

Special functions. Problems for seminar 15

1. Multiple Γ function $\Gamma_r(z|\omega_1, \dots, \omega_r)$ satisfies the relations

$$\frac{\Gamma_r(z + \omega_r|\omega_1, \dots, \omega_r)}{\Gamma_r(z|\omega_1, \dots, \omega_r)} = \Gamma_{r-1}^{-1}(z|\omega_1, \dots, \omega_r)$$

It is known that $\Gamma_0(z) = \frac{1}{z}$ and Strling formula for $\Gamma_1(z|\omega)$ is

$$\log \Gamma_1(z|\omega) = \left(\frac{z}{\omega} - \frac{1}{2} \right) \log z - \frac{z}{\omega} + \frac{\omega}{12z} + O(z^{-2}), \quad z \rightarrow \infty, \operatorname{Re} z > 0$$

. Prove that

$$\Gamma_1(z|\omega) = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{z}{\omega}\right) \left(\frac{z}{\omega}\right)^{\frac{z}{\omega} - \frac{1}{2}}$$

2. The double sine function $S_2(z|\omega_1, \omega_2)$ is defined by the relation

$$S_2(z|\omega_1, \omega_2) = \Gamma_2^{-1}(z|\omega_1, \omega_2) \Gamma_2(\omega_1 + \omega_2 - z|\omega_1, \omega_2)$$

Find out:

- (a) functional relation on $S_2(z|\omega_1, \omega_2)$;
- (b) $S_2(\omega_i|\omega_1, \omega_2)$, $i = 1, 2$;
- (c) $\operatorname{Res}_{z=\omega_1+\omega_2} S_2(z|\omega_1, \omega_2)$;
- (d) $\ast \operatorname{Res}_{z=n_1\omega_1+n_2\omega_2} S_2(z|\omega_1, \omega_2)$, $n_1, n_2 > 0$.