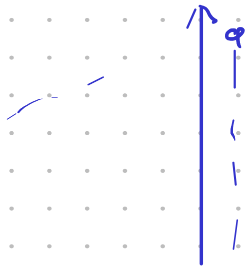
$f(t)$ $t>0$

$$\check{f}(s)=\int_0^\infty f(x)\,x^{s-1}\,dx \qquad \text{If } |f(x)|<Ce^{ax}$$

converges for $\operatorname{Re} s>a$ $\Rightarrow$ $\hat{f}(s)$ is analytical in half plane $\operatorname{Re} s>a$

Inversion formula

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \check{f}(s)\, x^{-s}\, ds \qquad c > a$$



$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \qquad \Gamma(s) = \text{Mellin transform of } e^{-t}$$

inversion formula:

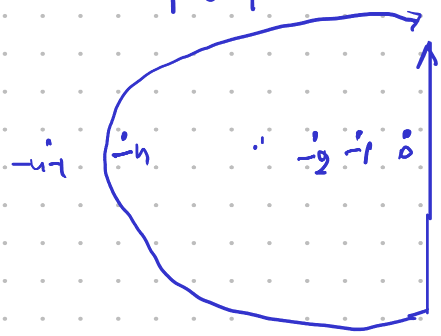
$$e^{-t} \cdot \chi_{[0,\infty]} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) t^{-s}\, ds = \sum_{0,-1,\dots} \mathrm{Res}\, \Gamma(s) t^{-s} = \sum_{n=0,1,\dots} \frac{(-1)^n}{n!} t^n = \exp(-t)$$

$\boxed{t>0}$

$$\chi_{[0,\infty]}(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

1) converges?

$$\Gamma(is) \sim e^{-\frac{\pi|x|}{2}} \cdot |x|^{-1/2}$$



$$|\Gamma(is)| = |\Gamma(is)\Gamma(-is)|^{1/2} =$$

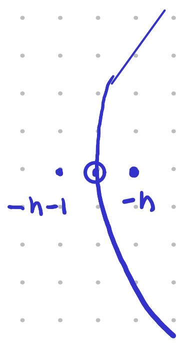
$$\Gamma(is)\Gamma(1-is) = \frac{\pi}{\sin \pi s}$$

$$\Gamma(1-is) = is \cdot \Gamma(is)$$

$$|\Gamma(is)|\, s^{1/2} = \left(\frac{\pi}{e^{\pi s} - e^{-\pi s}}\right)^{1/2} \sim \sqrt{\pi}\, e^{-\frac{\pi s}{2}}$$

$$|t^{is}| = |e^{is \log t}| = 1$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$



$$\Gamma\left(-n+\tfrac{1}{2}\right) = \frac{\pi}{\Gamma\left(n+\frac{1}{2}\right) \cdot \sin \pi\left(n+\frac{1}{2}\right)}, \qquad \sin \pi\left(n+\tfrac{1}{2}\right) = \pm 1$$

$$\Gamma\left(-n+\tfrac{1}{2}\right) \sim \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \qquad \left(n+\tfrac{1}{2}\right)^{n+\frac{1}{2}}$$

$n!$

Beta integral

1) $$\int_0^1 x^{s-1}(1-x)^{t-1}\, dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

$$\int_0^1 x^{s-1} \cdot f(x)\,dx \quad , \text{where} \quad f(x) = (1-x)^{t-1}\cdot \chi_{[0,1]}$$

Inversion formula:

2) $$\int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$(1-x)^{t-1}\cdot\chi_{[0,1]} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(s)\,\Gamma(t)}{\Gamma(s+t)} \cdot x^{-s}\, ds$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{\infty}\frac{\Gamma(s)}{\Gamma(s+t)}x^{-s} = \frac{1}{\Gamma(t)}(1-x)^{t-1}\cdot\chi_{[0,1]}$$

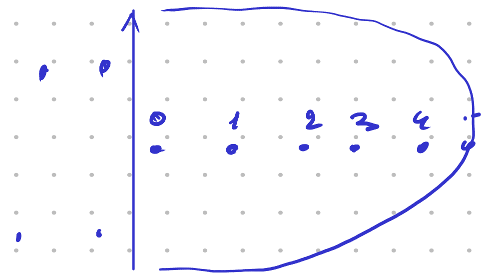
$$s=0,-1,-2,\dots \quad \frac{1}{\Gamma(t)}\sum \frac{(-1)^n\,\Gamma(t)}{n!\;\Gamma(t-n)} = \frac{1}{\Gamma(t)}\sum_{n\ge0} x^n\frac{(-1)^n}{n!}(t-1)(t-2)\dots(t-n) = \frac{1}{\Gamma(t)}\sum_{n\ge0}\cancel{\frac{(1-t)\dots(n+t)}{n!}x^n}$$

$$\Gamma(t-1) = \frac{\Gamma(t)}{t-1} \qquad \frac{1}{\Gamma(t-1)} = \frac{(t-1)}{\Gamma(t)} \qquad = \frac{1}{\Gamma(t)}(1-x)^{t-1}$$

Proof of Barnes theorem.

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}F(a,b;c;x) = \frac{1}{2\pi i}\int_{-i\infty+c}^{+i\infty+c}\frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(c+s)}(-x)^s\, ds$$

Residues:



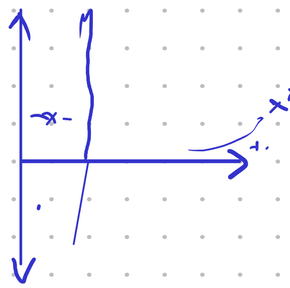
$s=0,1,\dots$

$$\frac{\Gamma(0)}{\Gamma(a)\Gamma(b)}\sum_{n\ge0}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)}\frac{(-1)^n}{n!}(-x)^n = \frac{\frac{\Gamma(a+n)}{\Gamma(a)}\cdot\frac{\Gamma(b+n)}{\Gamma(b)}}{\frac{\Gamma(c+n)}{\Gamma(c)}}\,\frac{x^n}{n!} = \frac{(a)_n\,(b)_n}{(c)_n\, n!}x^n$$

Diff. part.

1) convergence. $\Gamma(x+iy) \sim \sqrt{2\pi}\,|y|^{x-1/2}e^{-\pi|y|/2}$

Stirling formula.

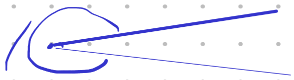


Integrand $s \to \pm i\infty$ $\qquad \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-x)^s$

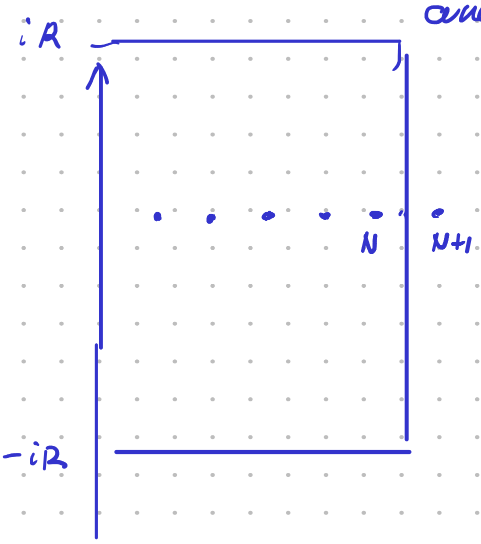
$$\frac{|s|^{a-1/2}|s|^{b-1/2}\cdot|s|^{-1/2}\, e^{-\frac{\pi|s|}{2}} e^{-\frac{\pi|s|}{2}} e^{-\frac{\pi|s|}{2}}}{|s|^{c-1/2}\, e^{-\frac{\pi|s|}{2}}}\; e^{i|s|(\log|x|+i\arg x)}$$

$$< C\cdot|s|^{a+b-c-1}\cdot e^{-\pi|s|}\cdot e^{\underbrace{|\arg(-x)|}_{\pi-\delta}\cdot|s|} < C\cdot|s|^{a+b+c-1}\cdot e^{-\delta|s|}$$

$|\arg(-x)| < \pi-\delta$

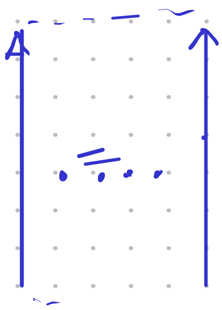


Conclusion integral presents function analytical in $|\arg x| > \delta$



step 1. Fix N

$\int I$ horizontal lines $\to 0$ when $R \to \infty$



$s = iR + \tau \qquad 0 < \tau < N + \frac{1}{2}$

$$I(s) = \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s)(-x)^s = \frac{\Gamma(a+s)}{\Gamma(c+s)}\,\frac{\Gamma(b+s)}{\Gamma(1+s)}\cdot\frac{(-x)^s}{\sin\pi s}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \qquad x\to\infty \quad |\arg x| < \pi - \delta$

$$\Rightarrow I(s) \sim \underbrace{|s|^{a-c}|s|^{b-1}}_{\|}\left(\frac{(-x)^s}{\sin\pi s}\right)$$

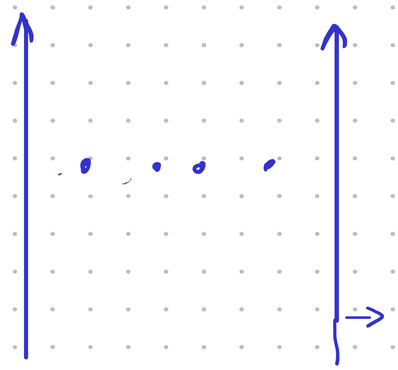
$$|s|^{a+b-c-1} = |iR+\tau|^{a+b-c-1} \sim R^{a+b-c-1}(1+O(R)).$$

$$|(-x)|^s = |e^{(iR+\tau)(\log x + i\arg(-x))}|\; e^{R\arg(-x)}\cdot x^{\tau}.$$

$|\sin \pi s| = |\sin(\pi(iR+\tau))| = |i\,\mathrm{sh}\,R\cos\tau + \mathrm{ch}\,R\cdot\sin\tau| > \mathrm{sh}\,R\underbrace{\sqrt{\cos^2\tau+\sin^2\tau}}_{1} >$

$\mathrm{ch}\,R > \mathrm{sh}\,R \qquad > \mathrm{sh}\,R > \varepsilon e^{\pi R},$

$$I < e^{R(\pi-\delta)}\cdot e^{-\pi R}\cdot \text{power}(R) = \underbrace{\text{power}(R)\cdot e^{-\delta R}}_{\to 0}$$



step 2 $\displaystyle\int_{N+\frac12-i\infty}^{N+\frac12+i\infty} I \Rightarrow 0 \qquad N\to\infty$

$$\left|\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)}\right| \underset{\text{Stirling}}{\sim} \left(N+\tfrac12+i\tau\right)^{a+b-c-1} = \left(N+\tfrac12\right)^{a+b-c-1}\cdot\left|1+\frac{i\tau}{N+\frac12}\right|^{a+b-c-1}$$

$\log|x|<-\alpha \qquad |x|<1$

$$|(-x)^s| = \left|e^{(N+\frac12+i\tau)(\log|x|+i\arg(-x))}\right| < \boxed{e^{-\alpha(N+\frac12)}}\cdot e^{(\pi-\delta)\cdot|\tau|}$$

$$\sin\pi\left((N+\tfrac12)+i\tau\right) = \cos i\tau = \mathrm{ch}\,\tau > e^{\pi\tau}$$

$$I < e^{-\delta|\tau|}\cdot \text{Power}(\tau)\cdot e^{-\alpha N}\cdot \text{Power}(N)$$

integral converges   tends to 0.

Proved for $|x|<1$. By analytical contin. argument is true without restriction $|x|<1$.

E. Rains   V. Spiridonov

Маринов.   Use convolution property of Mellin transform

Fourier transform

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx$$

$$f*g(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$$\widehat{f*g}(\lambda) = \hat{f}(\lambda)\cdot\hat{g}(\lambda)$$

$$\int_{-\infty}^{\infty} e^{-i\lambda x} f*g(x)dx = \int_{-\infty}^{\infty} e^{-i\lambda x} dx \int_{-\infty}^{\infty} f(y)g(x-y)dy =$$

$x, y \to x-y, y$

f, g are finit support

$$= \int_{-\infty}^{\infty} e^{-i\lambda(x-y)-i\lambda y} dx \int_{-\infty}^{\infty} f(y)g(x-y)dy =$$

$x-y \to z$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda z - i\lambda y} f(y)g(z)dz\,dy = \hat{f}(\lambda)\hat{g}(\lambda)$$

Part. case $\lambda=0$

$$\hat{f}*\hat{g}(0) = \int \hat{f}(\mu)\hat{g}(-\mu)d\mu = \int \hat{f}(\mu)\overline{\hat{\varphi}(\mu)}d\mu.$$

$$\frac{1}{\varphi_i}\quad \widehat{fg}(0) = \int f(x)g(\cancel{x})e^{i0x}dx = \int f(x)g(x)dx = \int f(x)\overline{\varphi}(x)dx$$

$\Downarrow$ $g(x) = \overline{\varphi(x)}$ $\qquad \hat{g}(-\mu) = \int e^{i\mu x}\overline{\varphi}(x)dx = \overline{\int e^{-i\mu x}\varphi(x)dx}$

Parceval $(\, , \,) \longrightarrow (\, , \,)$

Mellin transform

Fourier: $(\mathbb{R},+) \to (\mathbb{R},+)$
$e^{i\lambda x}$

Mellin:

$$\int f(x)e^{-i\lambda x}dx$$

$$\int f(t)t^{s-1}dt \qquad t=e^{x} \qquad dx \to \frac{dt}{t}$$

duality $(\mathbb{R}^+, \times) \to (\mathbb{C}, +)$
$\frac{dx}{x} \qquad dz$

$$f \dot{*} g(x) = \int_0^{\infty} f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}$$

$M(f\dot{*}g) = M(f)\cdot M(g)$ check it formally

$$M(fg) = \frac{1}{2\pi i} Mf * Mg$$

$$M(fg) = \int_0^{\infty} f(x)\left(\frac{1}{2\pi i}\int_{i\mathbb{R}+c} \check{g}(t)x^{-t}dt\right)x^{s}\frac{dx}{x} =$$

$$= \frac{1}{2\pi i}\int_{i\mathbb{R}+c} \check{g}(t)dt \int_0^{\infty} f(x)x^{-t+s}dx = \frac{1}{2\pi i}\int_{i\mathbb{R}+c} \check{g}(t)\check{f}(s-t)dt$$

Corollary $s = 0$

$$(f g)(0) = \int_0^\infty f(x) g(x) \frac{dx}{x} = \frac{1}{2\pi i} \int_{i\mathbb{R}+c} \check{f}(t)\, \check{g}(-t)\, dt$$

is used for calcul. of integrals.

$$g(t) = \overline{\varphi(t)}$$

$$(f, g) = \int f \bar{g} \frac{dx}{x} \qquad (\hat{f}, \hat{g}) = \frac{1}{2\pi i} \int f \bar{g}\, dt$$