Basic hypergeometric series (Heine series)

30 years after Gauss $\sim(1850)$ Heine introduced

$$1+\frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)}z^2+\dots$$

$c\neq 0,-1,-2,\dots$

It is a q-analog of hypergeom. series

converges $|q|<1$ $|z|<1$

$$\frac{A_{n+1}}{A_n}=\frac{z^{n+1}}{z^n}\frac{(1-q^{a+n})(1-q^{b+n})}{(1-q^{c+n})(1-q^{n+1})}\to|z|<1$$

$$(a)_q=\frac{1-q^a}{1-q}\underset{q\to1}{\to}a \qquad (n)_q=\frac{1-q^n}{1-q}=1+q+\dots+q^{n-1}\underset{q\to1}{\to}n$$

$$1+\frac{(a)_q(b)_q}{(c)_q(1)_q}z+\frac{(a)_q(a+1)_q(b)_q(b+1)_q}{(c)_q(c+1)_q(1)_q(2)_q}z^2+\dots \underset{q\to1-}{\to}F_{2,1}(a,b;c;z)$$

Notations $(a;q)_n=(1-a)(1-qa)\dots(1-q^{n-1}a)$ - q-Pochhammer

$\overset{\shortparallel}{(a|q)_n}$

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$$

q-hypergeom. function $\varphi_{2,1}(a,b;c;z|q)=\sum_{n\geq0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n$

$$(a,q)_n=(a^{-1};q^{-1})_n\cdot(-a)^n q^{\binom{n}{2}} \qquad \binom{n}{2}=\frac{n(n-1)}{2}$$

$$(1-a)(1-qa)\dots(1-q^{n-1}a)=(-a)^n(1-a^{-1})(q-a^{-1})(q^2-a^{-1})\dots(q^{n-1}-a)=$$
$$=(-a)^n q^{1+2+\dots+n-1}(1-a^{-1})(1-a^{-1}q^{-1})\dots(1-a^{-1}q^{-n+1})$$

$$\varphi_{2,1}(a,b,c;z|q)=\sum\frac{(a^{-1};q^{-1})_n(b^{-1};q^{-1})_n}{(c^{-1};q^{-1})_n(q^{-1};q^{-1})_n}\cdot\left(\frac{zab}{qc}\right)^n$$

converges for $|q|>1$

$\left|\frac{zab}{qc}\right|<1$ $\qquad |q|\neq1$

1) Newton binom formula

$$F_{1,0}(a;z)=\sum\frac{(a)_n}{n!}z^n=(1-z)^{-a}$$

$$(1-z)^{-a}=\sum_{n\geq0}\binom{-a}{n}(-z)^n=\sum\frac{(-a)(-a-1)\dots(-a-n+1)}{n!}(-1)^nz^n=$$

$$=\sum_{n\geq0}\frac{a(a+1)\dots(a+n-1)}{n!}z^n=\sum\frac{(a)_n}{n!}z^n$$

$f(z)=\sum \frac{(a)_n}{n!} z^n \qquad f'(z)=\sum_{n\geq 1} \frac{n(a)_n}{n!} z^{n-1} = \sum_{n\geq 1} \frac{(a)_n}{(n-1)!} z^{n-1} = \sum_{n\geq 0} \frac{(a)_{n+1}}{n!} z^n$

$\left(a+z\frac{d}{dz}\right)f(z) = \frac{d}{dz}f(z) \qquad f' = \frac{a}{1-z}f$

$a+n$; a ; $z\frac{d}{dz}$

$\frac{f'}{f} = \frac{a}{1-z} \Rightarrow f = C\cdot(1-z)^{-a} \qquad f(0)=1$

$\log f = a\log(1-z)+C \qquad f=(1-z)^{-a}$

q-binomial theorem.

$${}_1\varphi_0(a;z|q) = \frac{(az;q)_\infty}{(z;q)_\infty}$$

$\parallel$

$\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n$

$a=q^n \qquad a=q^n$

$$\frac{\cancel{(1-q^n z)(1-q^{n+1}z)\cdots}}{(1-z)(1-qz)\cdots\cancel{(1-q^n z)\cdots}} = \frac{1}{(1-z)(1-qz)\cdots(1-q^{n-1}z)}$$

$q\to 1 \quad \downarrow \quad \frac{1}{(1-t)^n}$

$a=q^{-n}$

$$\frac{(1-q^{-n}z)\cdots\quad \cancel{(1-z)\cdots}}{\cancel{(1-z)(1-qz)\cdots}} = (1-q^{-n}z)\cdots(1-q^{-1}z)$$

$\downarrow$

$(1-z)^n$

Proof $\quad f(z)=\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n$

$$f(z)-f(qz) = \sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n(1-q^n) = \sum_{n\geq 1}\frac{(a;q)_n}{(q;q)_{n-1}}z^n = \sum_{n\geq 0}\frac{(a;q)_{n+1}}{(q;q)_n}z^{n+1}$$

$$f(z)-af(qz) = \sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n(1-aq^n) = \sum_{n\geq 0}\frac{(a;q)_{n+1}}{(q;q)_n}z^n$$

$(1-aq^n)$

$z\cdot(f(z)-af(qz)) = f(z)-f(qz)$

$$f(z) = \frac{1-az}{1-z}f(qz) = \frac{(1-az)}{1-z}\frac{(1-aqz)}{(1-qz)}f(q^2z) = \dots \frac{(az;q)_n}{(z;q)_n}f(q^n z)$$

$|q|<1 \qquad n\to\infty$

$q^n z\to 0$

$$f(z) = \frac{(az;q)_\infty}{(z;q)_\infty}\cdot \underset{1}{\underset{\parallel}{f(0)}} = \frac{(az;q)_\infty}{(z;q)_\infty}$$

Corollary $\quad {}_1\varphi_0(a;z|q)\cdot{}_1\varphi_0(b;az|q) = {}_1\varphi_0(ab;z|q)$

$$(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$$

q-analysis (combinatorics) $\mathbb{F}_q$ Kac

$$(n)_q = \frac{1-q^n}{1-q} = 1+q+\dots+q^{n-1}; \quad (n)_q! = (1)_q(2)_q\cdots(n)_q$$

$$\binom{n}{k}_q = \frac{(n)_q!}{(k)_q!\,(n-k)_q!} \qquad \binom{n+1}{k}_q = \binom{n}{k}_q \cdot q^k + \binom{n}{k-1}_q$$

$$(x+y)^n = \sum_k \binom{n}{k}_q x^k y^{n-k}$$

x, y - variables q-commuting

$$\underline{xy} = q^{-1}yx$$

$$(x+y)(x+y) = x^2 + \underbrace{yx+xy}_{(q+1)xy} + y^2$$

$$\exp_q(x) = 1 + \frac{x}{(1)_q!} + \frac{x^2}{(2)_q!} + \dots$$

$$\exp_q(x+y) = \exp_q(x)\exp_q(y) \quad \text{if} \quad xy = q^{-1}yx$$

1st nontrivial formula <u>Heine transformation formula</u>

$$\varphi_{2,1}(a,b;c;z|q) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,\varphi_{2,1}(c/b,z,az;b|q)$$

$a \; b \; c \; z$

$c/b \; z \; az \; b$

$$(a_1,a_2\dots a_k|q)_\infty = (a_1;q)_\infty\cdot(a_2;q)_\infty\cdots(a_k;q)_\infty$$

<u>Proof</u> Recall q-binomial th.

$$\sum_{m\geq 0}\frac{(a;q)_m}{(q;q)_m}z^m = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad \text{substitute} \quad \begin{array}{l} z = bq^n \\ a = c/b \end{array}$$

$$\frac{(q^n c;q)_\infty}{(q^n b;q)_\infty} = \sum_{m\geq 0}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

$$\varphi_{2,1}(a,b;c;z|q) = \frac{(b,q)_\infty}{(c,q)_\infty}\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}\cdot\frac{(q^n c;q)_\infty}{(q^n b;q)_\infty}z^n =$$

$$= \frac{(b,q)_\infty}{(c,q)_\infty}\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}z^n\sum_{m\geq 0}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m =$$

for sum over n apply q-binomial

$$= \frac{(b,q)_\infty}{(c,q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m\, b^m}{(q;q)_m}\cdot\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} =$$

$$= \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \underbrace{\sum_{m\geq 0} \frac{(c/b;q)_m}{(q;q)_m} \frac{(z;q)_n}{(az;q)_m}}_{\varphi_{2,1}(c/b,z;az;b;q)}$$

Corollary q-Euler transformation f-la.

$$F_{2,1}(a,b;c;z) = (1-z)^{c-a-b} F_{2,1}(c-a,c-b,c;z)$$

$$\Downarrow$$

$$\varphi_{2,1}(a,b;c;z|q) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \varphi_{2,1}(c/a,c/b,c;\frac{abz}{c}|q)$$

Proof is ~~twice~~ 3 times Heine transform

$$\varphi_{21}(a,b;c;z|q) = \frac{(b,az|q)_\infty}{(c,z|q)_\infty} \cdot \varphi(c/b,z;az;b|q) =$$

$$= \qquad \varphi(\frac{abz}{c},b;bz;c/b|q) =$$

$$= \frac{(b,az|q)_\infty}{(c,z|q)_\infty} \cdot \frac{(bz,c/b;q)_\infty}{(az,b;q)_\infty} \frac{(\frac{abz}{c};q)(c;q)_\infty}{(c/b,bz;q)_\infty} \varphi(c/a,c/b;c;\frac{abz}{c}|q)$$

$$= \frac{(\frac{abz}{c};q)_\infty}{(z;q)_\infty} \varphi_{21}(c/a,c/b;c;\frac{abz}{c})$$

Corollary q-analog of Gauss summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (a,q)_\infty \sim \frac{1}{\Gamma(a)}$$

$$\varphi_{21}(a,b;c;\frac{c}{ab}|q) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}$$

Heine.

$$\varphi_{2,1}(a,b;c;z|q) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \varphi_{2,1}(c/b,z,az;b|q) \qquad z = \left(\frac{ab}{c}\right)^{-1}$$

$$\varphi_{2,1}(a,b;c\,\frac{c}{ab}|q) = \frac{(b,c/b;q)_\infty}{(c,\frac{c}{ab};q)_\infty} \varphi_{21}(c/b,c/ab,c/b;b|q) =$$

$$= \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty} \underbrace{\varphi_{10}(c/ab; b|q)}_{q\text{-binomial}} = \frac{(\cancel{b}, c/b; q)_\infty}{(c, c/ab; q)_\infty} \cdot \frac{(c/a; q)_\infty}{\cancel{(b; q)_\infty}} = \frac{(c/b, c/a; q)_\infty}{(c, c/ab; q)_\infty}$$

$\left|\frac{ab}{c}\right| > 1 \qquad \left|\frac{c}{ab}\right| < 1 \qquad a, b, c$

Particular cases.

$a = q^{-n}$ In this case Heine series terminates

$$\varphi_{2,1}(q^{-n}, b, c; \tfrac{c}{b} q^{n} | q) = \frac{(c/a, c/b | q)_\infty}{(c, c/ab | q)_\infty} \Bigg|_{a = q^{-n}} = \frac{(c/b | q)_n}{(c, q)_n}$$

$$\frac{(cq^n; \infty)}{(c; \infty)} = \frac{\prod_{k \geq 0} (1 - cq^{n+k})}{\prod_{k \geq 0} (1 - cq^k)} = \frac{1}{(1-c)(1-qc)\cdots(1-q^n c)}$$

$$\sum_{k=0}^{n} \frac{(q^{-n}, b; q)_k}{(c, q)_k} \left(\frac{c}{b}\right)^{k-n} = \frac{(c/b; q)_n}{(c; q)_n} \qquad q\text{-analog of Chu-Vandermonde}$$

$$F_{2,1}(-n, a, c; 1) = \frac{(c-a)_n}{(c)_n}$$

q- Pfaff transform.

$$F_{2,1}(a, b; c; z) = (1-z)^{-a} F_{2,1}\left(a, c-b, c, \frac{z}{z-1}\right)$$

$$\varphi_{21}(a, b; c; z | q) = \frac{(az; q)_\infty}{(z; q)_\infty} \varphi_{2,2}(a, c/b; c, az; bz | q)$$

$$\varphi_{2,2}(a_1, a_2; b_1, b_2; z) = \sum \frac{(a_1; q)_n (a_2; q)_n}{(b_1 q)_n (b_2; q)_n} \frac{z^n}{(q; q)_n} \cdot q^{\binom{n}{2}}$$

$$\varphi_{p,q}(a_1 \ldots a_p, b_1 \ldots b_q; z) = \sum \frac{(a_1, a_2 \ldots a_p; q)_n}{(b_1, \ldots b_q; q)_n} z^n \cdot q^{\binom{n}{2} \cdot (q-p+1)}$$

$p \neq q+1$