Introduction to KAM theory. Spring semester 2024/2025.

Problem List 3. Introduction to symplectic geometry.

Deadline: April 30.

Problem 1. Find Hamiltonian vector fields for the following functions on the standard symplectic space $\mathbb{R}^{2n}$:

$\sum_{j=1}^n p_j^2, \quad \sum_j q_j p_j \quad \exp(q_1 + p_2^2).$

Problem 2. How is the Hamiltonian vector field changed if we multiply the symplectic form by a constant c?

Problem 3. Prove that two Hamiltonian vector fields commute, if and only if their Poisson bracket is constant.

Definition. A subspace L of the standard symplectic space $(\mathbb{R}^{2n}_{q,p}, \omega_{st})$ is *isotropic,* if $\omega_{st}|_L = 0$; $\omega_{st} = dq \wedge dp = \sum_{j=1}^n dq_j \wedge dp_j$.

Problem 4. Prove that the maximal dimension of an isotropic subspace L is equal to n: in this case L is called *Lagrangian.*

Problem 5. Consider an arbitrary isotropic subspace $V \subset \mathbb{R}^{rn}$. Prove that its symplectic orthogonal $V^{\perp_\omega}$ contains V. Prove that V is contained in a Lagrangian subspace, and each Lagrangian subspace containing V lies in $V^{\perp}$.

Definition 1. A n-dimensional global or local subvariety is called *Lagrangian,* if its tangent spaces are Lagrangian.

Problem 6. Consider a Lagrangian submanifold $S \subset U \times V \subset \mathbb{R}^n_q \times \mathbb{R}^n_p$, $\omega = dq \wedge dp$. Let S be diffeomorphically projected to the q-space U, i.e., it is the graph of a vector function $p = v(q)$. Prove that S is Lagrangian, if and only if v is the gradient of a smooth function g:

$$v(q) = \nabla g(q) = \left(\frac{\partial g}{\partial q_1}, \dots, \frac{\partial g}{\partial q_n}\right).$$

Definition Let (M, ω) be a $2n$-dimensional symplectic manifold. A *Lagrangian fibration* is a splitting of the manifold M into disjoint Lagrangian submanifolds, called *fibers,* that is a locally trivial fibration. Namelt, there exists a projection $\pi : M \to N$ without critical points onto a N-dimensional manifold such that the fibers are preimages $\pi^{-1}(q)$, $q \in N$. *Example:* a foliation by invariant tori in an integrable system.

Problem 7. Consider a Lagrangian fibration $\pi : M \to N$ over a domain $N \subset \mathbb{R}^n_{q_1,\dots,q_n}$. Prove that

a) for every function $f : N \to \mathbb{R}$ the Hamiltonian vector field with the Hamiltonian function $F = f \circ \pi$ is tangent to the Lagrangian fibers;

b) every functions $F, G : M \to \mathbb{R}$ constant on fibers are in involution: $\{F, G\} = 0$;

c) for every function $F : M \to \mathbb{R}$ constant on fibers the Hamiltonian vector field X_F is a linear combination of the Hamiltonian vector fields X_{Q_j}, $Q_j = q_j \circ \pi$, with constant coefficients.

d) Deduce that in the case, when $n = 1$, i.e., fibers are curves, the fibration defines a family of length elements on fibers, in which the norm $||X_Q||$ is constant on each fiber. It is uniquely defined just by the fibration in the following sense: for two different projections defining the same fibration the corresponding length elements differ by multiplication by a factor that is constant on each fiber.

d) In the general case of arbitrary n deduce that a Lagrangian fibration uniquely determines an *affine structure* on fibers. This means that each fiber is equipped with an atlas where the transition functions between charts are affine transformations; two atlaces are said to be *equivalent,* if their charts differ by taking post-compositions with affine transformations.