Fermat manifolds in problems of algebraic topology, algebraic geometry and number theory

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Fermat hypersurfaces

Let

$$M_d^n = \{z_0^d + z_1^d + \cdots + z_{n+1}^d = 0\} \subset \mathbb{CP}^{n+1}$$

be a smooth Fermat hypersurface of complex dimension n and degree d.
Its normal bundle is

$$\nu(M_d^n \subset \mathbb{CP}^{n+1}) \cong \mathcal{O}(d).$$

The tangent bundle satisfies

$$c(TM_d^n) = \frac{(1+h)^{n+2}}{1+dh}, \; c_1(TM_d^n) = (n+2-d)h,$$

where $h = c_1(\mathcal{O}(1))|_{M_d^n} \in H^2(M_d^n; \mathbb{Z})$.

First Examples of Varieties M_d^n

We have the 2-parameter family of smooth, irreducible projective varieties:

$$M_d^n = \{z_0^d + \ldots + z_{n+1}^d = 0\} \subset \mathbb{C}P^{n+1}$$

$M_d^0 = d$;

$M_1^n = \mathbb{C}P^n, \ n \geqslant 1$;

M_d^1 is the curve of genus $(d-1)(d-2)/2, \ d \geqslant 1$;

$M_2^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$;

$M_2^4 = \mathbb{C}G(4,2)$;

M_{n+2}^n is the Calabi–Yau manifold, SU-manifold.

The Fermat quadric and the oriented Grassmannian

Let $M_2^n = Q^n = \{z_0^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{C}P^{n+1}$.
It is naturally diffeomorphic to the oriented Grassmannian of oriented real two-planes:

$$Q^n \cong \mathbb{R}G^+(n+2,2) = SO(n+2)/(SO(2) \times SO(n)).$$

The diffeomorphism is given by

$$\langle u, v\rangle \longmapsto [u+iv] \in \mathbb{C}P^{n+1},$$

where (u,v) is an oriented orthonormal basis of an oriented two-plane in $\mathbb{R}^{n+2}$.
Indeed,

$$(u+iv, u+iv) = |u|^2 - |v|^2 + 2i(u,v) = 0.$$

Conversely, if $z = x+iy \in Q^n$, then the equation $z_0^2 + \cdots + z_{n+1}^2 = 0$ implies

$$|x| = |y|, \qquad (x,y) = 0,$$

so x, y determine an oriented real two-plane.

Lefschetz hyperplane theorem

Let $h \in H^2(X_d^n, \mathbb{Z})$ be the restriction of the hyperplane class from $\mathbb{C}P^{n+1}$.

Theorem (S. Lefschetz)

For a smooth hypersurface $X_d^n \subset \mathbb{C}P^{n+1}$, the restriction map

$$H^k(\mathbb{C}P^{n+1}, \mathbb{Z}) \longrightarrow H^k(X_d^n, \mathbb{Z})$$

is an isomorphism for $k < n$ and is injective for $k = n$.
By Poincare duality the cohomology outside the middle degree is therefore the same as for projective space, up to the expected duality range.
The only new cohomology is the primitive middle cohomology

$$H^n_{\mathrm{prim}}(X_d^n) = \ker\{L : H^n(X_d^n) \to H^{n+2}(X_d^n)\},$$

where L is cup-product with the hyperplane class h.

Euler characteristic

Proposition.

The topological Euler characteristic of a smooth d-degree hypersurface $X_d^n \subset \mathbb{C}P^{n+1}$ is

$$\chi(X_d^n) = \frac{(1-d)^{n+2} - 1 + (n+2)d}{d}.$$

We have

$$c(TX_d^n) = \frac{(1+h)^{n+2}}{1+dh}.$$

Since $\int_{X_d^n} h^n = d$ and $\chi(X_d^n) = \int_{X_d^n} c_n(TX_d^n)$, we obtain the stated formula.

Corollary.

The middle Betti number is determined by $\chi(X_d^n)$ and the Lefschetz theorem. All dependence on the degree d beyond the ambient projective-space cohomology is concentrated in $H^n(X_d^n)$.

Cohomological projective spaces

Definition

A smooth manifold M^{2n} is called a cohomological $\mathbb{C}P^n$-manifold if $H^*(M;\mathbb{Z}) \cong H^*(\mathbb{C}P^n;\mathbb{Z})$ as Abelian groups.

The Lefschetz hyperplane theorem implies that for a smooth hypersurface $M_d^n \subset \mathbb{C}P^{n+1}$ all cohomology groups away from the middle dimension are the same as for projective space. The only possible difference occurs in degree n.

Theorem.

Among Fermat hypersurfaces M_d^n, $d > 1$, the cohomological $\mathbb{C}P^n$-manifolds are precisely the odd-dimensional quadrics

$$M_2^{2k+1} = Q^{2k+1}.$$

The case $d = 1$ gives the trivial example $M_1^n \cong \mathbb{C}P^n$, $n \in \mathbb{N}$.

Integral cohomology rings of Fermat quadrics

Theorem.

For odd-dimensional Fermat quadrics:

$$H^*(M_2^{2m+1};\mathbb{Z}) = \mathbb{Z}\{1, h, \dots, h^m, \alpha, h\alpha, \dots, h^m\alpha\},\ \ h^{m+1} = 2\alpha.$$

For even-dimensional Fermat quadrics:

$$H^*(M_2^{2m};\mathbb{Z}) = \mathbb{Z}\{1, h, \dots, h^{m-1}, a, b, h^{m+1}, \dots, h^{2m}\},\ \ h^m = a + b,$$

with middle products

$$\begin{cases} a^2 = b^2 = [\mathrm{pt}], \quad ab = 0, \quad m \text{ even}, \\ a^2 = b^2 = 0, \quad ab = [\mathrm{pt}], \quad m \text{ odd}, \end{cases}$$

where $[\mathrm{pt}] \in H^{2m}(M_2^{2m};\mathbb{Z}) = \mathbb{Z}$ is the fundamental cocycle.

The maximal $SO(n+2)$-torus action on M_2^n

Let $r = \left\lfloor \frac{n+2}{2} \right\rfloor$. A maximal torus $T^r \subset SO(n+2)$ acts by rotations in the coordinate real two-planes.
After complexification, the weights of the standard representation are

$$\pm u_1, \dots, \pm u_r,$$

and if $n+2$ is odd there is one additional zero weight.

The fixed points of the torus action on $Q^n \cong \mathbb{R}G^+(n+2,2)$ correspond to coordinate oriented two-planes.
Hence the fixed point set consists of two points for each coordinate two-plane:

$$p_1^+, \dots, p_r^+, \qquad p_1^-, \dots, p_r^-.$$

The sign records the choice of orientation of the corresponding coordinate two-plane.
The complexity of T^r-action on M_2^n is:

$$2m - r = m - 1 \text{ for } n = 2m, \qquad 2m + 1 - r = m \text{ for } n = 2m + 1.$$

$(\mathbb{C}^*)^r$-action on $\mathbb{C}P^{n+1}$

Set $r = \left\lfloor \frac{n+2}{2} \right\rfloor$ and let $T^r \subset SO(n+2)$ be a maximal torus.

For $n+2 = 2r$, the action of T^r on $\mathbb{C}P^{2r-1}$, induced by the action of $(\mathbb{C}^*)^r$, is

$$(t_1, \dots, t_r) \cdot [w_1^+ : w_1^- : \dots : w_r^+ : w_r^-] = [t_1 w_1^+ : t_1^{-1} w_1^- : \dots : t_r w_r^+ : t_r^{-1} w_r^-].$$

The quadric equation for $M_2^{2r-2} \subset \mathbb{C}P^{2r-1}$, where $w_k^+ = z_{k-1} + iz_k$, $w_k^- = z_{k-1} - iz_k$, $k = 1, \dots, r$, becomes $w_1^+ w_1^- + \dots + w_r^+ w_r^- = 0$.

The moment map $\mu : \mathbb{C}P^{2r-1} \to \mathbb{R}^r$ is

$$\mu([w]) = \frac{\sum_{j=1}^r (|w_j^+|^2 - |w_j^-|^2) e_j}{\sum_{j=1}^r (|w_j^+|^2 + |w_j^-|^2)}.$$

For $n+2 = 2r+1$: $w_0^2 + w_1^+ w_1^- + \dots + w_r^+ w_r^- = 0$ where an additional coordinate w_0 has weight 0. The moment map $\mu : \mathbb{C}P^{2r} \to \mathbb{R}^r$ is

$$\mu([w]) = \frac{\sum_{j=1}^r (|w_j^+|^2 - |w_j^-|^2) e_j}{|w_0|^2 + \sum_{j=1}^r (|w_j^+|^2 + |w_j^-|^2)}.$$

Equivariant embedding

Under the moments map the fixed points are mapped to $\pm e_1, \dots, \pm e_r$.
Hence the moment polytope is the r-dimensional cross-polytope

$$\Delta(Q^n) = \mathrm{conv}\{\pm e_1, \dots, \pm e_r\}.$$

With respect to the action of the maximal torus $T^r \subset SO(n+2)$ on $\mathbb{C}P^{n+1}$, the embedding $M_2^n \subset \mathbb{C}P^{n+1}$ is equivariant over the same moment polytope, the cross-polytope $\Delta(Q^n)$.
The standard moment polytope of $\mathbb{CP}^{N-1}$ is the simplex

$$\Delta_{N-1} = \mathrm{conv}\{\varepsilon_1, \dots, \varepsilon_N\}.$$

For $N = 2r$, define $\pi : \mathbb{R}^{2r} \to \mathbb{R}^r$ by

$$\pi(x_1^+, x_1^-, \dots, x_r^+, x_r^-) = (x_1^+ - x_1^-, \dots, x_r^+ - x_r^-).$$

Then

$$\pi(\varepsilon_j^+) = e_j, \qquad \pi(\varepsilon_j^-) = -e_j,$$

and therefore

$$\pi(\Delta_{2r-1}) = \mathrm{conv}\{\pm e_1, \dots, \pm e_r\}.$$

For $N = 2r + 1$, one can use projection $\mathbb{R}^{2r+1} \to \mathbb{R}^r$ for the similar result.

Formal group of geometric cobordisms

Theorem (S.P. Novikov, A.S. Mischenko, 1967)

The formal series $F_U(u,v) \in U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \subset \Omega[[u,v]]$

$$F_U(u,v) = c_1^U(\xi_1 \otimes_{\mathbb{C}} \xi_2) = u + v + \sum_{i,j} a_{i,j} u^i v^j, \ a_{i,j} \in \Omega_U^{-2(i+j-1)},$$

where $u = c_1^U(\xi_1), \ v = c_1^U(\xi_2)$, defines a formal group over Ω_U.

$F_U(u,v)$ is called the formal group of geometric cobordisms.

Theorem (A.S. Mischenko, 1967)

The logarithm of the group $F_U(u,v)$ is given by the series

$$g(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}, \quad \text{i.e. } g\big(F_U(u,v)\big) = g(u) + g(v).$$

$$\mathbb{C}P(u) \cdot \frac{\partial}{\partial v} F(u,v) \Big|_{v=0} = 1, \quad \text{where} \quad \mathbb{C}P(u) = 1 + \sum_{n=1}^{\infty} [\mathbb{C}P^n] u^n.$$

The addition law in the formal group of $U^*(\cdot)$

Theorem (V.M. Buchstaber, 1970)

The series $F_U(u,v)$ is given by the formula

$$F_U(u,v) = \frac{u+v+\sum_{i,j}[H_{i,j}]u^iv^j}{\mathbb{C}P(u)\mathbb{C}P(v)},\ i \geqslant 0,\ j \geqslant 0,\ i+j \geqslant 2,$$

where $\mathbb{C}P(u) = 1 + \sum_{n=1}^{\infty}[\mathbb{C}P^n]u^n$ and $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ are smooth irreducible algebraic Milnor hypersurfaces $H_{i,j} = \{\sum_{k \geqslant 0} z_k w_k = 0\}$.

Corollary.

The coefficients $a_{i,j}$ of the series $F_U(u,v)$ generate the ring Ω_U.

Multiplicative generators of the ring Ω_U

$$\text{Set} \quad m(k) = \begin{cases} p, & \text{if } k = p^q, \\ 1, & \text{if } k \neq p^q, \end{cases}$$

where p is a prime number and $q \in \mathbb{N}$. Let us introduce the polynomials

$$L_k(u, v) = \frac{1}{m(k)}[(u+v)^k - u^k - v^k].$$

Lemma

Let $\{a_1, \ldots, a_n, \ldots\}$ be some set of multiplicative generators of the ring Ω_U. Then

$$F(u, v) \approx u + v + \sum_{k \geqslant 2} a_{k-1} L_k(u, v),$$

where "$\approx$" is the equality symbol in the ring $\Omega_U/(\widetilde{\Omega}_U)^2$.

Corollary

$$F(u, u) \approx 2u + \sum_{k \geqslant 2} \frac{2}{m(k)}(2^{k-1} - 1) a_{k-1} u^k.$$

Generating series of classes $[M^n_d]$

Let's introduce a power system $\{\varphi_k(u) \in \Omega_U[[u]], k = 0, 1, \ldots\}$ such that $\varphi_0(u) = 0$, $\varphi_{k+1}(u) = F_U(u, \varphi_k(u))$, $k \geqslant 0$.
Note that $\varphi_k(u) = ku + \ldots$ and $\varphi_k(\varphi_l(u)) = \varphi_{kl}(u)$ for $k, l \in \mathbb{N}$.

Theorem (V.M. Buchstaber, 1970)

For $d \geqslant 1$: $$\sum_{n \geqslant 0} [M^n_d] u^{n+1} = \varphi_d(u) \mathbb{C}P(u).$$

For $d = 2$: $$\sum_{n \geqslant 0} [M^n_2] u^{n+1} = F(u, u) \mathbb{C}P(u).$$

Corollary

1. The set of classes $\{[M^n_1] = [\mathbb{C}P^n], n \neq 2^q - 1;\ [M^n_2], n = 2^q - 1, q \in \mathbb{N}\}$ multiplicatively generates the ring $\Omega_U \otimes \mathbb{Z}_{\mathrm{odd}}$ where $\mathbb{Z}_{\mathrm{odd}} \subset \mathbb{Q}$ is the subring with odd denominators.
2. The ring Ω_U is multiplicatively generated by the classes $[M^n_1]$ and $[M^n_p]$, where p runs over all prime numbers.

The Buchstaber–Panov–Ray localization formula

Let M^{2N} be a stably complex manifold with an effective action of a torus T and isolated fixed points. The universal toric genus is given by the localization formula

$$\Phi_T(M) = \sum_{p \in M^T} \sigma(p) \prod_{j=1}^{N} \frac{1}{[w_j(p)]_F},$$

where

- $w_1(p), \ldots, w_N(p)$ are the weights of the tangent representation at p,
- F is the formal group of geometric cobordism,
- $[w]_F$ denotes the first U-Chern class of the character of weight w,
- $\sigma(p) = \pm 1$ is the sign determined by the stable complex orientation.

For the complex quadric M_2^n with its standard complex structure $\sigma(p) = +1$.
After cancellation of poles, $\Phi_T(M_2^n)$ lies in $U^*(BT)$.
Forgetting the torus action gives the class $[M_2^n] \in \Omega_U^{-2n}$.

The even-dimensional quadric $M_2^{2m} = Q^{2m}$

Let $Q^{2m} = M_2^{2m} \subset \mathbb{C}P^{2m+1}$. Then $Q^{2m} \cong SO(2m+2)/(SO(2) \times SO(2m))$.

The maximal torus has rank $m+1$, and the standard weights are $\pm u_1, \ldots, \pm u_{m+1}$. The fixed points are $p_1^+, \ldots, p_{m+1}^+,\ p_1^-, \ldots, p_{m+1}^-$. Thus we have $2m+2$ fixed points.

At the fixed point p_j^+, the tangent weights are $u_k - u_j,\ -u_k - u_j,\ k \neq j$. There are $2m$ such weights, as required for a complex manifold of $\dim 2m$.

At the fixed point p_j^-, the tangent weights are obtained by replacing u_j by $-u_j$. Thus, we get: $u_k + u_j,\ -u_k + u_j,\ k \neq j$.

Therefore the universal toric genus of Q^{2m} is

$$\Phi_T(Q^{2m}) = \sum_{j=1}^{m+1} \left[\prod_{k \neq j} \frac{1}{F(u_k, \bar{u}_j) F(\bar{u}_k, \bar{u}_j)} + \prod_{k \neq j} \frac{1}{F(u_k, u_j) F(\bar{u}_k, u_j)} \right],\ F(u, \bar{u}) = 0.$$

After forgetting the equivariant parameters, this expression gives

$$[Q^{2m}] = [M_2^{2m}] \in \Omega_U^{-4m}.$$

Example: $M_2^4 \cong \mathbb{C}G(2,4)$

For $m = 2$, $M_2^4 = Q^4 \subset \mathbb{C}P^5$. This quadric is isomorphic to the complex Grassmannian $Q^4 \cong \mathbb{C}G(2,4)$ via the Plücker embedding.

The maximal torus has rank 3, and there are six fixed points.
Equivalently, for the standard torus of $U(4)$, the fixed points of $\mathbb{C}G(2,4)$ are indexed by coordinate two-planes

$$p_{ij} = \langle e_i, e_j \rangle, \quad 1 \leqslant i < j \leqslant 4.$$

At p_I, $I = \{i, j\}$, the weights are

$$u_b - u_a, \qquad a \in I, \quad b \notin I.$$

Thus the localization formula can also be written as

$$\Phi_T(\mathbb{C}G(2,4)) = \sum_{|I|=2} \prod_{a \in I, b \notin I} \frac{1}{F(u_b, \bar{u}_a)}, \; F(u, \bar{u}) = 0.$$

This is equivalent to the preceding orthogonal formula for Q^4.

The odd-dimensional quadric $M_2^{2m+1} = Q^{2m+1}$

Let $Q^{2m+1} = M_2^{2m+1} \subset \mathbb{C}P^{2m+2}$. Then $Q^{2m+1} \cong SO(2m+3)/(SO(2) \times SO(2m+1))$.

The maximal torus has rank $m+1$. The standard representation of $SO(2m+3)$ has weights $\pm u_1, \dots, \pm u_{m+1}$ and one additional zero weight. The fixed points are $p_1^+, \dots, p_{m+1}^+,\ p_1^-, \dots, p_{m+1}^-$. Thus the number of fixed points is $2m+2$.

At p_j^+, the weights are $u_k - u_j,\ -u_k - u_j,\ k \neq j,\ -u_j$. The last weight $-u_j$ comes from the additional zero weight in the standard representation of $SO(2m+3)$.

At p_j^-, the weights are $u_k + u_j,\ -u_k + u_j,\ k \neq j,\ u_j$. Hence

$$\Phi_T(Q^{2m+1}) = \sum_{j=1}^{m+1} \left[\frac{1}{\bar{u}_j \prod\limits_{k \neq j} F(u_k, \bar{u}_j) F(\bar{u}_k, \bar{u}_j)} + \frac{1}{u_j \prod\limits_{k \neq j} F(u_k, u_j) F(\bar{u}_k, u_j)} \right].$$

After forgetting equivariant parameters, this gives

$$[Q^{2m+1}] = [M_2^{2m+1}] \in \Omega_U^{-(4m+2)}.$$

Basics of projective duality

Consider a projective space $\mathbb{C}P = \mathbb{C}P(V_{\mathbb{C}})$.
Hyperplanes in $\mathbb{C}P$ form the dual projective space $\mathbb{C}P^* = \mathbb{C}P(V^*)$.

Let $X \subset \mathbb{C}P$ be a closed irreducible algebraic subvariety.
A hyperplane $H \subset \mathbb{C}P$ is said to be tangent to X if there exists a smooth point $x \in X$ such that $x \in H$ and the tangent space to H at x contains the tangent space to X at x, i.e. $\mathcal{T}_x X \subset \mathcal{T}_x H$.

Denote by $X^\vee \subset \mathbb{C}P^*$ the closure of the set of all hyperplanes tangent to X.
The variety $X^\vee$ is called projectively dual to X.

Basics of projective duality

- Let $H = \{\ell = 0\}$. Then

$$x \text{ is a singular point of } X \cap H \Leftrightarrow d\ell|_{\mathcal{T}_x X} = 0 \Leftrightarrow \mathcal{T}_x X \subset \mathcal{T}_x H.$$

  If X is smooth and does not lie in any hyperplane, then

$$H \in X^\vee \;\Leftrightarrow\; \exists x \in X \text{ with } \mathcal{T}_x X \subset \mathcal{T}_x H \;\Leftrightarrow\; H \cap X \text{ is singular.}$$

- For any projective variety $X \subset \mathbb{C}P$ we have $(X^\vee)^\vee = X$.
- If $z \in X$ and $H \in X^\vee$ are smooth points, then

$$H \text{ is tangent to } X \text{ at } z \Leftrightarrow \; z \subset \mathbb{C}P^* \text{ is tangent to } X^\vee \text{ at } H.$$

- If X is irreducible then $X^\vee$ is irreducible.

Basics of projective duality

For any smooth curve $X \subset \mathbb{C}P^2$ of degree d, according to Plücker formulas, we have $\deg X^\vee = d(d-1)$.

Hence, $X^\vee$ cannot be smooth for $\deg X > 3$ because $(X^\vee)^\vee = X$.

Example. For $M_2^1 = \{x^2 + y^2 = z^2\}$, we have

$$(M_2^1)^\vee = \{u^2 + v^2 + w^2 = 0\}.$$

Example. For the curve $M_3^1 = \{x^3 + y^3 = z^3\}$ we have

$$(M_3^1)^\vee = \{u^6 + v^6 + w^6 - 2u^3v^3 - 2u^3w^3 - 2v^3w^3 = 0\}.$$

The curve $(M_3^1)^\vee$ is not smooth since M_3^1 is smooth and $\deg M_3^1 = 3 > 2$.

n-valued groups: a brief history

In 1971, V.M. Buchstaber and S.P. Novikov proposed a construction motivated by the theory of characteristic classes.

This construction describes a multiplication in which the product of any pair of elements is a multiset of n points, see [Buchstaber, 2006].

An axiomatic definition of n-valued groups and the first results of their algebraic theory were obtained in a subsequent series of works by V.M. Buchstaber.

Currently, the theory of n-valued (formal, finite, discrete, topological, and algebro-geometric) groups and their applications in various areas of mathematics and mathematical physics are being developed by a number of authors.

n-valued monoids

An n-valued monoid is a space X equipped with an operation

$$* : X \times X \to \mathrm{Sym^n(X)}$$

where $\mathrm{Sym^n(X) = X^{\times n}/S_n}$ is the space of unordered n-tuples of elements of X:

- Associativity. The n^2-multisets

  $$[x * w \mid w \in y * z],$$
  $$[w * z \mid w \in x * y]$$

  coincide.
- Unit. There exists an element $e \in X$ such that

  $$e * x = x * e = [x, x, \dots, x]$$

  for every $x \in X$.

n-valued groups

An n-valued group X is an n-valued monoid equipped with an inverse map

$$\mathrm{inv} : X \to X$$

that is, a map such that for every $x \in X$

$$x * \mathrm{inv}(x) \ni e, \quad \mathrm{inv}(x) * x \ni e.$$

The notions of homomorphisms, commutativity, kernels, and related concepts admit natural generalizations from the context of 1-valued groups to that of n-valued groups.

Algebraic n-valued monoids and groups

An algebraic n-valued monoid is an algebraic variety X equipped with an associative n-valued multiplication given by a rational morphism $X \times X \to \mathrm{Sym}^{\mathrm{n}}(\mathrm{X})$ with a neutral element $e \in X$ such that

$$x * e \ = \ e * x \ = \ [x, x, \ldots, x] \quad \text{for every } x \in X.$$

An algebraic n-valued group is an algebraic n-valued monoid on X together with a regular morphism $\mathrm{inv} : X \to X$ such that for any $x \in X$ the following two conditions hold:

$$e \in x * \mathrm{inv}(x), \qquad x * \mathrm{inv}(x) = \mathrm{inv}(x) * x.$$

Polynomials $p_d(z; x, y)$

Let us introduce the following symmetric polynomials:

$$p_d(z; x, y) = \prod_{r,s=1}^{d} \left(\sqrt[d]{z} + \varepsilon^r \sqrt[d]{x} + \varepsilon^s \sqrt[d]{y} \right),$$

where $\varepsilon = e^{2\pi i/d}$ and $\sqrt[d]{-}$ denotes some branch of the root.

$$\begin{aligned}
p_1 &= \sigma_1, \\
p_2 &= \sigma_1^2 - 4\sigma_2, \\
p_3 &= \sigma_1^3 - 27\sigma_3, \\
p_4 &= \sigma_1^4 - 2^3\sigma_1^2\sigma_2 + 2^4\sigma_2^2 - 2^7\sigma_1\sigma_3, \\
p_5 &= \sigma_1^5 - 5^4\sigma_1^2\sigma_3 - 5^5\sigma_2\sigma_3, \\
p_6 &= \sigma_1^6 - 2^2 \cdot 3\sigma_1^4\sigma_2 - 2 \cdot 3^4 \cdot 17\sigma_1^3\sigma_3 + 2^4 \cdot 3\sigma_1^2\sigma_2^2 - \\
&\qquad - 2^3 \cdot 3^4 \cdot 19\sigma_1\sigma_2\sigma_3 - 2^6\sigma_2^3 + 3^3 \cdot 19^3\sigma_3^2,
\end{aligned}$$

where $\sigma_j's$ are elementary symmetric polynomials in x, y, z.

Groups $\mathbb{G}_d(\mathbb{C})$

The polynomial

$$p_d(z; (-1)^d x, (-1)^d y)$$

defines a commutative algebraic d-valued group $\mathbb{G}_d(\mathbb{C})$ on $\mathbb{C}$ with multiplication

$$x * y = [z \mid p_d(z; x, y) = 0],$$

neutral element 0, and inverse $\mathrm{inv}(x) = (-1)^d x$.

θ-circulants

For any elements $a_0, ..., a_{n-1}, \theta \in \mathbb{C}$, introduce a θ-circulant matrix:

$$\mathrm{Circ}_\theta(a_0, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & \theta a_1 & \theta a_2 & \cdots & \theta a_{n-1} \\ a_{n-1} & a_0 & \theta a_1 & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_0 & \ddots & \theta a_2 \\ \vdots & \ddots & \ddots & \ddots & \theta a_1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

Wendt matrices

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

The polynomial $p_d(z; x, y)$ defining the d-valued multiplication is the determinant of a y-circulant $d \times d$ matrix:

$$\mathrm{Circ}_y \left(w^d + (-1)^{d+1} x + y, \binom{d}{1} w, \ldots, \binom{d}{n-1} w^{d-1} \right)$$

where $w^d = z$.

These matrices generalize the Wendt matrices (the substitution $x = (-1)^d$, $y = z = 1$), see [E. Wendt, 1894].

$p_d(z;x,y)$ and Wendt Matrices

$$p_2 = \begin{vmatrix} w^2 - x + y & 2wy \\ 2w & w^2 - x + y \end{vmatrix}$$

$$p_3 = \begin{vmatrix} w^3 + x + y & 3yw & 3yw^2 \\ 3w^2 & w^3 + x + y & 3yw \\ 3w & 3w^2 & w^3 + x + y \end{vmatrix}$$

$$p_4 = \begin{vmatrix} w^4 - x + y & 4yw & 6yw^2 & 4yw^3 \\ 4w^3 & w^4 - x + y & 4yw & 6yw^2 \\ 6w^2 & 4w^3 & w^4 - x + y & 4yw \\ 4w & 6w^2 & 4w^3 & w^4 - x + y \end{vmatrix}$$

Polynomials $p_d(z; x, y)$

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

For prime $d \geqslant 5$, the polynomial

$$p_d(z; x, y) - (x + y + z)^d$$

is divisible by $d^4 xyz$.

This result follows from the above observations and from the following:

Theorem (J. Wolstenholme, 1862)

$$\binom{2d-1}{d-1} - 1$$

is divisible by d^3 for primes $d \geqslant 5$.

Fermat curves and $\mathbb{G}_d(\mathbb{C})$

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

Consider the Fermat curve $(d \geqslant 2)$

$$M_d^1 = \{x^d + y^d = z^d\}.$$

Then the dual curve $(M_d^1)^\vee \subset (\mathbb{C}P^2)^*$ is given by the equation

$$p_{d-1}(w^d; u^d, v^d) = 0.$$

Algebraic monoids $\mathbb{M}_d(\mathbb{C}P^1)$

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

The group $\mathbb{G}_d(\mathbb{C})$ extends (only) to an algebraic d-valued coset monoid $\mathbb{M}_d(\mathbb{C}P^1)$ on $\mathbb{C}P^1$ with

$$* : \ \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^n$$

$$(x_1 : x_0) * (y_1 : y_0) = (b_d : b_{d-1} : \cdots : b_0),$$

where $b_j = b_j(x, y)$ is the coefficient of $z_1^{d-j} z_0^j$ in the homogeneous polynomial

$$(x_0 y_0 z_0)^d \ p_d\left(\frac{z_1}{z_0};\ (-1)^d \frac{x_1}{x_0},\ (-1)^d \frac{y_1}{y_0}\right)$$

whenever $(x_1 : x_0)$ and $(y_1 : y_0)$ are not both equal to $(1 : 0)$.

Here the point ∞ is absorbing, i.e.

$$\infty * x = x * \infty = [\infty, \infty, \ldots, \infty]$$

for any $x \in \mathbb{C}P^1 \backslash \{\infty\}$, and the value $\infty * \infty$ is undefined.

Projective duality is a shift on $\mathbb{M}_d$'s

For each $d \geqslant 2$ consider a curve

$$X_d = \{p_d(z; x, y) = 0\}.$$

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

Under projective duality the curve X_d $(d \geqslant 2)$ goes to

$$X_d^\vee = \{\, (uvw)^{d-1}\, p_{d-1}(1/w; 1/u, 1/v) = 0 \,\} \subset (\mathbb{C}P^2)^*.$$

The composition of the duality $X_d \mapsto X_d^\vee$ with the subsequent Möbius transformation $(u, v, w) \mapsto (1/u, 1/v, 1/w)$ defines a shift operation

$$\mathbb{M}_d(\mathbb{C}P^1) \mapsto \mathbb{M}_{d-1}(\mathbb{C}P^1)$$

in the family of algebraic d-valued monoids.

Projective duality is a shift on $\mathbb{M}_d$'s

Example. For $X_2^\vee$ we have the parametrization

$$(u,v) = \left(-\frac{1}{1+t},\ \frac{1}{t}\right) \quad \text{or} \quad \frac{1}{u} + \frac{1}{v} = -1.$$

Taking the projective closure (homogenization), we find that $X_2^\vee$ is given by the zero locus of the polynomial

$$P_1 = uvw\, p_1(w^{-1}; u^{-1}, v^{-1}) = (u+v)w + uv.$$

Projective duality is a shift on $\mathbb{M}_d$'s

Example. For $X_3^\vee$:

$$(u, v) = \left(\frac{1}{(1+t)^2}, \ \frac{1}{t^2} \right) \quad \text{or} \quad \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} = 1.$$

The curve $X_3^\vee$ is given by the polynomial

$$P_2 = (uvw)^2 \, p_2(w^{-1}; u^{-1}, v^{-1}) = (uv - w(u+v))^2 - 4uvw^2,$$

$$P_2 = (uv)^2 + (vw)^2 + (uw)^2 - 2u^2vw - 2uv^2w - 2uvw^2.$$

Polynomials $p_{d,n}$

In connection with Bessel kernels for solutions of Picard–Fuchs differential equations for the kernel

$$K_d = \sum_{j,k} \binom{j+k}{k}^d \frac{x^j y^k}{z^{j+k}},$$

the iterated analogue of the polynomials $p_d(z; x, y)$ was considered for $x = (x_1, \ldots, x_n)$ in [I.Gaiur, V.Rubtsov, D.Straten, 2024]:

$$p_{d,n}(z; x) = \prod_{k_1,\ldots,k_n=1}^{d} \left(\sqrt[d]{z} + \varepsilon^{k_1} \sqrt[d]{x_1} + \cdots + \varepsilon^{k_n} \sqrt[d]{x_n}\right).$$

Operations $O_{d,n}(\mathbb{C}P^1)$

The polynomial $p_{d,n}(z;x)$ defines an n-ary d^{n-1}-valued algebraic operation

$$\mu(x_1, ..., x_n) = [z \mid p_{d,n}(z;x) = 0].$$

Denote by $O_{d,n}(\mathbb{C}P^1)$ the variety $\mathbb{C}P^1$ with the operation μ.

Let

$$X_d^{n-1} = \{\, p_{d,n} = 0 \,\}$$

be the hypersurface in $\mathbb{C}P^n$. For integers $d \geqslant 2$ and $n \geqslant 2$ define

$$P_{d,n} = (u_1 \cdots u_n w)^{d-1}\, p_{d-1}(w^{-1}; u_1^{-1}, \ldots, u_n^{-1})$$

the polynomial of degree d^{n-1}.

Operations $O_{d,n}(\mathbb{C}P^1)$

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

The composition of the duality ($d \geqslant 2$, $n \geqslant 2$)

$$X_d^{n-1} \mapsto (X_d^{n-1})^\vee = \{P_{d,n} = 0\} \subset (\mathbb{C}P^n)^*$$

with the subsequent Möbius transformation

$$(u_1, \ldots, u_n, w) \mapsto (1/u_1, \ldots, 1/u_n, 1/w)$$

defines a shift operation

$$O_{d,n}(\mathbb{C}P^1) \mapsto O_{d-1,n}(\mathbb{C}P^1)$$

in the family of n-ary d^{n-1}-valued algebraic structures $O_{d,n}(\mathbb{C}P^1)$.

Fermat hypersurfaces and $p_{d,n}$'s

Theorem (V.M. Buchstaber, M.I. Kornev, 2025)

Let M_d^{n-1} be a Fermat hypersurface

$$M_d^{n-1} = \{x_1^d + x_2^d + \cdots + x_n^d = z^d\}$$

in $\mathbb{C}P^n$ with coordinates $x_1, \ldots, x_n, z$. The dual hypersurface is defined by the equation

$$(M_d^{n-1})^\vee = \{p_{d-1,n}(w^d; u_1^d, \ldots, u_n^d) = 0\}$$

in $(\mathbb{C}P^n)^*$ with the dual coordinates $u_1, \ldots, u_n, w$.

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Thank You for your attention!