Algebraic Bethe ansatz – 2025

Problems 3

1. Consider the system of N "classical spins" σ_i (placed at sites $i = 1, \ldots, N$ of one-dimensional lattice) which take two values: $\sigma_i = \pm 1$. The energy of the configuration $\{\sigma_i\} = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ is

$$E(\{\sigma_i\}) = J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} + H \sum_{i=1}^{N} \sigma_i,$$

   and periodic boundary condition is assumed, i.e., $\sigma_{N+1} \equiv \sigma_1$. (This is the one-dimensional Ising model.) Assuming that the system is at temperature T find the partition function. (Hint: the problem is most elegantly solved by introducing a transfer matrix.)

2. In the linear space $(\mathbb{C}^2)^{\otimes N}$ consider the operators

$$\mathbf{H}_j = \sum_{k=1, \neq j}^{N} \frac{P_{jk}}{x_i - x_k}, \quad j = 1, \ldots, N,$$

   where x_i are arbitrary (complex) numbers and P_{jk} is the operator acting as permutation of the jth and kth tensor factors.

   a) Prove that these operators commute with each other: $[\mathbf{H}_j, \mathbf{H}_l] = 0$.

   b) Find their common eigenvectors and the corresponding eigenvalues.

   (Hint: This model is the simplest version of the Gaudin model. Consider it as a limiting case of the inhomogeneous XXX spin chain.)

3. Prove that the transfer matrix of the 6-vertex model commutes with the operator of cyclic shift $(n \to n+1)$ and the operator

$$S_z = \sum_j \sigma_z^{(j)}.$$

4. Consider the 6-vertex model on thee $N \times N$ lattice with the R-matrix

$$\mathsf{R} = \mathsf{R}(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

   where $a = \sinh(u + \eta)$, $b = \sinh u$, $c = \sinh \eta$ and the quantum monodromy matrix

$$\mathcal{T}(u) = \mathsf{R}_{10}(u) \mathsf{R}_{20}(u) \ldots \mathsf{R}_{N0}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

   Find the scalar product $\langle \Omega | C(v) B(u) | \Omega \rangle$. Here $|\Omega\rangle = |+++\ldots+\rangle$.

5. Let $\mathsf{R}_{0j}(u) = \mathbf{1}u + \mathsf{P}_{0j}$ be the R-matrix of the XXX model (P_{0j} is the permutation operator). Consider the transfer matrix of the XXX model

$$\mathsf{T}(u) = \mathrm{tr}_0 \Big(\mathsf{R}_{01}(u)\mathsf{R}_{02}(u)\ldots\mathsf{R}_{0N}(u)\Big).$$

Find $\partial_u^2 \log \mathsf{T}(u)\Big|_{u=0}$.

6. Find all commutation relations between the operators $A(u)$, $B(u)$, $C(u)$, $D(u)$ which are encoded in the intertwining relation $\mathsf{R}_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathsf{R}_{12}(u-v)$ with the 6-vertex R-matrix.

7. Prove that the quantity

$$\det_q \mathcal{T}(u) = A(u+\eta)D(u) - B(u+\eta)C(u)$$

is a central element of the algebra generated by elements of the quantum monodromy matrix, i.e., that $\det_q \mathcal{T}(u')$ (the quantum determinant) commutes with $A(u)$, $B(u)$, $C(u)$, $D(u)$ for all u, u'.