Special functions. Problems for seminar 12

1. Assume that $xy = qyx$. Prove the equalities

$$\exp_q(x+y) = \exp_q(y)\exp_q(x)$$

where

$$\exp_q(z) = \sum_{n\geq 0} \frac{z^n}{(n)_q!}, \qquad (n)_q = \frac{1-q^n}{1-q}, \quad (n)_q! = (1)_q(2)_q\cdots(n)_q.$$

2. Let $\mathrm{Ad}_A(B) = ABA^{-1}$ and $\mathrm{ad}_a(b) = ab - ba$. H'Adamard formula says

$$\mathrm{Ad}_{e^x} = e^{\mathrm{ad_x}}: \qquad e^x y e^{-x} = y + [x,y] + \frac{[x[x,y]]}{2} + \dots$$

a) Prove H'Adamard formula.
b) Suggest and prove q version of H'Adamard formula:

$$\mathrm{Ad}_{\exp_q(x)} = ?$$

3. Zaaltshutz summation formula reads as:

$$\sum_{j=0}^{n} \binom{n}{j} \cdot \frac{(a)_j(b)_j(c-a-b)_j}{(c)_j} = \frac{(c-a)_n(c-b)_n}{(c)_n}$$

a) Prove it comparing coefficiens at powers of a variable in Euler transformation formula

b) Suggest and prove a q version of Zaaltshutz summation formula