$$\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{z}{\omega}\right)\left(\frac{z}{\omega}\right)^{z/\omega-\frac12} = f(z) \qquad \frac{f(z+\omega)}{f(z)} = ?$$

$$\sim \Gamma(z)\, C$$

$$C = \frac{1}{\sqrt{2\pi}}$$

$$\ln\Gamma(z) = \left(z-\frac12\right)\ln z - z + \underbrace{\frac12\ln 2\pi}_{} + o\left(\frac1z\right)$$

$$\ln\left[\Gamma\left(\frac{z}{\omega}\right)\omega^{z/\omega-\frac12}\right]$$

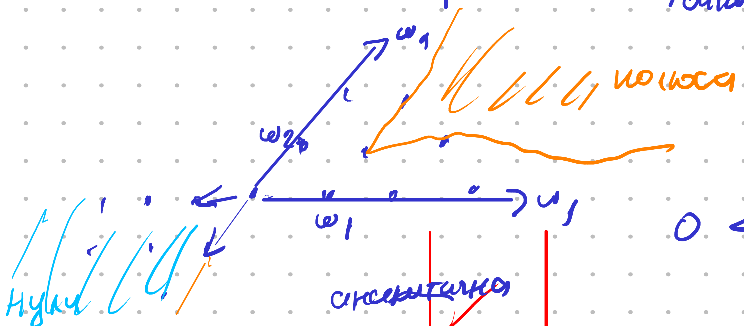
$$S_2(\omega_1) = \frac{\Gamma_2^{-1}(\omega_1|\omega_1,\omega_2)\cdot\Gamma_2(\omega_2|\omega_1,\omega_2)}{\Gamma_2(\omega_1+\omega_2)}$$

$$\begin{cases} \dfrac{S_2(z+\omega_1)}{S_2(z)} = \dfrac{1}{2\sin\frac{\pi}{\omega_2}z} \\ \dfrac{S_2(z+\omega_2)}{S_2(z)} = \dfrac{1}{2\sin\frac{\pi}{\omega_1}z} \end{cases}$$

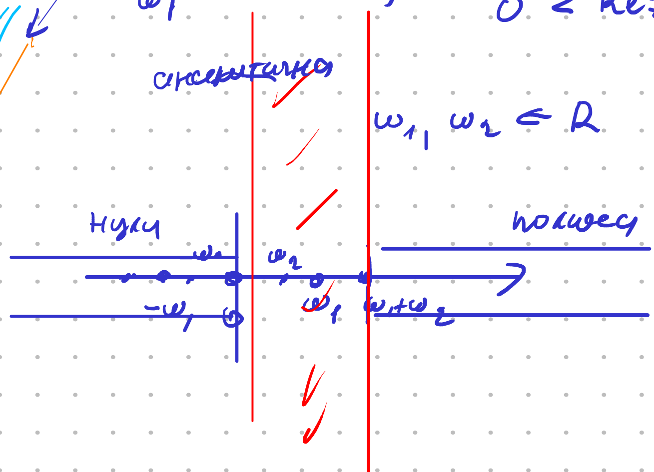
$S_2(z)$ имеет нули в точках

$$z = -n_1\omega_1 - n_2\omega_2$$

полюса в точках $z = (n_1+1)\omega_1 + (n_2+1)\omega_2$



$$0 < \operatorname{Re} z < \operatorname{Re}(\omega_1+\omega_2)$$



$\omega_1, \omega_2 \in \mathbb{R}$ $\quad \omega_1, \omega_2 > 0$

$\omega_1 > \omega_1$

Если $\dfrac{\omega_1}{\omega_2} \notin \mathbb{Q}$

то все нули и полюса простые

0. $S'(z)|_{z=0}$ $\qquad \lim\limits_{z\to0}\dfrac{S_2(z|\vec\omega)}{z}$

$$S_2(z+\omega_1) = \frac{S_2(z)}{2\sin\frac{\pi z}{\omega_2}} \qquad \frac{S_2(z)}{z} = S_2(z+\omega_1)\cdot\frac{2\sin\frac{\pi z}{\omega_2}}{z} \qquad z\to0$$

$$\lim_{z\to0}\frac{S_2(z)}{z} = S_2(\omega_1)\cdot\frac{2\pi}{\omega_2} = \frac{2\pi}{\sqrt{\omega_1\omega_2}}$$

$$\mathrm{Res}\, S_2(z)\Big|_{z=\omega_1+\omega_2} = \lim_{z\to\omega_1+\omega_2} (z-\omega_1-\omega_2) S_2(z) =$$

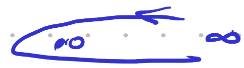
$$= \lim_{z\to 0} z\, S(z+\omega_1+\omega_2) = \lim_{z\to 0} z \frac{S_2(z+\omega_1)}{2\sin\frac{\pi(z+\omega_1)}{\omega_1}} = \frac{\cancel{z^2}\, S(z)/z}{2\sin\frac{\pi z}{\omega_2}\cdot -2\sin\frac{\pi z}{\omega_1}}$$

$\frac{2\pi}{\omega_1\omega_2}$

$$= \frac{2\pi}{\sqrt{\omega_1\omega_2}}\cdot\frac{\omega_1\omega_2}{-4\pi^2} = -\frac{\sqrt{\omega_1\omega_2}}{2\pi}$$

$$\mathrm{Res}_{z=(n_1+1)\omega_1+(n_2+1)\omega_2} S_2(z) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\cdot\frac{(-1)^{n_1 n_2+}}{\prod_{k=1}^{n_1} 2\sin\pi k\frac{\omega_1}{\omega_2}\cdot\prod_{l=1}^{n_2} 2\sin\pi l\frac{\omega_2}{\omega_1}}$$

Integral presentations $\Gamma_2 \Rightarrow$



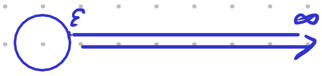
1. $\log S_2(z) = \int_C \frac{\sinh(z-\frac{\omega_1+\omega_2}{2})t}{2\sinh\frac{\omega_1 t}{2}\sinh\frac{\omega_2 t}{2}}\log(-t)\frac{dt}{2\pi i t}$ $\quad 0<\mathrm{Re}\, z<\mathrm{Re}\,\omega_1+\omega_2$

C Hankel

Ruijsenaars

2. $\log S_2(z) = \int_0^\infty \left(\frac{\sinh(z-\frac{\omega_1+\omega_2}{2})t}{2\sinh\frac{\omega_1 t}{2}\sinh\frac{\omega_2 t}{2}} - \frac{2}{t}B_{2,1}(z|\omega)\right)\frac{dt}{t}$

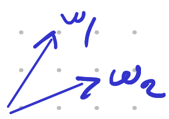
$$B_{2,1}(z,\omega) = \frac{z}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}$$



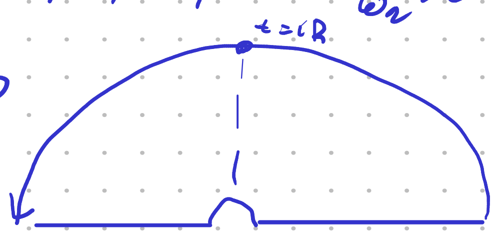
3. $\log S_2(z) = \frac{\pi i}{2}B_{22}(z|\omega) + \int_{\mathbb{R}+i0}\frac{e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)}\frac{dt}{t}$

Residue calculation

ω_1,ω_2 are complex, $\mathrm{Im}\frac{\omega_1}{\omega_2}>0$



$\mathrm{Im}\, z>0$



Jordan lemma

$\omega_1 t = 2\pi i n$
$\omega_2 t = 2\pi i n$

$z = a+ib$

$$e^{iR(a+ib)} = e^{-Rb+iRa} \xrightarrow{b>0} 0$$

$$\log S_2(z|\omega) = \frac{\pi i}{2}B_{2,2}(z|\omega) + \sum_{n\geq 1}\frac{1}{n}\left\{\frac{e^{\frac{2\pi i n z}{\omega_1}}}{e^{2\pi i n\omega_2/\omega_1}-1} + \frac{e^{\frac{2\pi i n z}{\omega_2}}}{e^{2\pi i n\omega_1/\omega_2}-1}\right\}$$

$\mathrm{Im}\frac{\omega_1}{\omega_2}>0$ $\qquad e^{2\pi i\omega_1/\omega_2} = q \quad |q|<1$

$$= \frac{\pi i}{2} B_{2,2}(z|\omega) + \sum_{n\geq 1}\left(-\frac{1}{n}\sum_{m\geq 0}\frac{e^{\frac{2\pi i n z}{\omega_2}}}{e^{2\pi i m n\omega_1/\omega_2}} + \sum_{m\geq 0}\frac{e^{2\pi i n(z-\frac{\omega_2}{\omega_1})}}{e^{-2\pi i m n\omega_2/\omega_1}} \right) \quad 1 - e^{-2\pi i\,\omega_2/\omega_1}$$

$$\tilde{q} = e^{-2\pi i\,\omega_2/\omega_1}$$

$$|\tilde{q}| < 1$$

$$= \frac{\pi i}{2} B_{2,2}(z|\omega) +$$

$$+ \log\prod_{m=0}^{\infty}\left(1-q^{2m}e^{2\pi i\frac{z}{\omega_2}}\right) - \log\prod_{m=1}^{\infty}\left(1-\tilde{q}^{2m}e^{+2\pi i\frac{z}{\omega_1}}\right)$$

$$S_2(z|\omega) = \frac{\prod_{m=0}^{\infty}\left(1-q^{2m}e^{2\pi i z/\omega_2}\right)}{\prod_{m=1}^{\infty}\left(1-\tilde{q}^{2m}e^{2\pi i z/\omega_1}\right)}\, e^{\frac{\pi i}{2}B_{2,2}(z|\omega)} =$$

$\operatorname{Im} z > 0$

$$= e^{\frac{\pi i}{2}B_{2,2}(z|\omega_2)}\,\frac{\left(e^{2\pi i z/\omega_2};q\right)_\infty}{\left(e^{2\pi i z/\omega_1}\tilde{q};\tilde{q}\right)_\infty}$$

$$q = e^{\pi i\frac{\omega_1}{\omega_2}}$$

$$\tilde{q} = e^{-\pi i\frac{\omega_2}{\omega_1}}$$

$$B_{22} = \frac{z^2}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{\omega_1\omega_2}z + \frac{\omega_1^2 - 3\omega_1\omega_2 + \omega_2^2}{\omega_1\omega_2}$$

$\operatorname{Im} z < 0$

$$S_2(z) = e^{-\frac{\pi i}{2}B_{2,2}(z|\omega_2)}\cdot\frac{\left(e^{-2\pi i z/\omega_1};\tilde{q}\right)_\infty}{\left(e^{-2\pi i z/\omega_2}q;q\right)_\infty}$$

$\omega_1, \omega_2 \in \mathbb{C}$

$$S_2(z) = e^{az^2+bz+c}\cdot\frac{\Gamma_q(z)}{\Gamma_{\tilde{q}}(z)}$$

elliptic Γ-function

$$\frac{\gamma_2(z+\omega_1|\omega_1,\omega_2,\omega_3)}{\gamma_2(z)} = G(z|\omega_2,\omega_3)$$

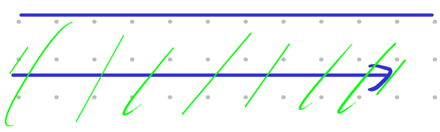
Relatives

Ruijsenaars $\quad G$ function

$$G(z|\omega) = S_2\left(iz + \frac{\omega_1+\omega_2}{2}\Big|\omega\right)$$

strip of analiticy

$$|\operatorname{Im} z| < \frac{1}{2}\operatorname{Re}(\omega_1+\omega_2)$$



$\frac{1}{2}\operatorname{Re}(\omega_1+\omega_2)$

$$\frac{G(z+\frac{i\omega_1}{2})}{G(z-\frac{i\omega_1}{2})} = 2\,\mathrm{ch}\,\frac{\pi z}{2} \qquad \frac{1}{\text{ris}}$$

3. Quantum dilogarithm by L. Faddeev

$$\varphi_b(z) = \exp\int_{\mathbb{R}+i0} \frac{e^{-izt}}{4\,\mathrm{sh}\frac{bt}{2}\,\mathrm{sh}\frac{t}{2b}}\,\frac{dt}{t} \qquad b>0$$

ω_1, ω_2

$\omega_1 = b$

$\omega_2 = 1/b$

$$S_2(\lambda z|\lambda\omega_1,\lambda\omega_2) = S_2(z|\omega_1,\omega_2)$$

$$\omega_1\omega_2 = 1$$

$$\frac{\varphi_b(z+ib^{\pm1}/2)}{\varphi_b(z-ib^{\pm1}/2)} = \frac{1}{1+e^{2\pi z b^{\pm1}}} \qquad \mathrm{ch}$$

$$\varphi_b(z) = S_2\left(\frac{b+b^{-1}}{2}-iz\,\Big|\,b,b^{-1}\right)\cdot\exp\left\{\pi i\left(z^2+\frac{b^2+b^{-2}}{12}\right)\right\}$$

$\Theta_1(z)$ $\quad$ $\wp(z)$

$\exp_q(x)$ $\qquad$ $\Gamma_q(x)$ $\qquad$ $|q|<1$

Physics asks for $|q|=1$

$$\varphi_b(z) \simeq \frac{\exp_q(z)}{\exp_{\tilde q}(z)} \qquad |q|\to 1$$

$$\exp_q(x+y) = \exp_q(y)\exp_q(x) \qquad xy = qyx$$

Pentagon identity $\qquad [p,x] = \frac{1}{2\pi i}$ $\qquad x \to x\cdot$, $\; p = \frac{1}{2\pi i}\frac{\partial}{\partial x}$

$$\varphi_b(p)\cdot\varphi_b(x) = \varphi_b(x)\,\varphi_b(p+x)\,\varphi_b(p)$$

$\varphi_b(x)$ – main object in "quantum" cluster geometry

$$\varphi_b^{-1}(x)\,e^{2\pi bp}\,\varphi_b(x) = e^{2\pi bp} + e^{2\pi b(p+x)}$$

$$\varphi_b(p)\,e^{2\pi bx}\,\varphi_b^{-1}(p) = e^{2\pi bx} + e^{2\pi b(p+x)}$$

$z(1-z)y''_{zz} + (c-(a+b+1)z)y'_z - aby = 0$ hypergeom. equation

change of variables $z = -sh^2 t$

$\frac{d}{dt} = -sh\,2t \frac{d}{dz}$; $\frac{d}{dz} = -\frac{1}{sh\,2t}\frac{d}{dt}$ $\quad \frac{d^2}{dz^2} = \left(-\frac{1}{sh\,2t}\frac{d}{dt}\right)\left(-\frac{1}{sh\,2t}\frac{d}{dt}\right) =$

$= \frac{1}{sh^2 2t}\frac{d^2}{dt^2} - \frac{2ch\,2t}{sh^3 2t}\frac{d}{dt}$

$-sh^2 t(1+sh^2 t)\left[\frac{1}{sh^2 2t}y''_{tt} - \frac{2ch\,2t}{sh^3 2t}y'_t\right] - \left(\frac{c+(a+b+1)sh^2 t}{sh\,2t}\right)y'_t - aby = 0$

$-\frac{sh^2 2t}{4}\left[\frac{1}{sh^2 2t}y''_{tt} - \frac{2ch\,2t}{sh^3 2t}y'_t\right] + \left(-\frac{c}{sh\,2t} + \frac{(a+b+1)}{2}th\,t\right)y'_t - aby = 0$

$\frac{1}{4}y''_{tt} + \left[-\frac{1}{2}cth\,2t + \frac{c}{sh\,2t} + \frac{(a+b+1)}{2}th\,t\right]y' - aby = 0$

$\frac{1}{sh\,2t} = \frac{ch^2 - sh^2}{2\,sh\,ch} = \frac{1}{2}cth\,t - \frac{1}{2}th\,t$

$cth\,2t = \frac{ch^2 + sh^2}{2\,sh\,ch} = \frac{1}{2}cth\,t + \frac{1}{2}th\,t$

$\frac{1}{4}y''_{tt} + \underbrace{\left[\frac{g}{2}cth\,t + \frac{g'}{2}th\,t\right]}\,y'_t + aby = 0$

$g = c - \frac{1}{2}$

$g' = a+b-c+\frac{1}{2}$

$y = W(t)\cdot\psi(t)$

$w = sh^{-g}t\cdot ch^{-g'}t \qquad w' = -(g\,cth\,t + g'\,th\,t)w$

$w'' = \left((g\,cth\,t + g'\,th\,t)^2 + \left(\frac{g}{sh^2 t} - \frac{g'}{ch^2 t}\right)\right)w$

$\frac{1}{4}\left(w''\psi + \cancel{2w'\psi'} + w\psi''\right) + \frac{1}{2}(g\,cth\,t + g'\,th\,t)(w'\psi + \cancel{w\psi'}) + \underset{ab}{\psi w} = 0 \,/ w$

$\frac{1}{4}\psi'' + (\overset{\text{"}V}{\qquad\qquad})\psi = 0$

$(\qquad\qquad) = \frac{1}{4}(g\,cth\,t + g'\,th\,t)^2 + \frac{1}{4}\left(\frac{g}{sh^2 t} - \frac{g'}{ch^2 t}\right) - \frac{1}{2}(g\,cth\,t + g'\,th\,t)^2$

$+ab$

$V = -\frac{1}{4}\left(\frac{g(g-1)}{sh^2 t} + \frac{g'(g'-1)}{4ch^2 t} - \frac{(g+g')^2}{4} - ab\right)$ $\qquad g + g' = a+b$

$$-\psi''_{tt} + \left[\frac{g(g-1)}{\operatorname{sh}^2 t} - \frac{g'(g'-1)}{\operatorname{ch}^2 t}\right]\psi = (a-b)^2\psi = -\lambda^2\psi$$

$$a = \frac{g+g'+i\lambda}{2} \qquad b = \frac{g+g'-i\lambda}{2}$$

Spectral problem for Schrödinger operator

$$\left| -\Delta\psi + V(t)\psi = -\lambda^2\psi \right| \qquad \Delta = \frac{\partial^2}{\partial t^2}$$

$$V(t) = \frac{g(g-1)}{\operatorname{sh}^2 t} - \frac{g'(g'-1)}{\operatorname{ch}^2 t}$$

BC_1 – Heckman – Opdam hypergeom. function

makes sense for any root systems

$A_n \quad (g\,e_n) \qquad V(t) = \frac{g(g-1)}{2} \sum_{i\neq j} \frac{1}{\operatorname{sh}^2(t_i - t_j)} \qquad \frac{1}{(t_i-t_j)^2}$

Calogero – Moser Sutherland system.

$B_n \quad C_n \qquad BC_n \qquad e_i \pm e_j,\ \pm e_i,\ \pm 2e_i \qquad \frac{1}{|t_i - t_j|}$

Geometric origin: for special values of coupling constants $g,\ g'$

zonal spherical function on symmetric spaces

$$\psi_\lambda(t) = |\operatorname{sh}^g t|\,|\operatorname{ch}^{g'} t|\; F_{21}\left(\frac{g+g'+i\lambda}{2}, \frac{g+g'-i\lambda}{2}; g+\frac{1}{2}; -\operatorname{sh}^2 t\right)$$

– wave function of BC_1 Calogero system.

Usually integrable systems admits bispectral solution.

Conjugancy relations.

$$\boxed{\begin{matrix} F(a,b;c;z) \\ F(a+1,b;c;z) \\ F(a,b,c+1,z) \end{matrix}} \qquad F(a+1, b, c; z) \atop b+$$

$\forall$ 3 conjug. F are linearly dependent $/\ \mathbb{C}[z]$

/ Разностн. ур-ние 2 порядка

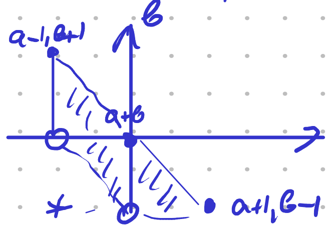
$$\frac{(g+g'+i\lambda)(\frac{1}{2}+g-g'-i\lambda)}{2i\lambda(2i\lambda-1)}\left(\psi_{i\lambda+1}(t) - \psi_{i\lambda}(t)\right) +$$

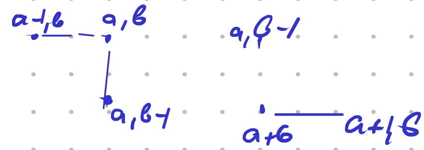
$$+ \ (\lambda \leftrightarrow -\lambda) = -\operatorname{sh}^2 x\, \psi'_{i\lambda}(t)$$

1) $F(a+) = F(a+1, b; c; z)$
$F(b-) \quad F(a, b-1, c; z)$

Difference equation – linear dependence of

$F(a+1, b-1; c, z)$; $F(a-1, b+1; c; z)$ and $F(a, b; c; z)$ / over $\mathbb{C}(z)$





1) $a\,F(a_+) - F = z\frac{d}{dz}F = b\,F(b_+) - F$

Double sine functions, dilogarithm...

Ruijsenaars:
$$\sqrt{1-\frac{v^2}{c^2}}$$

In relativistic model Schrödinger eq. turns to be a difference equation

the same for dual equation on λ

The solution is given by integrals over product of sine functions.

$$F = \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)} x^{-t}\,dt \qquad \Gamma(t) \rightsquigarrow S_2^{-1}(t|\omega_1,\omega_2)$$