V - вект. пр-во. vector space. X $Mat(V)$ $End(V)$ X – матрица

$$\dot{Y} = [X,Y] = ad_X(Y)$$

$ad_X: Mat \to Mat$ линейное linear map

$\dot{x} = Zx$ $x(t) = e^{tZ} x(0)$ $Y(t) = e^{t\, ad_X}(Y)$

Z - matrix

$$y(t) = e^{tX} \cdot y_0 \cdot e^{-tX} \qquad \frac{d}{dt} Y(t) = X \cdot Y(t) - Y(t) \cdot X$$

also solution with the same in. condition

$$(\exp_q(x))^{-1} \stackrel{?}{=} \exp_{q^{-1}}(-x)$$

$$\left(1 + x + \frac{x^2}{1+q} + \frac{x^3}{(1+q)(1+q+q^2)} + \dots\right)\left(1 - x + \frac{x^2}{1+q^{-1}} - \frac{x^3}{(1+q^{-1})(1+q^{-1}+q^{-2})} + \dots\right)$$

$$= 1 + (x-x) + x^2\left(\frac{1}{1+q} - 1 + \frac{1}{1+q^{-1}}\right) + x^3\left(\frac{1}{(1+q)(1+q+q^2)} - \frac{1}{(1+q)} + \frac{1}{1+q^{-1}} - \frac{1}{(1+q^{-1})(1+q^{-1}+q^{-2})}\right)$$

$$\frac{1}{1+q} - 1 + \frac{q}{1+q} = 0 \qquad \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{1+q} + \frac{q}{1+q} - \frac{q^3}{(1+q)(1+q+q^2)}$$

$$1-q^3 = (1-q)(1+q+q^2)$$

$$\exp_q(x)\, y \exp_{q^{-1}}(-x) = \left(1 + x + \frac{x^2}{1+q} + \dots\right) y \left(1 - x + \frac{x^2 q}{1+q} + \dots\right) =$$

$$= y + (xy - yx) + \frac{x^2 y}{1+q} - xyx + \frac{q y x^2}{1+q}$$

$$\frac{1}{1+q}\left(x^2 y - (1+q) xyx + q y x^2\right) = \frac{1}{1+q}\left(x^2 y - xyx + q(xyx - yx^2)\right)$$

$$= \frac{1}{1+q}\left(x(\underbrace{xy - yx}_{z}) - q(xy - yx)x\right) = \frac{1}{1+q}[x[x,y]]_q$$

$$xz - qzx$$

$$\exp_q(x)\, y \exp_{q^{-1}}(x) = y + [x,y] + \frac{1}{(2)_q}[x[x,y]]_q + \frac{1}{(3)_q!}[x[x[x,y]]_q]_{q^2} + \dots$$

$$Ad \exp_q(x) = \text{"}\exp_q(ad_q x)\text{"}$$

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; x)$$

$$(1-z)^{a+b-c} \cdot F = F(c-a, c-b; c; x) \qquad \frac{(c-a)_n (c-b)_n\, x^n}{(c)_n\, n!}$$

$$\left(\sum \frac{(c-a-b)_j}{j!} x^j\right) \sum \frac{(a)_k (b)_k}{(c)_k\, k!} x^k$$

$$\sum_{j=0}^{n} \frac{(a)_j (b)_j}{(c)_j\, j!} \cdot \frac{(c-a-b)_{n-j}}{(n-j)!} = \frac{(c-a)_n (c-b)_n}{n!\,(c)_n}$$

Recall:

1) q-binomial theorem

$$(1-z)^n \to (1-z)(1-qz)\dots(1-q^{n-1}z)$$

$$\varphi_{1,0}(a,b;z|q) := \sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (1-z)_q^n =$$

$$\frac{(a)_n}{n!} \qquad a = q^n$$

$$\frac{\cancel{(1-q^n z)(1-q^{n+1}z)\dots}}{(1-z)\dots(1-q^{n-1}z)(1-\dots}$$

2) Heine transf. formula

$$\varphi_{2,1}(a,b;c;z) = \frac{(b,az|q)_\infty}{(c,z|q)_\infty} \cdot \varphi_{2,1}(c/b, z, az; b|q)_\infty$$

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q-Gamma function

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma_q(x+1) = (x)_q \Gamma_q(x) \qquad (x)_q = \frac{1-q^x}{1-q} \quad q\to 1$$

$+$ analyt. prop. $\qquad$ $+$ anal. prop.

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} \cdot (1-q)^{1-x} \qquad \text{Main ingredient } (q^x;q)_\infty$$

$$|q|<1$$

$$(1-q^x)(1-q^{x+1})\dots$$
$$(1-q^{x+1})\dots$$

1. Check equation:

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x)} = \frac{(q^x;q)_\infty}{(q^{x+1};q)_\infty} \frac{(1-q)^{1-x-1}}{(1-q)^{1-x}} = \frac{1-q^x}{1-q} = (x)_q$$

2. Check limit $q \to 1-$

$$\Gamma_q(x+1) = \frac{(q,q)_\infty}{(q^{x+1};q)_\infty} (1-q)^{-x} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{(1-q^n)}{(1-q^{n+x})} \cdot \prod_{n=1}^{N} \frac{(1-q^{n+1})^x}{(1-q^n)^x}$$

$$\frac{1}{1-q^x} \cdot \frac{(1-q^2)^x}{(1-q^3)^x} \cdot \frac{(1-q^3)^x}{(1-q^4)^x} \dots \underbrace{(1-q^{N+1})^x}_{\to 1} \qquad |N| \to \infty$$

$$\lim_{q\to 1-} \lim_{N\to\infty} \left(\prod_{k=1}^{N} \frac{(1-q^k)}{1-q^{k+x}} \cdot \left(\frac{1-q^{n+1}}{1-q^n}\right)^x \right) =$$

subtle point

$$\lim_{N\to\infty} \lim_{q\to 1-} \left(\prod_{k=1}^{N} \frac{1-q^{k}}{1-q^{k+x}} \cdot \left(\frac{1-q^{n+1}}{1-q^{n}} \right)^{x} \right) =$$

Euler inf. product

$$= \lim_{N\to\infty} \underbrace{x\, x^{-1} \prod_{k=1}^{N} \left(1+\frac{x}{k}\right)^{-1} \cdot \left(1+\frac{1}{k}\right)^{x}} = x\,\Gamma(x) = \Gamma(x+1)$$

Analytical properties:

$\Gamma(x)$ is a merom. function with simple poles

$q^{x+n} = 1 \qquad x = -n + \frac{2\pi i k}{\log q} \qquad n = 0, 1, \ldots \quad k \in \mathbb{Z}$

$e^{(x+n)\log q} = e^{2\pi i k}$

Residue $x = -n + \frac{2\pi i k}{\log q}$ does not depend on k

$$x \to -n \qquad \lim_{x\to -n} (x+n)\,\Gamma_q(x) = \lim_{x\to n} \frac{\prod_{k\geq 0}(1-q^{k+1})}{\prod_{k\geq 0}(1-q^{k+x})} (1-q)^{1-x} \quad \downarrow (1-q)^{n+1}$$

$$= (1-q)^{n+1} \cdot \lim \frac{(x+n)}{1-q^{x+n}} \cdot \frac{\prod(1-q^{k+1})}{(1-q^{-n})(1-q^{1-n}) \cdots (1-q^{-1})} =$$

$$= \frac{(1-q)^{n+1}}{(q^{-1};q) \log q^{-1}} \qquad \frac{(-1)^n}{n!}$$

$$B_q(p,q) = \frac{\Gamma_q(x)\Gamma_q(y)}{\cancel{\Gamma_q(0)}\ \Gamma_q(x+y)} = (1-q) \frac{(q,q)_\infty (q^{x+y},q)_\infty}{(q^x;q)_\infty (q^y;q)_\infty} = (1-q) \frac{(q^{x+y}, q; q)_\infty}{(q^x, q^y; q)_\infty}$$

Jackson q-integral

$$\int_0^1 f(t)\, d_q t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n$$

it is a particular Riemann sums in log scale

$\Updownarrow$

$$\int_0^1 f(t) \frac{d_q t}{t} = (1-q) \sum_{n\geq 0}^{\infty} f(q^n)$$

$0 \; q^n \ldots q^3 \; q^2 \; q \; 1 \qquad x_k = q^k \quad \xi_k = q^k \qquad \sum \Delta_k f(\xi_k) \qquad \Delta_k = x_k - x_{k+1}$

$$J = \sum_{k\geq 0} (q^k - q^{k+1})\cdot f(q^k) = \sum_{k\geq 0} q^k (1-q) f(q^k)$$

$0 \qquad aq^2 \quad aq \quad a$

$$\int_0^a f(t)\, d_q t = a(1-q)\sum f(aq^n)\cdot q^n$$

$$a<b \qquad \int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t$$

$t = \frac{1}{\tau}$

$$\int_0^\infty f(t)\frac{d_q t}{t} = \int_0^1 f(t)\frac{d_q t}{t} + \int_1^\infty f(t)\frac{d_q t}{t}$$

$f(q^n)$ $\qquad$ $\| \quad \int_0^1 f\left(\frac{1}{\tau}\right)\frac{d_q \tau}{\tau}$

$f(q^{-n})$

$$\int_0^\infty f(t)\frac{d_q t}{t} = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)$$

$$B_q(x,y) = (1-q)\frac{(q, q^{x+y};q)_\infty}{(q^x, q^y;q)_\infty} = \frac{(q;q)_\infty}{(q^y;q)_\infty}\cdot\sum_{n=0}^\infty (1-q)\frac{(q^y;q)_n}{(q;q)_n}\cdot q^{nx} =$$

$$\frac{(q^{x+y};q)_\infty}{(q^y;q)_\infty} \leftarrow \text{q-binomial}$$

$$= \cancel{\frac{(q;q)_\infty}{(q^y;q)_\infty}}\cdot\frac{(q^y;q)_\infty}{(q;q)_\infty}\underbrace{\sum_{n=0}^\infty (1-q)\frac{(q^{1+n};q)_\infty}{(q^{y+n};q)_\infty}\cdot q^{nx}} = \int_0^1 \frac{d_q t}{t}\, t^x \frac{(tq;q)}{(tq^y;q)}$$

$\mathrm{Re}\, x>0$ $\qquad$ $t^{x-1}(1-t)^{y-1}$

Heine transf. formula

$\Updownarrow$ q-analog of Euler integral for $F_{2,1}$

Ramanujan ψ_{11} summation formula.

$$B_q(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}\,dx \qquad \frac{(q^a x;q)_\infty}{(q^n x;q)_\infty} \qquad (1-x)_q^{n\to}$$

$$\sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k} x^n = \frac{(ax, q/ax, b/a, q \mid q)_\infty}{(x, b, b/ax, q/a \mid q)_\infty}$$

$$\varphi_{1,0}(a,x) = \sum_{n=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k \qquad \psi(a_1 \ldots a_p; b_1 \ldots b_p; x) =$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_n)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_p)_n} \qquad (-1) q^{\binom{n}{2}}{}^{q-p}$$

$(1)_k$

$$(a;q)_{-1} \overset{?}{=} \frac{1}{1-aq^{-1}} \qquad (a;q)_{-n} = \frac{1}{(1-aq^{-1}) \cdots (1-aq^{-n})}$$

$$(a,q)_n = \frac{(a,q)_\infty}{(aq^n;q)_\infty} \qquad (a;q)_{-n} = (-1)^n a^{-n} q^{\binom{n}{2}} (a^{-1}q;q)^{-1}$$

<u>Proof</u> $\psi_{1,1}(a,b;x)$ as function on b $\quad f(b)$

$$f(b) = \underbrace{\sum_{n\geq 0} \frac{(a,q)_n}{(b,q)_n} x^n}_{f_+(b)} + \underbrace{\sum_{n\geq 1} \frac{(b^{-1}q;q)_n}{(a^{-1}q;q)_n} \left(\frac{b}{ax}\right)^n}_{f_-(b)} \qquad n=-1 \quad (1-q^{-1}b)^{\to 0}$$

$$\frac{f(b)\cdot\left(1-\frac{b}{ax}\right)}{1-\frac{b}{a}} \qquad \frac{1}{1-\frac{b}{a}} \cdot^{f_+} \left(\sum_{n\geq 0} \frac{(a,q)_n}{(b,q)_n} x^n - \sum_{n\geq 0} \frac{b}{a} \frac{(a,q)_n}{(b,q)_n} x^{n-1} \right) =$$

$n \to n+1$

$$= \frac{1}{1-\frac{b}{a}} \left(\sum_{n\geq 0} \frac{(a,q)_n}{(b,q)_n} x^n - \sum_{n\geq 0} \frac{b}{a} \frac{(a,q)_{n+1}}{(b,q)_{n+1}} x^n \right) =$$

$$= \frac{1}{1-\frac{b}{a}} \sum_{n\geq 0} x^n \frac{(a,q)_n}{(b,q)_{n+1}} \left(1 - \cancel{bq^n} - \frac{b}{a}(1 - \cancel{aq^n}) \right) =$$

$$= \sum_{n\geq 0} \frac{(a,q)_n}{(b,q)_{n+1}} x^n$$

$$\frac{f(qb)}{1-b} = \frac{1}{1-b} \sum \frac{(a,q)_n}{\underbrace{(qb,q)_n}_{1-qb}} x^n = \sum_{n\geq 0} \frac{(a,q)_n}{(b,q)_{n+1}} x^n$$

$$\frac{f(qb)}{1-b} = \frac{\left(1-\frac{b}{a}x\right)}{1-\frac{b}{a}} f(b)$$

$$f(b) = \frac{1-\frac{b}{a}}{(1-b)\left(1-\frac{b}{a}x\right)} f(qb) = \frac{\left(1-\frac{b}{a}\right)\left(1-q\frac{b}{a}\right)}{(1-b)(1-qb)\left(1-\frac{b}{a}x\right)\left(1-q\frac{b}{a}x\right)} f(q^2 b)$$

$$f(b) = \frac{(b/a;q)_n}{(b, b/a\,x;q)_n} \cdot f(q^n b) \searrow f(0) \qquad n \to \infty \qquad |q|<1$$

$$f(b) = \frac{(b/a;q)_\infty}{(b,q)_\infty \cdot (b/a\,x;q)_\infty} \cdot f(0)$$

$$f(q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \frac{(q/a;q)_\infty}{(q,q)_\infty (q/a\,x;q)_\infty} \cdot f(0) \implies \text{get the answer}$$