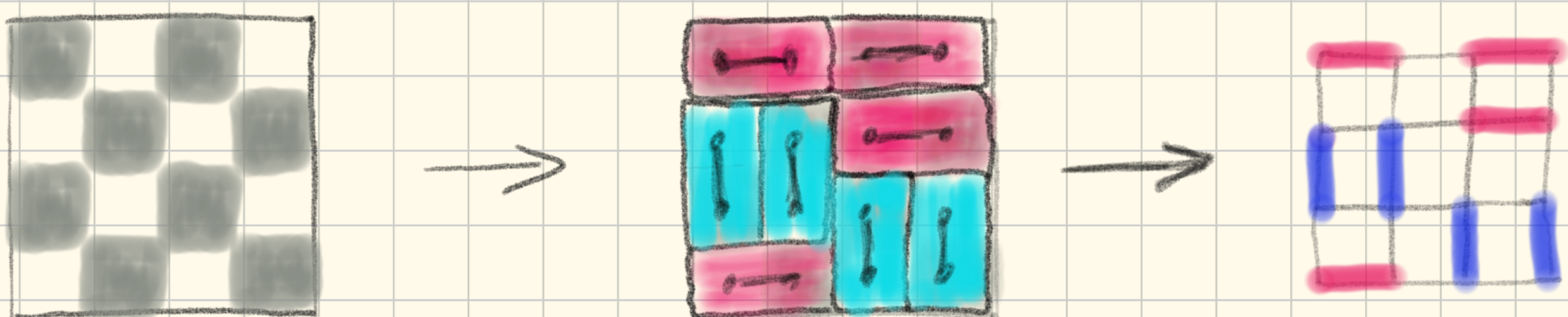
Dimers and tilings.

- In how many ways can one tile a chessboard with dominoes?



Chessboard — Domino tiling — Dimer packing

Entropy: $S = \ln(\#\,\text{tilings}) = ?$

Let $G = (V, E)$ - a simple graph $|V| = 2N$

Dimer configuration (perfect matching):

$G \supset d = (V, E^d)$ $E^d \subset E$ $\deg_d(v) = 1$ $\forall v \in V$

Let: $V = \{1, \dots, 2N\}$

$$E^d = \{(\sigma_1, \sigma_2), \dots, (\sigma_{2N-1}, \sigma_{2N})\}$$

$\sigma \in S_{2N}$ $\sigma_1 < \sigma_3 < \dots < \sigma_{2N-1}$, $\sigma_1 < \sigma_2, \dots, \sigma_{2N-1} < \sigma_{2N}$

$\mathcal{D}(G) = \{d\}_{d \subset G}$ - the set of dimer configurations.

We are interested in $|\mathcal{D}(G)| = ?$

Introduce edge weight: $w : V \times V \to \mathbb{R}_+$

For $i, j \in E$ $w((i,j)) = w((j,i)) \neq 0$ iff $(i,j) = (j,i) \in E$

$W : \mathcal{D}(G) \to \mathbb{R}_+$ $\mathcal{D}(G) \ni d \longrightarrow W(d) = \prod_{e \in E^d} w(e)$

$Z_G(w) = \sum_{d \in \mathcal{D}(G)} W(d)$ - partition function (generating)

$Z_G(1) = |\mathcal{D}(G)|$

Kasteleyn theory:

$$Z_G(w) = \sum_{\substack{\sigma \in S_{2N} \\ \sigma_1 < \sigma_3 \cdots < \sigma_{2N-1} \\ \sigma_1 < \sigma_2, \cdots, \sigma_{2N-1} < \sigma_{2N}}} w((\sigma_1, \sigma_2)) \cdots w((\sigma_{2N-1}, \sigma_{2N})) =$$

$$w((i,j)) = w((j,i)) \qquad i,j \in V$$

$$= \frac{1}{N!\, 2^N} \sum_{\sigma \in S_{2N}} w((\sigma_1, \sigma_2)) \cdots w((\sigma_{2N-1}, \sigma_{2N}))$$

Pfaffian:

Let $A = -A^T \in \mathbb{R}^{2N \times 2N}$ - skew symmetric matrix

Then, $\det A = (\mathrm{Pf}\, A)^2$, $\mathrm{Pf}(A)$ - polynomial in mat. coeff.

Def 1: $\mathrm{Pf}\, A = \mathrm{Pf}(\{a_{ij}\}, i<j) =$

$$= \sum_{\pi \in P_2(2N)} (-1)^{|\pi|} a_{\overline{\pi_1}\,\overline{\pi_2}} \cdots a_{\overline{\pi_{N-1}}\,\overline{\pi_N}}$$

$P_2(2N)$ - set of perfect matchings of $2N$ elements

$|\pi|$ - number of crossings in the link diagrams o π

π

$|\pi| = 1$

Def 2: $$\mathrm{Pf}(A) = \sum_{\substack{\sigma \in S_{2N} \\ \sigma_1 < \sigma_3 < \dots < \sigma_{2N-1} \\ \sigma_1 < \sigma_2, \dots, \sigma_{2N-1} < \sigma_{2N}}} (-1)^{|\sigma|} a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{2N-1}, \sigma_{2N}}$$

$$= \frac{1}{N!\, 2^N} \sum_{\sigma \in S_{2N}} (-1)^{|\sigma|} a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{2N-1} \sigma_{2N}}$$

If we define a skew symmetric matrix $K \in \mathbb{R}^{2N \times 2N}$ (Kastelegn matrix) such that $|K_{ij}| = w((i,j))$ and for $\forall\, \sigma \in S_{2N}$

$$\operatorname{sign}\left(\prod K_{\sigma_1 \sigma_2} \cdots K_{\sigma_{2N-1}\sigma_{2N}}\right) = (-1)^{|\sigma|} \left(\text{or } -(-1)^{|\sigma|}\right)$$

$$Z_G(w) = |\operatorname{Pf}(G)| = \sqrt{|\det K|}$$

Introduce orientation K on $G^K = (V, E^K)$

$$\varepsilon^K_{ij} = \begin{cases} 1 & \text{if } \exists e = (i,j) \in \vec{E} \\ -1 & \text{if } \exists e = (j,i) \in \vec{E} \\ 0 & \text{otherwise} \end{cases}$$

$\varepsilon^K : V \times V \to \{0, \pm 1\}$ $\quad \varepsilon^K : (i,j) \to \varepsilon^K_{ij}$

<u>Def.</u> (Kasteleyn matrix): $K_{ij} = \varepsilon^K_{ij} \cdot W((i,j))$

$$\operatorname{Pf} K = \sum_{d \in \mathcal{D}(G)} \varepsilon(d)\, W(d)$$

$\varepsilon(d) = \operatorname{sgn} \sigma \cdot \varepsilon^K_{\sigma_1 \sigma_2} \cdots \varepsilon^K_{\sigma_{2N-1} \sigma_{2N}}$ for $d = \{(\sigma_1, \sigma_2), \ldots, (\sigma_{2N-1}, \sigma_{2N})\}$

Let $d, d' \in \mathcal{D}'(G)$ $\quad d = ((\sigma_1, \sigma_2), \dots, (\sigma_{2w-1}, \sigma_{2w}))$

$\sigma = \tau \circ \sigma'$ $\quad d' = ((\sigma_1', \sigma_2'), \dots, (\sigma_{2w-1}', \sigma_{2w}'))$

τ decomposes into cycles of even length

$\tau = c_1 \cdots c_m \qquad \operatorname{sgn} c_i = -1 \qquad \operatorname{sgn} \tau = (-1)^m$

$$\varepsilon(d) \cdot \varepsilon(d') = \prod_{i=1}^{m} (-1)^{n^k(c_i) - 1}$$

$n^k(c_i)$ = # of edges of c_i of orientation opposite to the one in K

$n^K(C_i)$ depends only on orientation K and cycle C_i

If $n^K(C)$ is odd for any cycle of even length, such that $G \setminus C_i$ admits dimer cover, then $\varepsilon(d)$ have the same sign for $\forall d \in \mathcal{D}(G)$ and

$$\mathrm{Pf}\, K = \varepsilon(d_0) \cdot Z_G(w)$$

Theorem (Kasteleyn)

Let G - planar graph and K-orient. s.t. every face of G is odd.

Then for $\forall$ C - cycle $n^K(C)$

$$(-1)^{n^K(C)} = -(-1)^{v(C)}$$

$v(C) = \#$(sites inside C)

$$\Gamma_{n+1} = \Gamma_n \,\Delta\, f$$

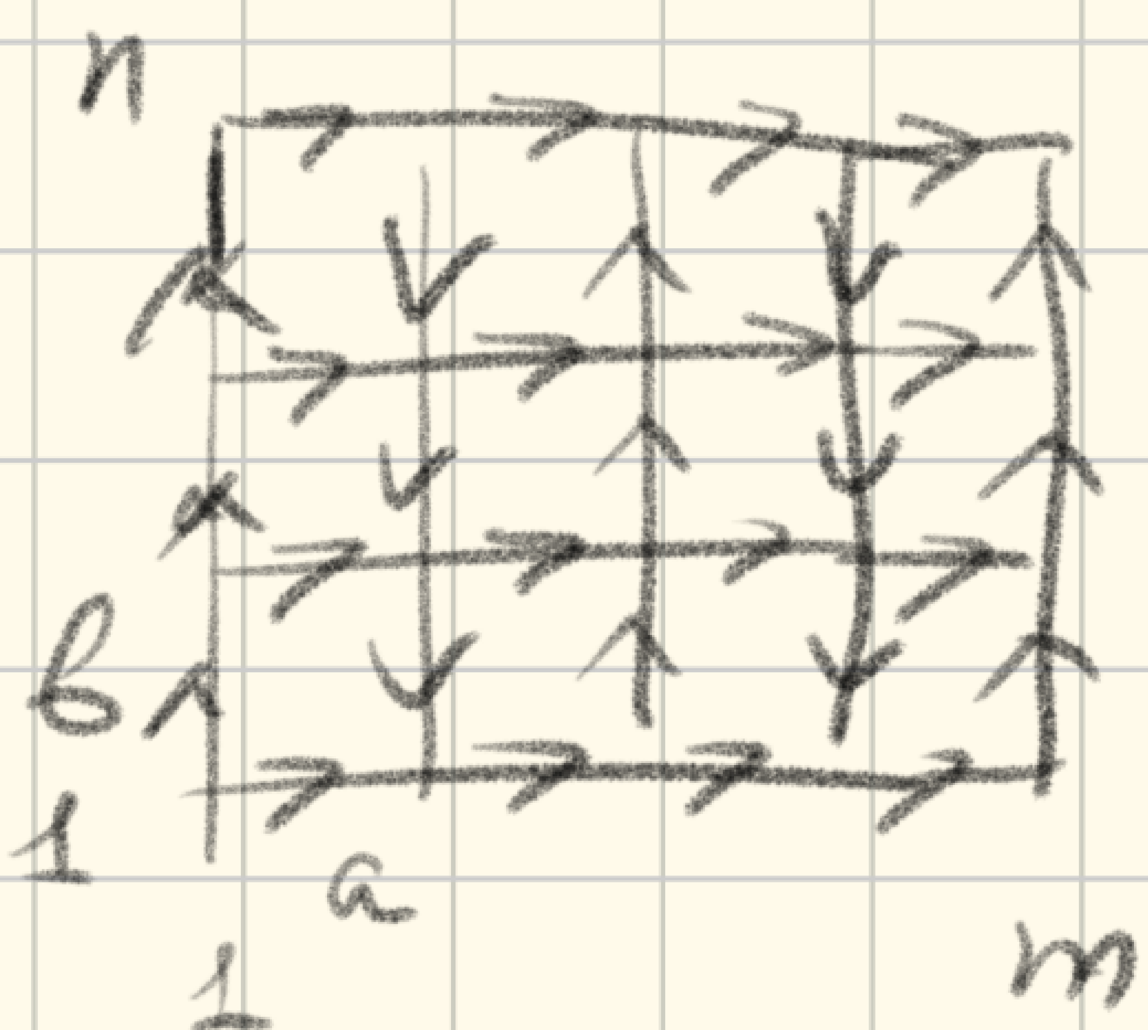
$$v(\Gamma_{n+1}) = v(\Gamma_n) + \beta - 1$$

$$n^k(\Gamma_{n+1}) = n^k(\Gamma_n) + n^k(f) - \beta$$

Since $v(\Gamma_n)$ и $n^k(\Gamma_n)$ have opposite parities and $n^k(f)$ - odd, the statement follows. $\square$

Consider a rect. subset of square lattice.

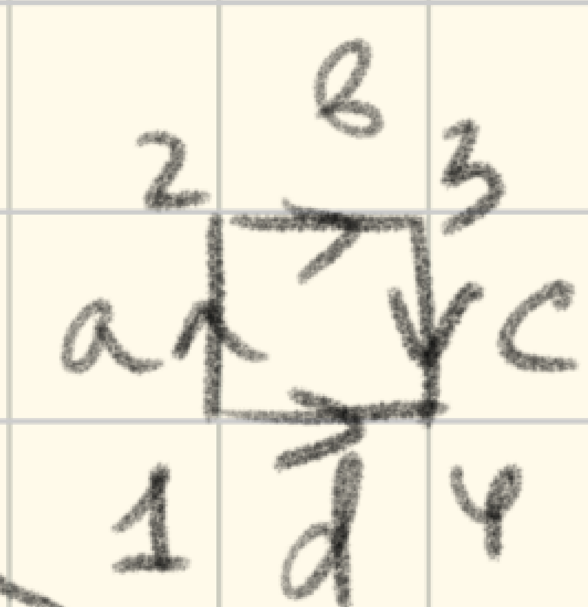
$V=\{1,\dots,n\}\times\{1,\dots,m\}$, s.t. $n\times m\in 2\mathbb{N}$



$m=2\tilde{m}$

$$\det K=\prod_{k_1=1}^{m/2}\prod_{k_2=1}^{n}\left(4a^2\cos^2\frac{\pi k_1}{m+1}+4b^2\cos^2\frac{\pi k_2}{n+1}\right)$$

basic example: $m=n=2$



$$K=\begin{array}{c|cccc} & 1 & 2 & 3 & 4\\ \hline 1 & \cdot & a & \cdot & d\\ 2 & -a & \cdot & b & \cdot\\ 3 & \cdot & -b & \cdot & c\\ 4 & -d & \cdot & -c & \cdot \end{array}$$

$$\underline{\det K=(ac+bd)^2}$$

$$Z=bd+ac$$

$f(a,b) =$

$$= \lim_{n,m\to\infty} \frac{1}{nm} \ln Z_{n,m}(a,b) = \frac{1}{\pi^2} \int_0^{\pi/2}\int_0^{\frac{\pi}{2}} \ln\left(4a^2\cos^2\varphi + 4b^2\cos^2\psi\right) d\varphi\, d\psi$$

Residual entropy of dimers per site:

$$s = f(1,1) = e^{\frac{G}{\pi}} \qquad G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

$\uparrow$

Catalan constant.

Let G - bipartite planar graph.

$V = W \sqcup B \quad |W| = |B| = \frac{N}{2} \quad E = \{(w_i, b_i)\}$

Random dimers: $\quad a = b = 1$

$$\mathbb{P}(d) = \frac{W(d)}{Z(w)}$$

$w \equiv 1$ - uniform measure

Kasteleijn matrix: $K = \begin{array}{c|cc} & B & W \\ \hline B & \cdot & \tilde{K} \\ W & -\tilde{K}^T & \cdot \end{array}$

$\det K = (\det \tilde{K})^2$

$Z = \det \tilde{K}$

Prob. of local subconf.:

$$\mathbb{P}\left(\xi = \{(b_1, w_1), \dots, (b_k, w_k)\} \subset d\right) =$$

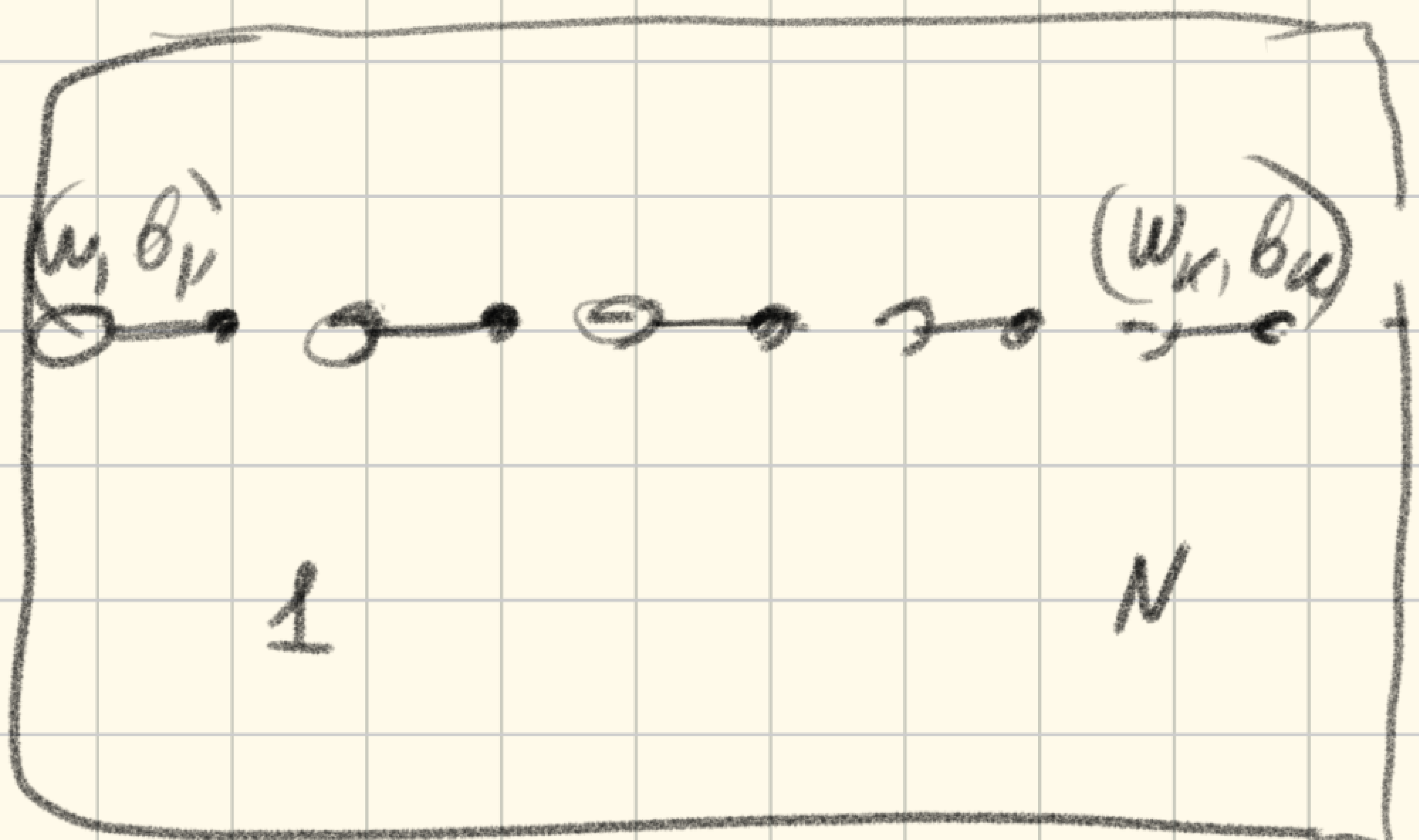
Jacoby formula $\quad \left(\det A^{-1}\right)_J^I = (-1)^{\Sigma I + \Sigma J} \dfrac{\det A_{\bar{I}}^{\bar{J}}}{\det A}$

$$= \frac{\det \tilde{K}\big|_{G \setminus \xi} \cdot \prod_i w(b_i w_i)}{\det \tilde{K}} = \det\left(\tilde{K}^{-1}\right)\Big|_{\xi} \prod w(b_i, w_i) =$$

$$= \det\left(\tilde{K}^{-1}(w_i, b_j)\, w(b_i, w_i)\right)_{i,j=1}^{k}$$

$$\mathbb{P}((b,w)\subset d) = W((b,w))\cdot K^{-1}(w,b)$$

$$A_{ij} = W(b_i w_j)\cdot K^{-1}_{b_i w_j}$$



$$k_{max} = \max\{k : (w_k, b_k)\in d\}$$

$$\mathbb{P}(k_{max}<a) = 1 - \sum_{i>a} A_{ii} + \frac{1}{2}\sum_{i,j>a} \det\begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj}\end{pmatrix} - \cdots$$

$$= \det(1-A)\big|_{i,j>a}$$

Let $W \equiv 1$, G - $n \times n$ - squar lattice

$$P\left(\frac{k_{max}}{n} < a\right) \to 0 \qquad n \to \infty$$

$n \times n$ Aztec diamond: $P\left(\frac{k_{max}}{n} < a\right) \to \theta(a - \sqrt{2})$

$$\mathbb{P}\left(\frac{k_{max} - \sqrt{2}\, n}{6\, n^{1/3}} < a\right) \to F_2(a) = \det(1 - K_{Airy})_{L_2(\mathbb{R}_a)}$$

↑ Tracy-widom.