Introduction to KAM theory. Spring semester 2024/2025.

Problem List 1. Normalization of formal power series in one variable.

Deadline: March 18.

Problem 1. Find explicit formula for the conjugate $h^{-1} \circ f \circ h(z)$ and for its Taylor series, where

$$f(z) = \lambda z, \quad h(z) = \frac{z}{1+az}, \quad a \in \mathbb{C}.$$

Problem 2. Prove Formal Linearization Theorem: *if θ is irrational then every formal power series $f(z) = \lambda z + \sum_{k=2}^{+\infty} a_k z^k$, $\lambda = e^{2\pi i\theta}$, is formally conjugated to its linear part λz, i.e., conjugated to it by a formal power series $h(z) = z + \sum_{j=2}^{+\infty} b_k z^k$:*

$$h^{-1} \circ f \circ h(z) = \lambda z.$$

Hint. a) Show that for every n in each series $f_n(z) = \lambda z + a_n z^n + \sum_{k>n} a_k z^k$ the term $a_n z^n$ can be killed by conjugation by a monomial variable change $h_n(z) = z + d_n z^n$ so that $h_n^{-1} \circ f_n \circ h_n(z) = \lambda z + \sum_{k>n} c_k z^k$ for some $c_k \in \mathbb{C}$. The coefficient d_n is found from a linear equation called *homological equation.* **Write down** the homological equation explicitly.

b) Construct the normalizing series h as infinite composition of the above monomial changes.

Problem 3. Show that if $\theta \notin \mathbb{Q}$, then the power series $h(z) = z + \dots$ conjugating the above $f(z)$ to its linear part is unique.

Problem 4. Particular resonant case. Prove that $f(z) = -z + z^3 + \dots$ cannot be conjugated to its linear part by a formal power series.

Problem 5. General resonant case. Let $\lambda^n = 1$, i.e., $\theta = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and let $\lambda^k \neq 1$ for $k \in [0, n-1]$. Prove that no series then $f(z) = \lambda z + a_n z^{n+1} + \dots$ with $a_n \neq 0$ can be formally conjugated to its linear part by a series $h(z) = z + \dots$.

Hint. (i) Prove that if such h exists, then it cannot contain nonlinear terms of degrees less than $n+1$. Prove this by contradiction, assuming that it contains a monomial $b_k z^k$, $k \geq 2$, of degree less than $n+1$ (let k denote the minimal degree of monomial) and showing that then the conjugate $h^{-1} \circ f \circ h$ will contain a monomial of degree k. Prove this by applying the homological equation argument from Problem 2 to degree k.

(ii) Prove that the homological equation on the $n+1$-th Taylor coefficient (mentioned in Problem 2) has no solution and thus, the term $a_n z^{n+1}$ cannot be killed by any formal series conjugation without creating new terms of smaller degrees.