Introduction to KAM theory. Spring semester 2024/2025.

Problem List 3. Introduction to symplectic geometry.

Deadline: May 23 (exam date).

Problem 1. Prove that for every $n \in \mathbb{N}$, $n \geq 2$, and $\gamma > n-1$ the set of γ-Diophantine n-dimensional vectors has full measure.

Problem 2. a) Prove that every symplectic vector field, i.e., a field whose flow preserves a symplectic form, is locally Hamiltonian: each point of the ambient manifold has a neighborhood where the field is Hamiltonian.

b)* Prove that in general a symplectic vector field on $\mathbb{T}^n \times \mathbb{R}^n$ is not necessarily Hamiltonian.

Problem 3. Prove that the KAM theorem is not true for perturbations of a non-degenerate integrable Hamiltonian vector field in the class of symplectic fields.

Hint. Consider perturbations of a standard integrable Hamiltonian $\frac{|p|^2}{2}$ on $\mathbb{T}^n_\phi \times \mathbb{R}^n_p$, i.e., harmonic oscillator.

Problem 4. Consider the above Hamiltonian function $K = \frac{p^2}{2}$ on 2-dimensional cylinder $C := S^1_\phi \times [-1, 2]$. Prove that for every analytic function H on $S^1 \times [-1, 2]$ close enough to K there is no trajectory of the perturbed Hamiltonian H going from the circle $\{p = 0\}$ to the circle $\{p = 1\}$. Here "analytic" and "close" means analytic on a complex s-neighborhood of the real cylinder C and close enough to K depending on s.

Hint. Use Kolmogorov–Arnold Invariant Tori Theorem.

Remark In the above problem it is essential that the dimension is equal to two: one degree of freedom. In higher dimensions the famous Arnold Diffusion Conjecture, with a lot of very strong results by many mathematicians including J.Mather, V.Kaloshin et al., is a kind of opposite statement.

Problem 5. (Herman's lemma)*. Let a manifold M be equipped with an *exact* symplectic form: $\omega = d\alpha$. Let a Hamiltonian system on M have an invariant torus T on which its flow is conjugated to a constant quasiperiodic flow $\dot{\phi} = w$, $< m, w > \neq 0$ for every non-zero integer vector m. Prove that the torus T is isotropic, i.e., $\omega|_T = 0$.