Algebraic Bethe ansatz – 2025

Problems 1

1. Let the number of sites N of the Heisenberg XYZ spin chain be even. Show that
$$H^{\mathrm{xyz}}(J_x, J_y, -J_z) = -\mathcal{U} H^{\mathrm{xyz}}(J_x, J_y, J_z)\mathcal{U}^{-1},$$
where $\mathcal{U} = \sigma_z^{(2)}\sigma_z^{(4)}\sigma_z^{(6)}\ldots\sigma_z^{(N)}$.

2. Find the spectrum of the XYZ Heisenberg spin chain for $N=2$:
$$H = J_x\sigma_x^{(1)}\sigma_x^{(2)} + J_y\sigma_y^{(1)}\sigma_y^{(2)} + J_z\sigma_z^{(1)}\sigma_z^{(2)}.$$
Separately consider the cases of the XXX ($J_x = J_y = J_z$) and XXZ ($J_x = J_y \neq J_z$) models and compare with the solution obtained through the Bethe ansatz.

3. Find the spectrum of the XXX model on 3 sites with periodic boundary conditions:
$$H^{\mathrm{xxx}} = -\frac{1}{2}\sum_{k=1}^{3}\left(\vec{\sigma}^{(k)}\vec{\sigma}^{(k+1)} - 1\right), \qquad \vec{\sigma}^{(4)} \equiv \vec{\sigma}^{(1)}.$$
Find multiplicities of the energy levels.

4. Consider the XXX model on N sites with periodic boundary conditions:
$$H^{\mathrm{xxx}} = -\frac{1}{2}\sum_{k=1}^{N}\left(\vec{\sigma}^{(k)}\vec{\sigma}^{(k+1)} - 1\right), \qquad \vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}.$$
Let $\vec{S}$ be the vector of the total spin: $\vec{S} = \sum_j \vec{\sigma}^{(j)}$. Let $|\Psi_m\rangle$ be Bethe states with m magnons (i.e. eigenstates of the Hamiltonian such that $S_z|\Psi_m\rangle = (N-2m)|\Psi_m\rangle$ and the corresponding Bethe roots $\lambda_i < \infty$). Prove that $S_+|\Psi_m\rangle = 0$ for $m = 1, 2$. Try to prove this for $m > 2$ (this is a rather difficult problem).

5. For the Bethe states $|\Psi_m\rangle$ from the previous problem prove that any two different such states are orthogonal ($\left\langle \Psi'_{m'} \middle| \Psi_m \right\rangle = 0$ for all possible m, m' and find squares of their norms $\left\langle \Psi_m \middle| \Psi_m \right\rangle$ for $m = 1, 2, 3$.

6. Consider the "open" XXX spin chain with the Hamiltonian
$$H = \sum_{k=1}^{N-1}\vec{\sigma}^{(k)}\vec{\sigma}^{(k+1)}.$$
Show that the state with all spins up is an eigenstate. Find the eigenstates with m inverted spins for $m = 1, 2, 3$ (and try to do this for any m) and see how the Bethe equations are modified in this case.

7. Consider the XXX model on N sites with quasiperiodic (twisted) boundary conditions:
$$H = \sum_{k=1}^{N}\vec{\sigma}^{(k)}\vec{\sigma}^{(k+1)}, \qquad \sigma_\alpha^{(N+1)} \equiv U\sigma_\alpha^{(1)}, \quad \alpha = 1, 2, 3,$$
where U is a 2×2 diagonal matrix. Find Bethe states with the corresponding eigenvalues and see how the Bethe equations are modified in this case.