

Algebraic Bethe ansatz – 2026

List of problems

1. 1 point

Find the spectrum of the XYZ Heisenberg spin chain for $N = 2$:

$$H = J_x \sigma_x^{(1)} \sigma_x^{(2)} + J_y \sigma_y^{(1)} \sigma_y^{(2)} + J_z \sigma_z^{(1)} \sigma_z^{(2)}.$$

Separately consider the cases of the XXX ($J_x = J_y = J_z$) and XXZ ($J_x = J_y \neq J_z$) models and compare with the solution obtained through the Bethe ansatz.

2. 1 point

Find the spectrum of the XXX model on 3 sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^3 \left(\vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \quad \vec{\sigma}^{(4)} \equiv \vec{\sigma}^{(1)}.$$

Find multiplicities of the energy levels.

3. 2+4 point

Consider the XXX model on N sites with periodic boundary conditions:

$$H^{\text{xxx}} = -\frac{1}{2} \sum_{k=1}^N \left(\vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)} - 1 \right), \quad \vec{\sigma}^{(N+1)} \equiv \vec{\sigma}^{(1)}.$$

Let $\vec{S}$ be the vector of the total spin: $\vec{S} = \sum_j \vec{\sigma}^{(j)}$. Let $|\Psi_m\rangle$ be Bethe states with m magnons (i.e. eigenstates of the Hamiltonian such that $S_z |\Psi_m\rangle = (N - 2m) |\Psi_m\rangle$ and the corresponding Bethe roots $\lambda_i < \infty$). Prove that $S_+ |\Psi_m\rangle = 0$ for $m = 1, 2$.

Try to prove this for $m > 2$ (this is a rather difficult problem).

4. 3 points

For the Bethe states $|\Psi_m\rangle$ from the previous problem prove that any two different such states are orthogonal i.e. $\langle \Psi_{m'} | \Psi_m \rangle = 0$ for all possible m, m' and find squares of their norms $\langle \Psi_m | \Psi_m \rangle$ for $m = 1, 2$.

5. 2 points

Consider the “open” XXX spin chain with the Hamiltonian

$$H = \sum_{k=1}^{N-1} \vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)}.$$

Show that the state with all spins up is an eigenstate. Find the eigenstates with m inverted spins for $m = 1, 2, 3$ (and try to do this for any m) and see how the Bethe equations are modified in this case.

6. 2 points

Consider the XXX model on N sites with quasiperiodic (twisted) boundary conditions:

$$H = \sum_{k=1}^N \vec{\sigma}^{(k)} \vec{\sigma}^{(k+1)}, \quad \sigma_{\alpha}^{(N+1)} \equiv U \sigma_{\alpha}^{(1)} U^{-1}, \quad \alpha = 1, 2, 3,$$

where U is a 2×2 diagonal matrix. Find Bethe states with the corresponding eigenvalues and see how the Bethe equations are modified in this case.

7. 2 points

Consider the Bose gas of two identical particles with a point-like interaction on the segment $[0, L]$ with periodic boundary conditions. The Hamiltonian is

$$H = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2)$$

with the coupling constant $c < 0$ (attraction). Find the wave function (eigenfunction of this Hamiltonian) of the bound state of the particles in the limit $L \rightarrow \infty$. (In the bound state, the wave function $\Psi(x_1, x_2)$ should decay exponentially as $|x_1 - x_2| \rightarrow \infty$).

8. 2+2 points

Find eigenstates and the energy spectrum of identical Bose particles with the Hamiltonian

$$\hat{H}_N = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k) \quad (N = 1, 2)$$

for $N = 1$ (one free particle) and $N = 2$ (two particles interacting via the delta-functional potential) on the segment $[0, L]$ with impenetrable walls (i.e. such that the wave function vanishes if at least one particle is at the endpoints of the segment). How the Bethe equations are modified in this case?

Proof it for $m > 2$ particles

9. 2 points

Consider the system of N particles on the segment $[0, L]$ with periodic boundary conditions. Let the particles be different: i th particle carries its own charge e_i . The Hamiltonian is:

$$\hat{H}_N = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} e_j e_k \delta(x_j - x_k)$$

Find the energy spectrum (i.e., solve the stationary Schrodinger equation) for $N = 2$, and then try to do this for $N = 3$. If you fail, try to understand why: what is the key difficulty comparing to the Bethe ansatz solution for identical particles.

10. 2+3 points

In the linear space $(\mathbb{C}^2)^{\otimes N}$ consider the operators

$$\mathbf{H}_j = \sum_{k=1, \neq j}^N \frac{P_{jk}}{x_i - x_k}, \quad j = 1, \dots, N,$$

where x_i are arbitrary (complex) numbers and P_{jk} is the operator acting as permutation of the j th and k th tensor factors.

a) Prove that these operators commute with each other: $[\mathbf{H}_j, \mathbf{H}_l] = 0$.

b) Find their common eigenvectors and the corresponding eigenvalues.

(Hint: This model is the simplest version of the Gaudin model. Consider it as a limiting case of the inhomogeneous XXX spin chain.)

11. 3 points

Consider a Nonlinear Schrödinger equation with the Hamiltonian

$$H_2 = \int dx (\Psi_x^\dagger(x) \Psi_x(x) + c(\Psi^\dagger(x))^2 (\Psi(x))^2) \quad (0.1)$$

where $[\Psi(x), \Psi(y)] = [\Psi^\dagger(x), \Psi^\dagger(y)] = 0$ and $[\Psi(x), \Psi^\dagger(y)] = \delta(x - y)$

Consider a Hamiltonian H_3

$$H_3 = \int dx (\Psi^\dagger(x) \Psi_{xxx}(x) + \kappa (\Psi^\dagger(x))^2 (\Psi(x))^2_x) \quad (0.2)$$

Find κ for which H_3 and H_2 commute and find corresponding second quantized version of this Hamiltonian. What are its eigenvalues?

12. 1 point

Prove that the transfer matrix of the 6-vertex model commutes with the operator of cyclic shift ($n \rightarrow n + 1$) and the operator

$$S_z = \sum_j \sigma_z^{(j)}.$$

13. 2 points

Consider the 6-vertex model on the $N \times N$ lattice with the R -matrix

$$R = R(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

where $a = \sinh(u + \eta)$, $b = \sinh u$, $c = \sinh \eta$ and the quantum monodromy matrix

$$\mathcal{T}(u) = R_{10}(u) R_{20}(u) \dots R_{N0}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Find the scalar product $\langle \Omega | C(v) B(u) | \Omega \rangle$. Here $|\Omega\rangle = |++++\dots\rangle$.

14. **2+2 points**

Let $R_{0j}(u) = \mathbf{1}u + P_{0j}$ be the R -matrix of the XXX model (P_{0j} is the permutation operator). Consider the transfer matrix of the XXX model

$$T(u) = \text{tr}_0 \left(R_{01}(u) R_{02}(u) \dots R_{0N}(u) \right).$$

a) Find $\partial_u^2 \log T(u) \Big|_{u=0}$.

b) Find eigenvalues of this operator using coordinate Bethe ansatz

15. **2 points**

Prove that the operator

$$R(x) = xI + P$$

acting in the tensor product of two linear spaces $V_1 \otimes V_2$ ($V_1 = V_2 = V$), where P is the permutation operator and x is any complex number satisfies the Yang-Baxter equation

$$R_{12}(x_1 - x_2) R_{13}(x_1 - x_3) R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3) R_{13}(x_1 - x_3) R_{12}(x_1 - x_2).$$

The both sides are operators in $V_1 \otimes V_2 \otimes V_3$ ($V_1 = V_2 = V_3 = V$) and $R_{ij}(x)$ means the matrix $R(x)$ acting as $R(x)$ in the space $V_i \otimes V_j$ and as the identity matrix in the rest space.

16. **2 points**

Find all commutation relations between the operators $A(u)$, $B(u)$, $C(u)$, $D(u)$ which are encoded in the intertwining relation $R_{12}(u - v) \mathcal{T}_1(u) \mathcal{T}_2(v) = \mathcal{T}_2(v) \mathcal{T}_1(u) R_{12}(u - v)$ with the 6-vertex R -matrix.

17. **3 points**

Prove that the quantity

$$\det_q \mathcal{T}(u) = A(u + \eta) D(u) - B(u + \eta) C(u)$$

is a central element of the algebra generated by elements of the quantum monodromy matrix, i.e., that $\det_q \mathcal{T}(u')$ (the quantum determinant) commutes with $A(u)$, $B(u)$, $C(u)$, $D(u)$ for all u, u' .

18. **2 points**

Let $\mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix from the group $GL(2)$. In the notation of the previous problem, prove that

$$R_{12}(x) \mathbf{g}_1 \mathbf{g}_2 = \mathbf{g}_1 \mathbf{g}_2 R_{12}(x).$$

19. **1 point**

Let $T_{\pm}$ be the shift operator acting to functions of a complex variable z as $T_{\pm}f(z) = f(q^{\pm 1}z)$. Prove that the operators

$$A = q^{-\ell} T_+,$$

$$B = \frac{z}{q^{-1} - q} (q^{-2\ell} T_+ - q^{2\ell} T_-),$$

$$C = \frac{z^{-1}}{q^{-1} - q} (T_- - T_+),$$

satisfy the commutation relation

$$[B, C] = \frac{A^2 - A^{-2}}{q - q^{-1}}$$

for all ℓ .

20. **2 points**

Let

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix},$$

where

$$a(u) = \sin(u + \eta), \quad b(u) = \sin u, \quad c(u) = \sin \eta, \quad d(u) = \gamma \sin \eta \sin(u + \eta) \sin u$$

be the R -matrix of the 7-vertex model, and

$$T(u) = \text{tr}_0 (R_{01}(u) R_{02}(u) \dots R_{0N}(u))$$

be the transfer matrix. Find

$$H = \partial_u \log T(u) \Big|_{u=0}.$$