Algebraic Bethe ansatz – 2025

Problems 2

1. For the system of three identical Bose particles with the Hamiltonian

$$\hat{H}_3 = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1\le j<k\le 3} \delta(x_j - x_k)$$

   a) construct common eigenstates of the Hamiltonian and the total momentum

$$\hat{P} = -i \sum_{j=1}^{3} \frac{\partial}{\partial x_j}$$

   b) impose periodic boundary conditions on the segment $[0, L]$ and obtain the system of Bethe equations.

2. Find eigenstates and the energy spectrum of identical Bose particles with the Hamiltonian

$$\hat{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1\le j<k\le N} \delta(x_j - x_k) \qquad (N = 1, 2)$$

   for $N = 1$ (one free particle) and $N = 2$ (two particles interacting via the delta-functional potential) on the segment $[0, L]$ with impenetrable walls (i.e. such that the wave function vanishes if at least one particle is at the endpoints of the segment). How the Bethe equations are modified in this case?

3. Consider the system of N particles on the segment $[0, L]$ with periodic boundary conditions. Let the particles be different: ith particle carries its own charge e_i. The Hamiltonian is:

$$\hat{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{1\le j<k\le N} e_j e_k \delta(x_j - x_k)$$

   Find the energy spectrum (i.e., solve the stationary Schrodinger equation) for $N = 2$, and then try to do this for $N = 3$. If you fail, try to understand why: what is the key difficulty comparing to the Bethe ansatz solution for identical particles.

4. For the Bethe states of the system of two identical Bose particles with the Hamiltonian

$$\hat{H} = -\sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2} + 2c\delta(x_1 - x_2)$$

   with periodic boundary conditions on the segment $[0, L]$ find the square norm of the eigenfunctions and prove that any two different eigenfunctions are orthogonal.