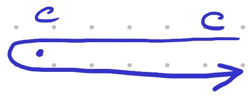
Problem 1

$$\zeta(s/z\,|\,w_1 \dots w_r) = -\Gamma(1-s)\int_C \frac{e^{-zt}(-t)^{s-1}}{\prod(1-e^{-w_k t})}\frac{dt}{2\pi i}$$



$$\zeta(s,z|\omega) = \sum_{n_1 \dots n_r = 0}^{\infty} \frac{1}{(n+z)^s} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-zt}t^{s-1}}{\prod(1-e^{-\omega_k t})}\,dt$$

Explanation of Hankel contour

$$\int_0^\infty f(t)t^{s-1}dt \longrightarrow \frac{1}{2i\sin\pi s}\int_C f(t)(-t)^{s-1}dt$$

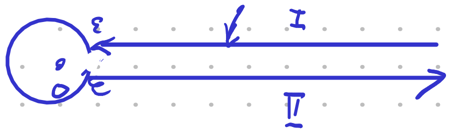
$f(t)\to 0$, $t\to\infty$

$f(t)\to 1$, $t\to 0$

$\int_0 t^{s-1}dt$

$\boxed{\operatorname{Re} s>0}$

$$\int_C f(t)(-t)^{s-1}dt.$$



$$\int_C f(t)\,e^{(s-1)(\log t + i\arg(-t))}dt$$

$$\int_\infty^\varepsilon f(t)\,t^{s-1}\cdot e^{-(s-1)\pi i}dt$$

$$+$$

$$\int_\varepsilon^\infty f(t)\,t^{s-1}\cdot e^{(s-1)\pi i}dt$$

$$\left(\frac{-e^{\pi i s}+e^{\pi i s}}{2i}\right)\int_\varepsilon^\infty f(t)t^{s-1}dt + \oint \int_{-\pi}^{\pi} f(\varepsilon e^{i\pi})\,e^{-i\varphi(s-1)}\,\varepsilon^{s-1}\varepsilon\cdot i\,d\varphi$$

$\varepsilon^s \to 0$, $s>0$

$$= -2i\sin\pi s\int_0^\infty f(t)t^{s-1}dt \quad \text{если } \operatorname{Re} s>0$$

$$\frac{+2i}{\Gamma(s)\sin\pi s}\int_C \frac{e^{-zt}(-t)^{s}}{\prod(1-e^{-t\omega_k})}\frac{dt}{t}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$

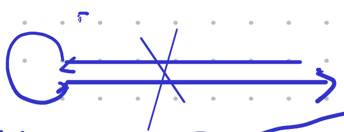
$$\zeta(s,z) = -\Gamma(1-s)\int_C \frac{e^{-zt}(-t)^s}{\prod(1-e^{-t\omega_k})}\frac{dt}{t}$$

$s = 1, 2, 3, \dots$

$s=-N$

$(-t)^s$

$(-t)^N = (-1)^N\cdot t^N$ — univalued function



$\sum c_k t^k$ $\quad c_0$

$$N! = +\Gamma(N+1)$$

$$= -\Gamma(1-s)\oint \frac{e^{-zt}(-t)^{-N}}{\prod(1-e^{-t\omega_k})}\frac{dt}{2\pi i t}$$

$s=-N$

t^{-N}

t^2

Bernoulli gener. function

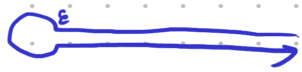
$$\frac{e^{zt}t^{r}}{\prod_{k=1}^{r}(e^{t\omega_k}-1)} = \sum_{k=0}^{\infty} B_{r,k}(z|\omega)\frac{t^k}{k!}$$

$$N!\int_{\odot} \frac{e^{zt}\,\cancel{t}\,dt}{\prod(1-e^{t\omega_k})\,2\pi i t^{N+1}} = \frac{(-1)^r N!}{2\pi i}\int_{\bigcirc}\frac{e^{zt}\,dt\,t^{2}}{\prod(1-e^{t\omega_k})\,t^{N+r+2}}$$

$$= (-1)^r N!\,\frac{1}{2\pi i}\int_{\odot}\frac{t^r e^{zt}\,dt}{\prod(e^{t\omega_k}-1)}\cdot\frac{1}{t^{N+r+2}} =$$

$$(-1)^r \frac{N!}{(N+r)!} B_{r,N+r}(z|\omega)$$

Problem 2



$$\int_C f(t)\log(-t)\frac{dt}{2\pi i t} = \frac{1}{2\pi i}\int_{\infty}^{\varepsilon}\frac{f(t)}{t}(\log t-\pi i)\,dt +$$

$$t = \varepsilon e^{i\varphi}$$

$$\frac{+1}{2\pi i}\int_{\varepsilon}^{\infty}\frac{f(t)}{t}(\log t+\pi i)\,dt + \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(\varepsilon e^{i\varphi})\frac{\varepsilon i e^{i\varphi}}{\varepsilon e^{i\varphi}}\,d\varphi\,(\log\varepsilon+i\varphi)$$

$$\int_{-\pi}^{\pi}\varphi\,d\varphi = 0$$

$$= \int_{\varepsilon}^{\infty}\frac{f(t)}{t}\,dt + \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left(f(0)+\cancel{f'(0)\cdot\varepsilon e^{i\varphi}}\right)(\log\varepsilon+i\varphi)\,d\varphi \to f(0)\log\varepsilon$$

$+$

$$\int_{\varepsilon}^{\infty}\frac{f(t)}{t}\,dt = \int_{\varepsilon}^{\infty} f(t)\,d\log t = -f(\varepsilon)\log\varepsilon - \int_{\varepsilon}^{\infty} f'(t)\log t\,dt$$

$$-f(\varepsilon)\log\varepsilon \to -f(0)\log\varepsilon$$

$$= -\int_0^{\infty} f'(t)\log t\,dt$$

$$\int f(t)\log(-t)\frac{dt}{2\pi i t^2} \qquad \int_{\varepsilon}^{\infty} f(t)\frac{dt}{t^2}$$

Stirling formula

$$z^r\cdot\log z \qquad z^2$$

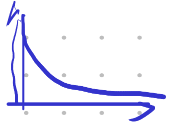
$$\log\Gamma_r(z|\omega) = (-1)^{r+1}\sum_{k=0}^{r} B_{r,r-k}(\omega)\frac{z^k}{k!(r-k)!}\left(\log z+\gamma-\sum_{m=1}^{k}\frac{1}{m}\right)+$$

$$+\gamma\frac{(-1)^r}{r!}B_{r,r}(z|\omega)+\sum_{k=1}^{N}(-1)^{r+k}\frac{(k-1)!}{(r+k)!}B_{r,r+k}(\omega)z^{-k}+O(z^{-N})$$

$$\log\Gamma(z) = \underbrace{\left(z-\frac{1}{2}\right)\log z-\frac{z}{2}+\frac{1}{2}\log 2\pi}_{\text{leading term}} + \frac{B_{2k}}{(2k)(2k+1)}\frac{1}{z^k}$$

$$\Gamma(z) \sim z^{z+\frac{1}{2}}\cdot e^{-z}\sqrt{2\pi/z}+\dots$$

Laplace method $F(z) = \int_0^\infty e^{-zt} f(t)\, dt$



$z \to \infty$

$f(t) \approx \sum a_k t^k$, $t \to 0$ $\quad F(z) \sim \sum \frac{a_k \Gamma(k+1)}{z^{k+1}}$, $z \to \infty$ $\quad f(t) > \frac{1}{t^{1+\epsilon}}$

$$\log \Gamma_r(z) = \underbrace{\gamma \frac{(-1)^r}{r!} B_{r,r}(z|\omega)}_{} + \int_C \frac{e^{-zt} \log(-t)}{\prod_{k=1}^{r} (1-e^{-\omega_k t})} \frac{dt}{2\pi i t}$$

$\operatorname{Re} z > 0$

$z \to \infty$

$$\int_C \frac{e^{-zt} \log(-t)}{\prod_{k=1}^{r} (1-e^{-\omega_k t})} \frac{dt}{2\pi i t} \qquad \frac{1}{\prod_{k=1}^{r}(1-e^{-\omega_k t})} = \sum B_{r,k}(\bar\omega) \cdot \frac{t^{k-r}}{k!}$$

$\Rightarrow$ Need to calculate

$$\int_C e^{-zt} \frac{dt \cdot \log(-t)}{2\pi i \; t^{r-k+1}} \qquad \frac{(-z)^k}{k!} \int_0^\infty e^{-zt} \log t \, dt = \frac{(-z)^k}{k!}(-\gamma - \log z)$$

$$\int_C f(t) \log(-t) \frac{dt}{2\pi i t^{k+1}} = \frac{f^{(k)}(0)}{k!}\left(\sum_{m=1}^{k} \frac{1}{m}\right) - \frac{1}{k!} \int_0^\infty f^{(k+1)}(t) \log t \, dt$$

Hankel $\qquad k \geq 0$

$k \leq 0$

$$\int_C f(t) \log(-t) \frac{dt}{2\pi i} = \int_0^\infty f(t)\, dt$$

Infinite product

$\frac{z}{n} - \frac{z}{n} \quad \frac{z^2}{2n^2}$

1) $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n \geq 1} \left(1 + \frac{z}{n}\right) e^{-z/n}$ $\qquad -\ln \Gamma(z) = \sum \ln\left(1 + \frac{z}{n}\right) + \gamma + \ln z$

2) $$\frac{1}{\Gamma_r(z|\omega_1 \ldots \omega_r)} = z e^{P_r(z)} \prod_{n_1, n_2 \ldots n_r = 0}^{\infty}{}' \left(1 + \frac{z}{n_1\omega_1 + \ldots + n_r\omega_r}\right) e^{-\frac{z}{n_1\omega_1 + \ldots + n_r\omega_r} + \frac{1}{2}\left(\frac{z}{n_1\omega_1 + \ldots + n_r\omega_r}\right)^2 + \ldots + \frac{(-1)^r}{r}\left(\frac{z}{n_1\omega_1 + \ldots + n_r\omega_r}\right)^r}$$

$P_r(z)$ - polynomial of degree r, coeff. via $\zeta^{(m)}(n|\bar\omega)$

$z = -n_1\omega_1 - n_2\omega_2 \cdots - n_r\omega_r$

why should improve convergence by pol. of deg r

$$\sum_{n,m} \qquad \sum \frac{z}{n_1\omega_1 + n_2\omega_2 + \ldots + n_r\omega_r}$$

$$\int_{\mathbb{R}_+^r} \frac{dx_1 \ldots dx_r}{|x|^s} \qquad \int \frac{r^{r-1}}{r^s} dr \quad \int \ldots$$

$r - s < 0$

$s > r$

$\Gamma_2(z_{\pm}|\bar\omega)$ appears in combinations.

Multiple sine function

Double sine function $S_2(z|\omega_1,\omega_2)$ Shintani ~1970

relatives: quantum dilogarithm (Faddeev)

Ruijenaars G-function

Dfn $S_2(z|\omega_1,\omega_2) = \Gamma_2^{-1}(z|\omega_1,\omega_2)\cdot\Gamma_2(\omega_1+\omega_2-z|\omega_1,\omega_2)$

More general $S_r(z|\omega_1,\omega_2) = \Gamma_r^{-1}(z|\bar\omega)\,\Gamma_r^{(-1)^r}(\sum\omega_i - z|\bar\omega)$

Ex. $S_1(z|\omega) = \Gamma_1^{-1}(z|\omega)\cdot\Gamma_1^{-1}(\omega-z|\omega) =$

$$= \Gamma^{-1}\left(\frac{z}{\omega}\right) w^{-z/\omega+\frac12}\cdot\sqrt{2\pi}\cdot\Gamma^{-1}\left(1-\frac{z}{\omega}\right) w^{w-z/\omega+\frac12}\sqrt{2\pi} =$$

$$= \frac{\sin\frac{\pi z}{\omega}}{\pi}\cdot 2\pi = 2\sin\frac{\pi}{\omega}z \qquad \frac{2\sin\frac{\pi}{\omega}(z+\omega)}{2\sin\frac{\pi}{\omega}z} = -1$$

Functional equations,

$S_0 = -1$

(1)
$$\begin{cases} \dfrac{S_2(z+\omega_1|\bar\omega)}{S_2(z|\omega)} = \dfrac{1}{2\sin\frac{\pi}{\omega_2}z} \\[2ex] \dfrac{S_2(z+\omega_2|\omega)}{S_2(z|\omega)} = \dfrac{1}{2\sin\frac{\pi}{\omega_1}z} \end{cases}$$

(2) $S_2\left(z+\frac{\omega_1+\omega_2}{2}\Big|\bar\omega\right)S_2\left(-z+\frac{\omega_1+\omega_2}{2}\Big|\omega\right) = 1 \qquad S_2\left(\frac{\omega_1+\omega_2}{2}\right) = 1$

$$\cancel{\Gamma_2^{-1}\left(z+\frac{\omega_1+\omega_2}{2}\right)}\cdot\Gamma_2\left(\cancel{\omega_1+\omega_2-\left(z+\frac{\omega_1+\omega_2}{2}\right)}\right)\cdot\Gamma_2^{-1}\left(\cancel{-z+\frac{\omega_1+\omega_2}{2}}\right)\cdot\Gamma_2\left(\cancel{\omega_1+\omega_2-\left(-z+\frac{\omega_1+\omega_2}{2}\right)}\right) = 1$$

$$\Rightarrow S_2(z|\omega)S_2(-z|\omega) = -4\sin\frac{\pi z}{\omega_1}\sin\frac{\pi z}{\omega_2}$$

$$S_2(w_1) = \Gamma_2^{-1}(\omega_1)\cdot\Gamma_2(\omega_2) = \sqrt{\frac{\omega_2}{2\pi}}\Big/\sqrt{\frac{\omega_1}{2\pi}} = \sqrt{\frac{\omega_2}{\omega_1}}$$

Integral
$$\begin{cases} -\log\Gamma_2(z) = \frac{\gamma}{2}\cancel{B_{22}(z|\omega)} + \displaystyle\int_0 \frac{e^{-zt}\log(-t)}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{dt}{2\pi i t} \\ \qquad\qquad\qquad\qquad\quad\ \ \| \\ +\log\Gamma_2(\omega_1+\omega_2-z) = \frac{\gamma}{2}\cancel{B_{22}(\omega_1+\omega_2-z)} + \displaystyle\int_C \frac{e^{(z-\omega_1-\omega_2)t}\log(-t)}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{dt}{2\pi i t} \end{cases}$$

$$\log S_2(z) = \int_C \frac{\operatorname{sh}\left(z-\frac{\omega_1+\omega_2}{2}\right)t}{2\operatorname{sh}\frac{\omega_1 t}{2}\operatorname{sh}\frac{\omega_2 t}{2}}\log(-t)\frac{dt}{2\pi i t}$$

$S_2(z) = \Gamma_2^{-1}(z)\,\Gamma_2(\omega_1+\omega_2-z)$

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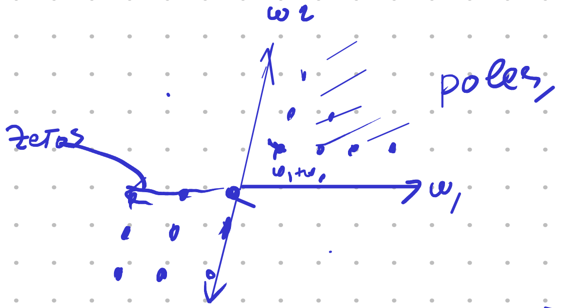
zeros: $z = -n_1\omega_1 + n_2\omega_2$, $n_1, n_2 \geq 0$

poles $z = (n_1+1)\omega_1 + (n_2+1)\omega_2$



Res $S_2(z)$ at $z=(n_1+1)\omega_1+(n_2+1)\omega_2$:

$$\operatorname{Res}_{z=(n_1+1)\omega_1+(n_2+1)\omega_2} S_2(z) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\,\frac{(-1)^{n_1n_2+n_1+n_2}}{\prod_{k=1}^{n_1} 2\sin\pi k\frac{\omega_1}{\omega_2}\prod_{m=1}^{n_2} 2\sin\pi m\frac{\omega_2}{\omega_1}}$$

zeros

$$S_2'(z)\Big|_{z=-n_1\omega_1-n_2\omega_2} = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\,\frac{(-1)^{n_1n_2+n_1+n_2}}{\prod_{k=1}^{n_1} 2\sin\pi k\frac{\omega_1}{\omega_2}\prod_{m=1}^{n_2} 2\sin\pi m\frac{\omega_2}{\omega_1}}$$