$F(a,b;c;z)$ $\qquad z(1-z)\dfrac{d^2F}{dz^2}+(c-(a+b+1)z)\dfrac{dF}{dz}-abF=0$

regular singularities $\quad 0,\ 1,\ \infty$ $\qquad P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{pmatrix}$

1) $0 \qquad g(x) \qquad x^{1-c}g(x)$

2) $1 \qquad g(x-1) \qquad (x-1)^{c-a-b}\tilde{g}(x-1)$

3) $\qquad \left(\frac{1}{x}\right)^a \varphi\left(\frac{1}{x}\right) \qquad \left(\frac{1}{x}\right)^b \psi\left(\frac{1}{x}\right)$

1) $F(a,b;c;z)=y_1 \qquad y_2=z^{1-c}F(a-c+1,b-a+1,2-c;z)$

2) $F(a,b,a+b-c,1-x)=y_3 \qquad y_4=(1-x)^{c-a-b}F(c-b,c-a,c-a-b+1,1-x)$

3) $(-x)^{-a}F(b,b-c+1;b-a+1;\frac{1}{x})=y_5 \quad y_6=(-x)^{-b}F(a,a-c+1;a-b+1;\frac{1}{x})$

1 ———— ∞

ex. $F(a,b;c;z)=A\cdot(-x)^{-a}F(b,b-c+1;b-a+1;\frac{1}{x})+B(-x)^{-b}F(a,a-c+1,a-b+1;\frac{1}{x})$

$0 \quad 1$ ———— ∞ $\qquad \operatorname{Re} b>\operatorname{Re} a \qquad x\to\infty$

Asymptotics $\quad$ RHS $\quad A\cdot(-x)^{-a}$

Asymp. of LHS: $\quad$ Pfaff identity

$$F(a,b,c;x)=\underset{\displaystyle (-x)^{-a}}{\underset{\wr}{(1-x)^{-a}}}\ F(a,c-b,c;\underset{\downarrow 1}{\tfrac{x}{x-1}})$$

$$F(a,c-b,c;1) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}$$

$\operatorname{Re} b>\operatorname{Re} a$

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$A=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} \qquad B=\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}$$

$F(a,b;c;z)=A\cdot(-x)^{-a}F(b,b-c+1;b-a+1;\frac{1}{x})+B(-x)^{-b}F(a,a-c+1,a-b+1;\frac{1}{x})$

$x^{1-c}F(a-c+1,b-c+1,2-c,z)=C\cdot(-x)^{-a}F(b,b-c\ \ b-a+1;\frac{1}{x})+$
$\qquad +D(-x)^{-b}F(a,a-c+1,a-b+1;\frac{1}{x})$

$\|$

$x^{1-c}(1-x)^{-a+c-1}F(a-c+1,-b+1,2-c;\frac{x}{x-1})\sim \underset{\displaystyle (-x)^{1-c}\cdot e^{\pi i(1-c)}}{\overset{}{x^{1-c}}}(-x)^{-a+c-1}F(a-c+1,-b+1,2-c;1)$

$$= (1-x)^{-a} \cdot e^{\pi i(1-c)} \cdot \frac{\Gamma(2-c)\cdot\Gamma(b-a)}{\Gamma(b-c+1)\,\Gamma(1-a)} = C \qquad D \qquad a \leftrightarrow b$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Pfaff identity $\quad t = \frac{x}{x-1} \quad \begin{matrix} 0 \to 0 \\ 1 \to \infty \\ \infty \to 1 \end{matrix}$

$$F \quad P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & a & a & x \\ 1-c & c-a-b & b & \end{pmatrix} = P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & a & 0 & t=\frac{x}{x-1} \\ 1-c & b & c-a-b & \end{pmatrix} =$$

$$= (1-t)^{+a} = \left(1-\frac{x}{x-1}\right)^{+a} = (1-x)^{-a} \cdot P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & b-a & c-b & \end{pmatrix}$$

look for regular sol. at 0 with coeff 1.

$$F(a,b;c;x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right)$$

game with integral repr.

$$F = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{b} (1-t)^{c-b-1} (1-tz)^{-a} \frac{dt}{t}$$

check that it satisfies Gauss d. eq

$$t = \frac{1}{s} \qquad \int_1^\infty s^{-b} \left(1-\frac{1}{s}\right)^{c-b-1} \left(1-\frac{z}{s}\right)^{-a} \frac{ds}{s} =$$

$$I = \int_1^\infty \underline{(s-1)^{c-b-1} s^{a-c} (s-z)^{-a}} ds$$

$$L_z = z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab\cdot 1 \qquad L_z \cdot I = 0$$

$$L_z \cdot I = \int_1^\infty L_z \left\{ (s-1)^{c-b-1} s^{a-c} (s-z)^{-a} \right\} ds = 0$$

$$L_z \cdot \left\{ (s-1)^{c-b-1} s^{a-c} \cdot (s-z)^{-a} \right\} = \frac{d}{ds} V(z,s)$$

$$V(z,s) = a\, s^{a-c+1} \cdot (s-1)^{c-b} (s-z)^{-a-1}$$
$$\qquad +1 \qquad +1 \qquad -1$$

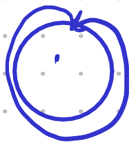
$$\frac{d}{ds} V = \frac{a-c+1}{s} V + \frac{(c-b)}{s-1} V - \frac{(a+1)}{s-z} V = \left(\frac{a-c+1}{s} + \frac{c-b}{s-1} - \frac{a+1}{s-z}\right) V$$

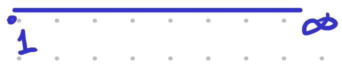
$$L_z \tilde{F} = \left[\frac{z(1-z)\,a(a+1)}{(s-z)^2} + \frac{[c-(a+b+1)z]}{s-z} - ab \right] \tilde{F}$$

$\tilde{F} = (s-z)\cdot\frac{1}{s}\cdot\frac{1}{s-1}\cdot V$

<u>conclusion</u>

$$\int_C w^{a-c}(w-1)^{c-b-1}(w-z)^{-a}\frac{dw}{w}$$ – solution of hyper. eq-n for any closed contour C



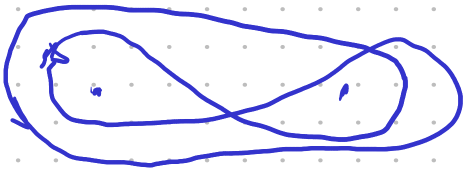


$$\int_0^1 w^{a-c}(w-1)^{c-b-1}(w-z)^{-a}\frac{dw}{w} =$$

$$= (-z)^{-a}\int_0^1 w^{a-c}(w-1)^{c-b-1}\left(1-\frac{w}{z}\right)^{-a}\frac{dw}{w} = (-z)^{-a}F\left(a, a+1-c, a+1-b; \frac{1}{z}\right)$$

· ∄ const

closed.



$d^{-1}F$

$$\int_1^\infty d_w\left(w^{a-c}(w-1)^{c-b-1}(w-z)^{-a}\right) = 0$$

$$\parallel$$

$$\int_1^\infty \frac{d}{dw}\left(w^{a-c}(w-1)^{c-b-1}(w-z)^{-a}\right)dw =$$

$$= a-c\int_1^\infty w^{a-c-1}(w-1)^{c-b-1}(w-z)^{-a}dw +$$

$$+ (c-b-1)\int_1^\infty w^{a-c}(w-1)^{c-b-2}(w-z)^{-a}dw +$$

$$+ -a\int_1^\infty w^{a-c}(w-1)^{c-b-1}(w-z)^{-a-1}dw =$$

$$= (a-c)\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z) + (c-b-1)\frac{\Gamma(b)\Gamma(c-b-1)}{\Gamma(c-1)}F(a,b;c-1;z) +$$

$$+ -a\cdot\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a+1,b,c;z)$$

$$\Gamma(c-b) = (c-b)\Gamma(c-b-1)$$

$$\frac{(a-c)(c-b)}{c}F + (c-b-1)F(a,b;c-1;z) - \frac{a(c-b)}{c}F(a+1,b;c;z) = 0$$

Conjugacy relation.

$F(a,b;c;z)$ — Gauss terminology: $F(a+1,b;c;z)$ is conjugate to F

$F(a-1,b;c;z)$ conj to F

6 conjugent to F. Gauß theorem: any two conj. to F functions and F itself are linearly dependend over $\mathbb{C}(x)$

$$\alpha(z)\, F(a+1, b; c; z) + \beta(z)\, F(a, b-1, c; z) + \gamma(z)\, F(a, b; c; z) = 0 \quad \text{ex.}$$

Proof. Idea: relate any conj. F_1 and $x\frac{d}{dx}F$

$$x\frac{d}{dx}F = \sum \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!}\cdot n = \sum_{n\geq 0}\frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{(n-1)!}$$

$F(a_+) := F(a+1, b; c; x)$

$F(a_-) = F(a-1, b, c; x)$

$$F(a_+) - F = \sum_{n\geq 0}\frac{(a+1)_n (b)_n}{(c)_n}\frac{x^n}{n!} - \sum \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!} =$$

$$(a+1)_n = (a+1)\dots(a+n) = \frac{(a+n)}{a}(a)_n$$

$$(a)_n = a \dots (a+n-1)$$

$$= \frac{1}{a}\sum_{n\geq 0}\frac{n}{a}\frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!} = \frac{1}{a}\, x\frac{d}{dx}F$$

$$a\left(F(a_+) - F\right) = x\frac{d}{dx}F \qquad \text{differ. - difference equation}$$

$$b\left(F(b_+) - F\right) = x\frac{d}{dx}F$$

$$(c-1)\left(F(c_-) - F\right) = x\frac{d}{dx}F$$

$$x(1-x)\frac{d}{dx}F = (c-a)F(a_-) - (a-c+bx)F \quad *$$

$$x(1-x)\frac{d}{dx}F = (c-b)F(b_-) - (a-b+ax)F$$

$$x(1-x)\frac{d}{dx}F = (c-a)(c-b)F(c_+) + c(a+b-c)F$$

$$(c-a)F(a_-) = (a-c+bx)F + x(1-x)\frac{d}{dx}F$$

$$L_z = z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab\cdot 1$$

Question: find out the measure on $(0,1)$ to s.that L_z is symmetric w.r. to it

$$(f,g) = \int_0^1 f(x)g(x)\,\underbrace{\mu(x)\,dx}$$

$$(L_2 f, g) = (f, L_2 g)$$

symmetric

$$(L_2 f, g) = \int_0^1 \left(x(1-x) f'' \cdot g \cdot \mu + (c-(a+b+1)x) f' \cdot g\mu - \cancel{fg\mu} \right) dx$$

$$\int_0^1 \frac{d}{dx}(f') \cdot (x(1-x) g \mu)\,dx = - f' \cdot (x(1-x)g\mu)\Big|_0^1 -$$

$$- \int f' \cdot [(x(1-x))' \cdot g\mu + \cancel{x(1-x)g' \cdot \mu} + (x(1-x)g \cdot \mu'] \,dx$$

symm.

$$\int_0^1 f' \cdot [(c-(a+b+1)x)\mu - (x(1-x))'\mu - x(1-x)\cdot\mu'] g\,dx$$

$$\| \; 0$$

$$x(1-x)\mu' = ((c-(a+b+1)x) + 2x - 1)\mu$$

$$\frac{\mu'}{\mu} = \frac{c-(a+b+1)x+2x-1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$x=0$ $\quad A = c-1$

$x=1$ $\quad B = c-a-b$

$$\boxed{\mu = x^{c-1}(1-x)^{a+b-c}}$$

$$L = \mu^{-1}(x) \frac{d}{dx} \mu(x) x(1-x) \frac{d}{dx} - ab$$

$$\int Lf g \mu\,dx$$

$$\int \frac{d}{dx} \mu(1-x)x \frac{d}{dx} f \cdot g$$

$$\frac{d}{dx} \mu x(1-x) = \frac{d}{dx} x^c (1-x)^{a+b-c+1} =$$

$$= x^c(1-x)^{a+b-c+1}\frac{d}{dx} + \left(\frac{c}{x} - \frac{a+b-c+1}{1-x}\right)\mu x(1-x)$$

$$Lf = x(1-x)\frac{d^2}{dx^2} + x(1-x)\left(\frac{c}{x} - \frac{a+b-c+1}{1-x}\right)\frac{d}{dx} - ab$$

$$\| \quad (1-x)\cdot c - x(a+b-c+1)$$

Polynomial degen. of F

$$F = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n$$ when it is a finite series?

$a = -N$ or $b = -N$ $\quad N \in \mathbb{N}$.

$F(-n, b, c; x)$ is a polynomial of degree n $\qquad F(a, b; c; x)$

Rodrigues formula. $\qquad \mu = x^{c-1}(1-x)^{(a+b-c)}$

$$\mu^{-1}\frac{d}{dx}x(1-x)\mu\frac{d}{dx}F = abF \qquad \boxed{\frac{d}{dx}\Delta\mu\frac{d}{dx}F = ab\mu F} \quad \text{hypergeom eq.}$$

$$\Delta = x(1-x)$$

$y = F(a,b;c;x)$

$y' = F' = \frac{ab}{c}F(a+1, b+1, c+1, x)$ $\qquad (a)_{m+1} = a\cdot(a+1)\dots(a+m) = a\cdot(a+1)_m$

$$F' = \sum \frac{(a)_n (b)_n}{(c)_n}\frac{x^{n-1}}{(n-1)!} = \sum \frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}}\frac{x^m}{m!} = \frac{ab}{c}\sum\frac{(a+1)_m (b+1)_m}{(c+1)_m}\frac{x^m}{m!}$$

$n-1 = m$

$$\frac{d}{dx}\left|\frac{d}{dx}\Delta\underset{\underset{\Delta\cdot\mu}{\parallel}}{\mu(a+1,b+1,c+1)}\frac{d}{dx}y' = (a+1)(b+1)\mu(a+1,b+1,c+1)y' = (a+1)(b+1)\cancel{\mu(a+1)}\right.$$

$$\frac{d}{dx}(\Delta\mu y')$$

$$\frac{d}{dx}\Delta^2\mu\frac{d}{dx}y' = (a+1)(b+1)\Delta y' \qquad = (a+1)(b+1)\frac{d}{dx}\Delta\mu y' = (a+1)(b+1)a\cdot b\,\mu y$$

$$\frac{d}{dx}(\Delta^2\mu y'') = \frac{(a)_2(b)_2}{c}\mu y \qquad \frac{d^2}{dx^2}(\mu\Delta^2 y'') = (a)_2(b)_2\mu y$$

$$\frac{d^n}{dx^n}\left(\mu\Delta^n\frac{d}{dx}\right)y \qquad \boxed{\frac{d^n}{dx^n}\left(\Delta^n\mu\,F^{(n)}(x)\right) = \mu F\cdot(a)_n\cdot(b)_n}$$

$a = -n$ $\qquad y = F(-n, b, c; x)$ polyn. of degree n

$$y^{(n)} = ? \qquad \left(\frac{a(a+1)\dots(a+n-1)\cdot(b)_n}{(c)_n}\frac{x^n}{n!}\right)^{(n)} \qquad a = -n$$

$$\frac{(-1)^n n!\,(b)_n}{(c)_n}$$

Rodriguez formula

$$F(-n, b, c; x) = \frac{x^{1-c}(1-x)^{c+n-b}}{(c)_n}(b)_n\frac{d^n}{dx^n}\left[x^{c+n-1}(1-x)^{b-c}\right]$$