$$\varphi_{21}(q^a, q^b; q^c; z|q) = \underbrace{\frac{(q^b, \cancel{q^a z}|q)_\infty}{(q^c, \cancel{z}|q)_\infty}}_{A} \varphi_{21}(q^{c-b}, z; q^a z; q^b|q)$$

$(1-a)(1-qa)\ldots\cancel{(1-q^n a)}$

$$(a,q)_n = \frac{(a,q)_\infty}{(aq^n;q)_\infty}$$

$$= A \sum_{n\geq 0} \frac{(q^{c-b};q)_n (z;q)_n}{(q^a z;q)_n (q;q)_n} \cdot q^{bn} = A \frac{(q^{c-b};q)_\infty \cdot (\cancel{z};q)_\infty}{(\cancel{q^a z};q)_\infty (q;q)_\infty} q^{nb}$$

$$\cdot \underbrace{\sum_{n\geq 0} \frac{(q^{a+n}z;q)_\infty (q^{n+1};q)_\infty}{(q^{c-b+n};q)_\infty \cdot (zq^n;\infty)} q^{nb}}_{\sum_{n\geq 0} f(q^n) = \int_0^1 f(t) \frac{d_q t}{t}} = \int_0^1 \frac{(q^a zt;q)_\infty (q\cdot t;q)_\infty}{(q^{c-b}t;q)_\infty (zt;q)_\infty} t^b \frac{d_q t}{t}$$

$$= \frac{(q^b, q^{c-b};q)_\infty}{(q^c, q;q)_\infty} \int_0^1 \frac{(q^a zt;q)_\infty}{(zt;q)_\infty} \cdot \frac{(qt;q)_\infty}{(q^{c-b}t;q)_\infty} t^{b-1} d_q t$$

$$\frac{(q^b, q^{c-b};q)_\infty}{(q^c, q;q)_\infty} = \frac{\Gamma_q(c)\cdot \cancel{\Phi_q(b)}}{\Gamma_q(b)\Gamma_q(c-b)}, \qquad \frac{(q^a zt;q)_\infty}{(zt;q)_\infty} = (1-zt)_q^{-a}, \qquad \frac{(qt;q)_\infty}{(q^{c-b}t;q)_\infty} = (1-t)_q^{c-b-1}$$

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} \cdot (1-q)^{1-x}$$

$$(1-q)\sum_{n=-\infty}^{+\infty} f(q^n) = \int_0^\infty f(t)\frac{d_q t}{t} \qquad \int_0^1 + \int_1^\infty$$

$$\sum_{n\geq 0} \qquad \sum_{n<0}$$

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n \stackrel{?}{=} \qquad q^{n\log_q x} \qquad t^{\log_q x}$$

$$= \frac{(a,q)_\infty}{(b;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} \cdot q^{n\log_q x} = \frac{1}{1-q}\int_0^\infty \frac{(bt;q)_\infty}{(at;q)_\infty} t^{\log_q x} \frac{d_q t}{t}$$

$$x = q^\alpha;\quad a = -q^{\frac{1-\alpha}{2}},\quad b\ -q^{\beta+\frac{\alpha-1}{2}}$$

$$\int_0^\infty t^{\alpha-1} \frac{(-q^{\beta+\frac{\alpha-1}{2}}t;q)}{(-q^{\frac{1-\alpha}{2}}t;q)} d_q t = \frac{\cancel{(b;q)_\infty}}{(a;q)_\infty} \cdot \frac{(ax;q/ax, b/a|q)}{(x, q/a, b/ax, b)} =$$

$$\frac{(-q^{\beta+\frac{\alpha-1}{2}}t;q)}{(-q^{\frac{1-\alpha}{2}}t;q)} = (1+t)_q^{-\alpha-\beta}$$

$$B(\alpha,\beta) = \int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} dx \qquad = \frac{(q^{\alpha+\beta}, q|q)_\infty}{(1-q)(q^\alpha, q^\beta|q)_\infty} = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

Ramanujan also found contour integral

$$f(a)=\int_0^\infty x^{\alpha-1}\frac{(-ax;q)_\infty}{(-x;q)_\infty}\,dx=\int_0^\infty x^{\alpha-1}\frac{(-aqx;q)_\infty}{(-x;q)}\,dx\cdot\underbrace{(1+ax)}_{[(1-a)+a(1+x)]}=$$

$(1+ax)$

$$=(1-a)f(aq)+a\int_0^\infty x^\alpha\frac{(-aqx;q)_\infty}{(-xq;q)_\infty}\frac{dx}{x}$$

$xq=y$

$$a\int_0^\infty\frac{(xq)^\alpha}{q^\alpha}\frac{(-aqx;q)_\infty}{(-xq;q)_\infty}\frac{dqx}{qx}=a\cdot q^{-a}f(a)$$

$$f(a)=(1-a)f(qa)+aq^{-\alpha}f(a)\qquad |q|<1$$

$$f(a)=\frac{1-a}{1-aq^{-\alpha}}f(qa)=\frac{(1-a)(1-qa)}{(1-aq^{-\alpha})(1-aq^{1-\alpha})}f(q^2a)=\dots$$

$$=\frac{(a;q)_\infty}{(aq^{-\alpha};q)_\infty}\cdot\underline{f(0)}$$

$$f(q)=\int_0^\infty x^{\alpha-1}\frac{\cancel{(-qx;q)_\infty}}{\underbrace{(-x;q)_\infty}_{1+x}}dx=\int_0^\infty\frac{x^{\alpha-1}}{1+x}dx=B(\alpha,1-\alpha)=\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}=\frac{\pi}{\sin\pi\alpha}$$

$$\frac{\pi}{\sin\pi\alpha}=f(q)=\frac{(q;q)_\infty}{(q^{1-\alpha};q)_\infty}\cdot f(0)$$

$$\int_0^\infty x^{\alpha-1}\frac{(-ax;q)_\infty}{(-x;q)_\infty}dx=\frac{(a,q)_\infty}{(q;q)_\infty}\cdot\frac{(q^{1-\alpha};q)_\infty}{(aq^{-\alpha};q)_\infty}\cdot\frac{\pi}{\sin\pi\alpha}\,!$$

$\|$

Преобр. Меллина от $\dfrac{(-ax;q)_\infty}{(-x;q)_\infty}$ Формула обр-ния.

$f(t)$

$$\check{f}(s)=\int_0^\infty f(t)t^{s-1}dt\Rightarrow f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\check{f}(s)t^{-s}ds$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(a,q)_\infty}{(q;q)_\infty}\cdot\frac{(q^{1-s};q)_\infty}{(aq^{-s};q)_\infty}\cdot\frac{\pi}{\sin\pi s}x^{-s}ds=\frac{(-ax;q)_\infty}{(-x;q)_\infty}$$

Multiple Гамма-functions by Barnes

Multiple Bernoulli polynomials

$\underline{\text{Dfy.}}$ $\omega_1, \dots \omega_r \in \mathbb{C}$

$$\frac{t^r e^{zt}}{\prod_{k=1}^{r} (e^{\omega_k t} - 1)} = \sum_{k=0}^{\infty} B_{r,k}(z | \omega_1 \dots \omega_r) \frac{t^k}{k!}$$

Bernoulli polynom.

$\bar{\omega} = (\omega_1 \dots \omega_r)$

$\bar{\omega}(j) = (\omega_1 \dots \cancel{\omega_j} \dots \omega_r)$

$\underline{\text{Properties}}$ 1. $B_{r,k}(\lambda z | \lambda\omega_1 \dots \lambda\omega_r) = \lambda^{k-r} B_{r,k}(z | \omega_1 \dots \omega_r)$

$z \to \lambda z$, $\omega_k \to \lambda \omega_k$, $t \to \frac{t}{\lambda}$

2. $B_{r,k}(z + \omega_j / \bar{\omega}) - B_{r,k}(z / \bar{\omega}) = k B_{r-1,k-1}(z / \bar{\omega}(j))$

$$\frac{t^r e^{(z+\omega_j)t}}{\prod_{k=1}^{r} (e^{\omega_k t} - 1)} - \frac{t^r e^{zt}}{\prod_{k=1}^{r} (e^{\omega_k t} - 1)} = \frac{t^r e^{zt}}{\prod_{k \neq j} (e^{\omega_k t} - 1)}$$

$B_k(z) = B_{1,k}(z/1)$

3. $|\omega| = \omega_1 + \dots + \omega_r$

$B_{r,k}(|\omega| - z | \bar{\omega}) = (-1)^k B_{r,k}(z | \bar{\omega})$

$$\frac{t^r e^{(\Sigma\omega_k - z)t}}{\prod (e^{\omega_k t} - 1)} = \frac{t^r e^{-zt}}{\prod (1 - e^{-\omega_k t})} \overset{t \leftrightarrow -t}{=} \frac{(-t)^r e^{-zt}}{\prod (e^{-\omega_k t} - 1)} = \sum \frac{(-t)^k}{k!} B_{r,k}(z | \bar{\omega})$$

$\underline{\text{Examples}}$ $B_{1,0}(z|\omega_1) = \frac{1}{\omega_1}$ $\quad B_{1,1}(z|\omega_1) = \frac{z}{\omega_1} - \frac{1}{2}$ $\quad B_{1,2} = \frac{z^2}{\omega_1} - z + \frac{\omega_1}{6}$

$B_{2,0}(z|\omega_1, \omega_2) = \frac{1}{\omega_1 \omega_2}$, $\quad B_{2,1}(z|\omega_1, \omega_2) = \frac{z}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1 \omega_2}$

$B_{2,2}(z|\omega_1 \omega_2) = \frac{z^2}{\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} z + \frac{\omega_1^2 + 3\omega_1 \omega_2 + \omega_2^2}{6 \omega_1 \omega_2}$

Multiple Hurwitz ζ-function

$\int_{\mathbb{R}^r} \frac{1}{|x|^s} |x|^{r-1} d|x| \, d\varphi$

$\zeta(s, z) = \sum_{n=0,1,2\dots} \frac{1}{(z+n)^s}$ $\qquad \operatorname{Re} s > 1$

$\zeta(s, z | \omega_1 \dots \omega_r) = \sum_{n_1, n_2 \dots n_r \geq 0} \frac{1}{(z + n_1\omega_1 + n_2\omega_2 + \dots + n_r\omega_r)^s}$ $\qquad \operatorname{Re} s > r$

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ $\qquad \operatorname{Re} s > 0$

$$\sum_{n \geq 0} \int_0^\infty e^{-(z+n)t} t^{s-1} dt = \int_0^\infty e^{-(z+n)t} \frac{((z+n)t)^s}{(z+n)^s} \frac{dt}{t} = \frac{1}{(z+n)^s} \Gamma(s)$$

$$\zeta(s, z) = \sum \frac{1}{(z+n)^s} = \Gamma^{-1}(s) \int_0^\infty \frac{e^{-zt}}{1 - e^{-t}} t^{s-1} dt \qquad \operatorname{Re} z > 0$$

$e^{-(z-1)t}$

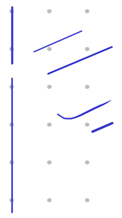
$$\frac{1}{(z + n_1\omega_1 + \dots + n_r\omega_r)^s} = \Gamma^{-1}(s) \int_0^\infty e^{-(z + n_1\omega_1 + \dots + n_r\omega_r)t} t^s \frac{dt}{t}$$

$$\zeta(s,z|\omega_1\ldots\omega_r)=\Gamma^{-1}(s)\int_0^\infty \frac{e^{-zt}}{\prod\limits_{k=1}^{r}(1-e^{-\omega_k t})}\,t^{s-1}\,dt \qquad \mathrm{Re}\, s > r$$

2. Analytical continuation Hankel contour

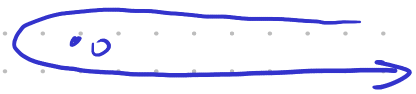
$$\int_0^\infty t^{s-1} f(t)\,dt$$ преобр. Меллина

$f(t)$ не растет быстрее экс.



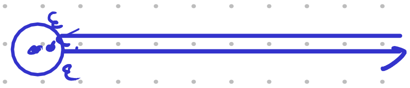
Re $s>0$

$$\int_C f(t)(-t)^s\,dt$$



Re $s>0$

$$\int_C f(t)\,e^{(s-1)(\log|t|+i\arg(-t))}$$



$$\int_\infty^\varepsilon + \int_\varepsilon^\infty + \int_{\circ\,\varepsilon}$$

$$\int_\infty^\varepsilon f(t)\,t^{s-1}\cdot e^{-(s-1)i\pi}dt + \int_\varepsilon^\infty f(t)t^{s-1}e^{(s-1)\pi}dt + \int f(\varepsilon e^{i\varphi})\varepsilon^{s-1}\varepsilon e^{(s-1)i\varphi}d\varphi$$

$\downarrow f(0)\,\varepsilon^s$ Re $s>0$ $\downarrow 0$

$\|$

$-2i\sin\pi s$

$$\int_{\hookleftarrow} = -2i\sin\pi s\int_0^\infty$$

$$\check{f}(s)=\int_0^\infty f(t)\,t^{s-1}dt$$ получили аналит. прод. на $\mathbb{C}$

$\Gamma(s)\Gamma(1-s)$

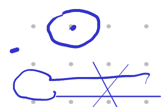
$\|$

$\sin\pi s\,\check{f}(s)$

$s\in\mathbb{Z}$

Ex: $$\Gamma(s)\zeta(s,z)=-\frac{\pi}{\sin\pi s}\int_C \frac{e^{-zt}}{1-e^{-t}}\frac{dt}{2\pi i t}$$

$$\zeta(s,z)=+\Gamma(1-s)\int_C\frac{e^{-zt}(-t)^s}{1-e^{-t}}\frac{dt}{2\pi i t} \qquad s=1,2,3,4\ldots$$



$$\zeta(s,z|\omega_1\ldots\omega_r)=+\Gamma(1-s)\int_C\frac{e^{-zt}(-t)^s}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t} = -\Gamma(1-s)\int_C\frac{e^{-zt}(-t)^{s-1}}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i}$$

For usual $\zeta(s)=\zeta(s,1|1)$

$$\zeta(1-2k)=-\frac{1}{2k}B_{2k},\quad \zeta(0)=-\frac{1}{2},\ \zeta(-2k)=0$$

$s \in \mathbb{Z}$ $\qquad s = 0,-1,-2,\dots$ $\qquad \int_{(\ )} e^{-z}\frac{dt}{t^k}$

Multiple Gamma function

$$\Gamma_r(z|\omega_1,\dots,\omega_r) = \exp\left\{\frac{\partial}{\partial s}\zeta(s,z|\omega_1\dots\omega_r)\Big|_{s=0}\right\}$$

$$\log\Gamma_r(z|\omega_1\dots\omega_r) = \frac{\partial}{\partial s}\zeta(s,z|\omega_1\dots\omega_r)\Big|_{s=0} =$$

$$\frac{\partial}{\partial s}\Gamma(1-s)\Big/_{s=0} = \gamma$$

$$\Gamma'(s)\Big|_{s=1} = -\gamma$$

$$= \frac{\partial}{\partial s}\left(-\Gamma(1-s)\int_C \frac{e^{-zt}(-t)^{s-1}}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i}\right)\Bigg|_{s=0}$$

$$+\gamma\int_0 \frac{e^{-zt}}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t} + \int_C \frac{e^{-zt}\log(-t)}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t}$$

$$\|$$

$$\gamma\int_0 \frac{e^{-zt}\,t^{r}}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t\cdot t^r}$$

$\circlearrowleft$ коэф-т при t^r

$$\gamma\cdot\frac{(-1)^r}{r!}B_{r,r}(z|\omega)$$

Ответ:

$$\log\Gamma_r(z|\omega_1\dots\omega_r) = \gamma\cdot\frac{(-1)^r}{r!}B_{r,r}(z|\omega_1\dots\omega_r) + \int_C \frac{e^{-zt}\log(-t)}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t}$$

$$\frac{\Gamma_r(z+\omega_j|\omega_1\dots\omega_r)}{\Gamma_r(z|\omega_1\dots\omega_r)} = \Gamma_{r-1}^{-1}(z|\overline{\omega}(j))$$

$$\log\Gamma(z+\omega_j|\omega_1\dots\omega_r) - \log\Gamma(z|\omega_1\dots\omega_r) =$$

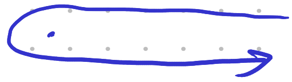
$$= \gamma\frac{(-1)^r}{r!}\left(B_{r,r}(z+\omega_j|\omega_1\dots\omega_r) - B_{r,r}(z|\omega_1\dots\omega_r)\right) +$$

$$+\int_C \frac{e^{-zt}(e^{-\omega_j t}-1)\log(-t)}{\prod(1-e^{-\omega_k t})}\frac{dt}{2\pi i t} =$$

$$= \gamma\frac{(-1)^{\cancel{r}\;r-1}}{\cancel{r!}}\cancel{r}\,B_{r-1,r-1}(z|\overline{\omega}(j)) - \int_C \frac{e^{-zt}\log(-t)\,\cancel{dt}}{\prod_{k\neq j}(1-e^{-\omega_k t})}\frac{dt}{2\pi i t}$$

$\Gamma_0(z) = ?$ $\qquad \log\Gamma_0(z) = \gamma\int_C e^{-zt}\log(-t)\frac{dt}{2\pi i t}$

$$\int_C f(t)\log(-t)\frac{dt}{2\pi i t^k}$$



$k=0$ $$\int_\infty^\varepsilon f(t)(\cancel{\log t}-\pi i)\frac{dt}{2\pi i} + \int_\varepsilon^\infty f(t)(\cancel{\log t}+\pi i)\frac{dt}{2\pi i} +$$

$$+\int_0^{2\pi} f(\varepsilon e^{\theta})\underbrace{[\log\varepsilon + i(\theta-\pi)]\varepsilon\, d\theta}_{\to 0} = \int_0^\infty f(t)\,dt$$

$k=1$ $$\int_C f(t)\log(-t)\frac{dt}{2\pi i t} =$$

$$\int_\infty^\varepsilon f(t)(\log t-\pi i)\frac{dt}{2\pi i t} + \int_\varepsilon^\infty f(t)(\log t+\pi i)\frac{dt}{2\pi i t} +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} f(\varepsilon e^{\theta})(\log\varepsilon + i(\theta-\pi))\,\bar{e}^{i\theta}d\theta$$

$$= \int_\varepsilon^\infty \frac{f(t)\,dt}{t} + \cancel{f(0)\log\varepsilon}$$

$$\|$$

$$\int_\varepsilon^\infty f(t)\,d\log t = \underbrace{f(t)\log\Big|_\varepsilon^\infty}_{\cancel{f(0)\log\varepsilon}} - \int_\varepsilon^\infty f'(t)\log t\,dt$$

$k=1$ $$\int_C f(t)\log(-t)\frac{dt}{2\pi i t} = -\int_0^\infty f'(t)\log t\,dt$$

Более общая ф-ла:

$$\int_C f(t)\log(-t)\frac{dt}{2\pi i t^{k+1}} = \frac{f^{(k)}(0)}{k!}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right) - \frac{1}{k!}\int_0^\infty f^{(k+1)}(t)\log t\,dt$$

$k \geq 1$

$$\log \Gamma_0(z) = \gamma + \int_C e^{-zt}\log(-t)\frac{dt}{2\pi i t} = \gamma + z\int_0^\infty e^{-zt}\log t\,dt$$

$$\| \qquad -\int_0^\infty (e^{-zt})'_t \log t\,dt$$

$$\Gamma'(s) = \int_0^\infty e^{-t}t^{s}\frac{dt}{t} = \int_0^\infty e^{-t}\log t\cdot t^{s-1}dt$$

$s=0$

$$\Gamma'(s)\Big|_{s=1} = \int_0^\infty \log t\, e^{-t}dt = -\gamma$$

$$z\int_0^\infty e^{-zt}(\log(zt)-\log z)\frac{d(zt)}{z} = -\gamma - \log z\int_0^\infty e^{-t}dt = -\gamma-\log z$$

$\log \Gamma_0(z) = -\log z$ $\qquad$ $\boxed{\Gamma_0(z) = \frac{1}{z}}$

$\Gamma_1(z|\omega) = ?$ $\qquad$ $\frac{\Gamma_1(z+\omega|\omega)}{\Gamma_1(z)} = \Gamma_0^{-1}(z) = z$

$\Gamma_1(z|\omega) = \Gamma\left(\frac{z}{\omega}\right)$ $\qquad$ $\Gamma\left(\frac{z+\omega}{\omega}\right) = \Gamma\left(\frac{z}{\omega}+1\right) = \frac{z}{\omega}\Gamma\left(\frac{z}{\omega}\right)$

$$\Gamma_1(z|\omega) = \Gamma\left(\frac{z}{\omega}\right)\cdot \omega^{z/\omega}\cdot C(\omega)$$

$$\Gamma\left(\frac{z+\omega}{\omega}\right)\cdot \omega^{z+\omega/\omega}\cdot C(\omega) = \frac{z}{\cancel{\omega}}\Gamma\left(\frac{z}{\omega}\right)\omega^{z}\cancel{\omega}\, C(\omega)$$

$$\boxed{\Gamma_1(z|\omega) = \Gamma\left(\frac{z}{\omega}\right)\cdot \omega^{z/\omega}\cdot\frac{1}{\sqrt{2\pi\omega}}}$$

$\Gamma_1(\omega|\omega) = \Gamma(1)\frac{\omega}{\sqrt{2\pi\omega}} = \sqrt{\frac{\omega}{2\pi}}$

Poles at $z = 0, -\omega, -2\omega, \dots$ $\qquad$ $\Gamma(1) = 1$

$\lim_{z\to 0} z\Gamma_1(z)$ $\qquad$ $\Gamma_1(z) = \frac{\Gamma_1(z+\omega)}{z}$ $\qquad$ $z\Gamma_1(z) = \Gamma_1(z+\omega)$

$z\to 0$

$\operatorname{Res}\Gamma_1(0)$ $\qquad$ $\Gamma_1(\omega|\omega) = \sqrt{\frac{\omega}{2\pi}}$

$$\lim_{z\to-\omega}(z+\omega)\Gamma(z) = (z+\omega)\frac{\Gamma_1(z+\omega)}{z} = \frac{\cancel{(z+\omega)}\Gamma_1(z+2\omega)}{z\cancel{(z+2\omega)}} \qquad \frac{\Gamma_1(\omega)}{-\omega}$$

$$\operatorname*{Res}_{-\omega}\Gamma_1(z|\omega) = \frac{-1}{\omega}\sqrt{\frac{\omega}{2\pi}}$$

$$\operatorname*{Res}_{-n\omega}\Gamma_1(z|\omega) = \frac{(-1)^n}{n!\,\omega^n}\sqrt{\frac{\omega}{2\pi}}$$

$\Gamma_r(z|\omega_1\dots\omega_r)$ - poles at $z = -n_1\omega_1 - n_2\omega_2 \dots - n_r\omega_r$

If $\omega_1\dots\omega_r$ $\quad$ $\omega_i - \omega_j \notin \mathbb{Q}$ $\Rightarrow$ simple poles.

$\Gamma_2(\omega_1+\omega_2|\omega_1,\omega_2) = \rho_2.$

$z = 0$

$$\Gamma_2(z+\omega_1+\omega_2|\bar\omega) = \frac{\Gamma_2(z+\omega_1|\bar\omega)}{\Gamma_1(z+\omega_1|\omega_1)} = \frac{\Gamma_2(z|\bar\omega)}{\Gamma_1(z|\omega_2)\cdot\Gamma_1(z+\omega_1|\omega_2)} =$$

$$= \frac{\cancel{\Gamma_2(z|\bar\omega)}}{\Gamma_1(z|\omega_2)}\frac{\cancel{\Gamma_0(z)}}{\Gamma_1(z|\omega_1)} \qquad \cancel{\Gamma_2(z|\omega)\cdot z = \Gamma_2(z+\omega_1+\omega_2|\omega)}\cdot\Gamma_1(z+\omega_1$$

$$= \frac{\Gamma_2(z|\omega)}{\Gamma_1(z+\omega_2|\omega_2)\cdot\Gamma_0(z)^{1/2}\,\Gamma_1(z+\omega_1|\omega_1)} \qquad z\to 0$$

$$\Gamma_2(z+\omega_1+\omega_2|\overline{\omega})\cdot\Gamma_1(z+\omega_1|\omega_1)\cdot\Gamma_1(z+\omega_2|\omega_2) = z\cdot\Gamma_2(z|\overline{\omega})$$

$$\rho_2 = \Gamma_2(\omega_1+\omega_2) \qquad \sqrt{\frac{\omega_1}{2\pi}}\cdot\sqrt{\frac{\omega_2}{2\pi}} \qquad \mathrm{Res}$$

$$\mathrm{Res}_0\,\Gamma_2 = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\,\Gamma_2(\omega_1+\omega_2)$$