Introduction to KAM theory. Spring semester 2024/2025.

Problem List 2. Circle homeomorphisms and rotation number.

Deadline: April 2.

Problem 1. Prove that for $\gamma > 2$ the complement of the set of γ-Diophantine numbers has Lebesgue measure zero, i.e., for $\gamma > 2$ typical numbers are γ-Diophantine.

Problem 2. Prove that θ is Diophantine, if and only if it satisfies the Siegel condition

$$\sup_{n\in\mathbb{N}} \frac{\ln q_{n+1}}{\ln q_n} < +\infty.$$

Hint. Use the following classical properties of the continued fractions for every $n \in \mathbb{N}$:

- θ lies between successive approximating fractions $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$,
- $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}}$;
- one has $q_n = a_n q_{n-1} + q_{n-2}$.

Problem 3. Prove that the rotation number depends on a circle homeomorphism

a) continuously in the C^0-topology;

b) monotonously: if $f_1 \leq f_2$ then $\rho(f_1) \leq \rho(f_2)$.

Problem 4. Prove that if $F_1 < F_2$ and F_1 is a translation (i.e., f_1 is a rotation, then $\rho(F_2) > \rho(F_1)$.

Consider a family of circle diffeomorphisms $f_u : S^1 \to S^1$ depending on a parameter u lying in a domain $U \subset \mathbb{R}^n$. Consider its rotation number as a function $\rho(u)$ of parameter u. Its level set $L_r := \{u \in U \mid \rho(u) = r\}$ is called a *phase-lock area*, if its interior $Int(L_r)$ is non-empty.

Problem 5. Prove that if a family f_u contains a diffeomorphism f with rotation number $\frac{m}{n}$ that has a n-periodic point ϕ_0 where $(f^n)'(\phi_0) \neq 1$, then $L_{\frac{m}{n}}$ is a phase-lock area.

Problem 6. (optional). Consider Arnold family of circle diffeomorphisms:

$$f_{a,b}(\phi) = \phi + a + b\sin\phi.$$

a) Find Taylor coefficients at b and at b^2 of the iterate $f^n_{a,b}(\phi)$ (a is fixed, b is variable).

b)*** Using result of a) and solution of the above problem for $a = 2\pi\frac{m}{n}$, for every irreducible fraction $\frac{m}{n}$ prove existence of a phase-lock area in $\mathbb{R}^2_{a,b}$ (called *Arnold tongue*) corresponding to the rotation number $\frac{m}{n}$, accumulating to the point $(2\pi\frac{m}{n}, 0)$.

Problem 7. *** (an optional small research project) Construct the Denjoy example of a C^1-smooth circle diffeomorphism with an arbitrary given irrational rotation number ρ that has an invariant Cantor set, i.e., non-dense orbits, as follows:

1) Take the rotation $R : \phi \mapsto \phi + 2\pi\rho$. Fix its orbit, say, $\phi_0 = 0$, $\phi_1 = 2\pi\rho$, $\phi_2 = 4\pi\rho, \dots$. Cut the circle at the points ϕ_n and insert intervals I_n at the cuts so that the sum of lengths of the inserted intervals be finite.

2) Consider the new circle obtained from the initial one by insertion of the above intervals. Extend the homeomorphism of the initial circle to the homeomorphism of the new circle thus constructed by homeomorphisms $I_n \to I_{n+1}$.

3) Show that for appropriate choice of interval lengths the above extension can be chosen to be C^1-smooth on the whole new circle.