Electrical Lie algebras

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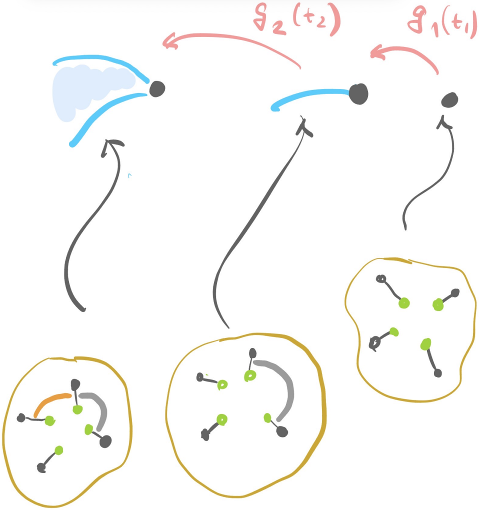
Definition

This work is joint with Arkady Berenstein

The electrical Lie algebras were introduced by T. Lam and P. Pylyavskyy and further studied by Yi Su. An infinitely generated electrical Lie algebra of type A also appeared in the context of categorification in representation theory in a paper by V.Serganova, C. Stroppel and others.
In our story the electrical Lie algebras appear as a way to decompose a complicated network, which is a graph with an extra structure, into elementary pieces. This idea appeared in the work of A. Postnikov and proved to be a very powerful tool for tackling many problems in the network theory.



In general networks $g(t) \in Sl(n)$

In electrical networks $g(t) \in Sp(n)$

T. Lam and P. Pylyavskyy have found a remarkable presentation for the symplectic group while studying the electrical networks. They called this group the **electrical Lie group** and its Lie algebra the **electrical Lie algebra**.

Theorem

The Lie algebra sp_n has a presentation

- $[u_i, u_j] = 0$ if $|i - j| > 1$
- $[u_i, [u_i, u_j]] = 2u_i$ if $|i - j| = 1$

This representation of sp_n preserves a special, non-standard, symplectic form.

We generalize the definition of the electrical Lie algebras. To each Kac-Moody Lie algebra $\mathfrak{g}$ we associate the generalized electrical algebra which is always a subalgebra of $\mathfrak{g}$ and is a deformations of the nilpotent Lie subalgebra of $\mathfrak{g}$.

Definition

Let $A = (a_{ij})$ be a (generalized, not necessarily symmetrizable) $I \times I$ Cartan matrix and let $\mathfrak{g} = \mathfrak{g}_A =< e_i, f_i, i \in I >$ be the corresponding Kac-Moody Lie algebra. For any family $\mathbf{a} = (a_i, i \in I) \in \mathbb{C}^I$ let $\mathfrak{g}^{(\mathbf{a})}$ be a Lie subalgebra of $\mathfrak{g}$ generated by

$$u_i := e_i + a_i[e_i, f_i] - a_i^2 f_i \ , i \in I. \tag{1}$$

We call $\mathfrak{g}^{(\mathbf{a})}$ a **generalized electrical Lie algebra of type** $\mathfrak{g}$.

In particular, $sl_2^{(\mathbf{a})}$ is generated by a single nilpotent

$$u = \begin{pmatrix} a & 1 \\ -a^2 & -a \end{pmatrix} \in sl_2$$

Locally the generators u_i are related to the usual nilpotents by conjugation

$$u_i = e^{-a_i f_i}\, e_i\, e^{a_i f_i}.$$

Theorem

For any Kac-Moody Lie algebra $\mathfrak{g}$ and any $\mathbf{a} \in \mathbb{C}^I$ the generalized electrical Lie algebra $\mathfrak{g}^{(\mathbf{a})}$ of type $\mathfrak{g}$ satisfies

$$(ad\ u_i)^{1-a_{ij}}(u_j) = -2\delta_{a_{ij},-1}a_{ji}a_i a_j u_i \tag{2}$$

for all distinct $i, j \in I$.
The algebra $\mathfrak{g}^{(\mathbf{a})}$ is **a flat deformation** *of the nilpotent subalgebra of $\mathfrak{g}$.*

Now we construct a real form $\mathfrak{g}^{(\mathbf{b})}$ for the electrical Lie algebra $sl_n^{(\mathbf{b})}$.

Theorem

Let $\mathfrak{g} = sl_n$. Then $g_{\mathbf{a}} u_i g_{\mathbf{a}}^{-1} = e_i + b_{i-1} f_{i-1}$ for $i = 1, \ldots, n-1$ in the notation above (with the convention $f_0 = 0$, $a_0 = 0$), where we abbreviated $b_j := -a_j a_{j+1}$ and $g_{\mathbf{a}} := e^{a_{n-1}f_{n-1}} \ldots e^{a_2 f_2} e^{a_1 f_1} \in SL_n$.

Studying Poisson Lie properties and quantization of these groups seems to be an interesting problem.

Theorem

$sl_n^{(\mathbf{a})}$ preserves a particular simplectic form $\omega_{\mathbf{a}}$ in $\mathbb{C}^n = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$. If n is even then this form is non-degenerate. In this case $sl_n^{(\mathbf{a})}$ is isomorphic to sp_{n-1} when $a_1 \cdots a_{n-2} \neq 0$. If n is odd, then the form $\omega_{\mathbf{a}}$ has a one-dimensional kernel that is an invariant of the action of $sl_n^{(\mathbf{a})}$.

Remark

$v_i = e_i + b_i f_i$ for $i = 1, \ldots, n-1$
is a deformation of so_n Lie algebra.

Theorem

- For any $\mathbf{b} \in \mathbb{C}^{n-1}$ one has:

$$sp_{2n}^{(\mathbf{b})} \cong sl_n^{(b_1,\dots,b_{n-2})} \oplus sl_{n+1}^{(b_1,\dots b_{n-1})}$$

- Let $\mathfrak{g} = \widehat{sl}_n$, the untwisted affine Lie algebra of type $\hat{A}_{n-1}$ with $I = \{0,\dots,n-1\}$. Then the assignments $u_i \mapsto e_i + b_{i-1,i}f_{i-1}$, where $i-1$ is calculated modulo n for $i \in I$, define an injective homomorphism of the Lie algebras $\widehat{sl}^{(\mathbf{b})} \hookrightarrow \widehat{sl}_n$.

Let B_- be the Borel subgroup of low-triangular matrices, $U_+(n)$ be the opposite unipotent subgroup of $Gl(n)$, and $U_+^{(a_1,\dots,a_{n-1})}(n)$ be its electrical deformation. It is generated by $\exp(tu_i)$.

Theorem

The map

$$B_-(n) \cdot U_+^{(\mathbf{a})}(n) \to Gl(n)$$

is a birational isomorphism. In other words $X \in Gl(n)$ admits a decomposition $X = b_- \cdot u_+^{(\mathbf{a})}$ iff the the deformed principle minors of X are not equal to zero. It generalizes the Gauss decomposition of $X \in Gl(n)$ which corresponds to the zero value of the parameter $\mathbf{a}$.

Consider the case when $n = 3$, then $U_+^{(a,b)}(3)$ is generated by the exponents of the generators of the electrical Lie algebra. Let $X = (x_{ij}) \in Gl(3)$ then

$$X = b_- \cdot u_+^{(a_1,a_2)}$$

where

$$b_- = \begin{pmatrix} h_1 & 0 & 0 \\ b_1 & h_2 & 0 \\ b_3 & b_2 & h_3 \end{pmatrix}$$

and the matrix coefficients of b_- are given by the formulas

$$h_1 = x_{11} - a_1 x_{12} + a_1 a_2 x_{13};\ h_2 = \frac{\Delta_{12}^{12} - a_2 \Delta_{13}^{12} + a_1 a_2 \Delta_{12}^{23}}{h_1};\ h_3 = \frac{\det X}{h_1 h_2};$$

$$b_1 = x_{21} - a_1 x_{22} + a_1 a_2 x_{23} + a_1 h_2;\ b_2 = \frac{\Delta_{13}^{12} - a_2 \Delta_{13}^{13} + a_1 a_2 \Delta_{13}^{23}}{h_1} + a_1 \frac{\det X}{h_1 h_2};$$

$$b_3 = x_{31} - a_1 x_{32} + a_1 a_2 x_{33} + a_1 \frac{\Delta_{13}^{12} - a_2 \Delta_{13}^{13} + a_1 a_2 \Delta_{13}^{23}}{h_1};$$

Using this it is easy to calculate the matrix coefficients of $u_+^{(a_1,a_2)}$ which will denote by u_{ij}.

$$u_{11} = \frac{x_{11}}{h_1},\ u_{12} = \frac{x_{12}}{h_1},\ u_{13} = \frac{x_{13}}{h_1}$$

$$u_{21} = \frac{x_{21} - b_1 x_{11}}{h_2},\ u_{22} = \frac{x_{22} - b_1 x_{12}}{h_2},\ u_{23} = \frac{x_{23} - b_1 x_{13}}{h_2}$$

$$u_{31} = \frac{x_{31} - b_2 x_{21} - b_3 x_{11}}{h_3},\ u_{32} = \frac{x_{32} - b_2 x_{22} - b_3 x_{12}}{h_3},$$

$$u_{33} = \frac{x_{33} - b_2 x_{23} - b_3 x_{13}}{h_3}$$

Conjecture

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and V_λ be a simple finite-dimensional $\mathfrak{g}$-module with highest weight λ. For any $\mathbf{a} \in \mathbb{C}^I$ there is a unique (up to scalar) nonzero $\mathfrak{g}^{(\mathbf{a})}$-invariant $v_\lambda^{(\mathbf{a})} \in V_\lambda$. Moreover,

$$v_\lambda^{(\mathbf{a})} = \sum_\beta (-\mathbf{a})^{\lambda-\beta} v_\lambda(\beta)$$

where $\mathbf{c}^\gamma := \prod_{i \in I} c_i^{(\gamma, \alpha_i^\vee)}$.

Example

The invariants of $sl_4^{(\mathbf{a})}$ in $\mathbb{C}^4$ and in $\wedge^2\mathbb{C}^4$

$$w = v_1 - a_1\, v_2 + a_1 a_2\, v_3 - a_1 a_2 a_3\, v_4.$$

$$\begin{aligned} w = \ & v_1 \wedge v_2 - a_2\, v_1 \wedge v_3 + a_2 a_3\, v_1 \wedge v_4 \\ & + \ a_1 a_2\, v_2 \wedge v_3 - a_1 a_2 a_3\, v_2 \wedge v_4 \\ & + \ a_1 a_2^2 a_3\, v_3 \wedge v_4. \end{aligned}$$

Example

If V_λ is minuscule, then $v_\lambda(w\lambda) = v_{w\lambda}$ is the standard extremal vector for all $w \in W$.
If $V = V_{\omega_1+\omega_2}$ is the adjoint sl_3-module, then $v_{\omega_1+\omega_2}(0) = \frac{2}{3}(h_1+h_2)$.
If $V = V_{2\omega_1}$ is the adjoint sp_4-module, then $v_{2\omega_1}(0) = h_2$.

For any Lie algebra $\mathfrak{g}$, its Lie subalgebra $\mathfrak{g}'$ and any $\mathfrak{g}$-module denote by $V^{\mathfrak{g}'}$ the space of $\mathfrak{g}'$-invariants in V.

Theorem

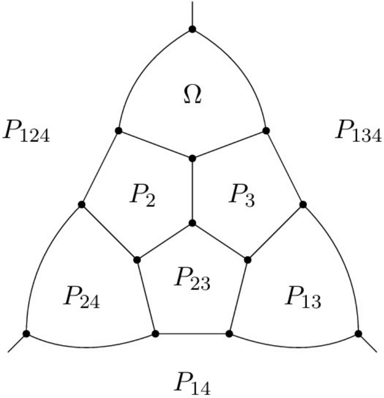
The conjecture stated above holds for all simple finite-dimensional modules V_λ over $\mathfrak{g} = sl_n(\mathbb{C}), so_n(\mathbb{C}), sp_{2n}(\mathbb{C})$, $n \geq 2$ and all minuscule and quasi-minuscule $\mathfrak{g}$-modules V_λ for all complex simple $\mathfrak{g}$.

Cluster algeras

Let U be a unipotent subgroup of SL_n which acts on it by multiplication on the right. Consider the algebra $\mathbb{C}[SL_n]^U$ of U invariant functions on SL_n under the above action. It is generated by flag minors and it carries a

cluster algebra structure

found by A. Zelevinsky and S. Fomin. There is a Poisson bracket on this algebra inherited from the Sklyanin bracket on $\mathbb{C}[SL_n]$. A remarkable result by M. Gehktman M. Shapiro and A. Vainshtein say that this bracket controls the edge structure of this graph-the bracket of the two functions is log-canonical, $\{X, Y\} = a_{xy}XY$, iff they belong to one cluster.



Ω
P_{124}
P_{134}
P_2
P_3
P_{23}
P_{24}
P_{13}
P_{14}

Denote the group with the Lie algebra $sl_n^{(\mathbf{a})}$ by $SL(n)^{\mathbf{a}}$. The Lie algebra $sl_n^{(\mathbf{a})}$ is coideal hence the algebra $\mathbb{C}[SL(n)]^{SL_n^{\mathbf{a}}}$ is a Poisson algebra. In fact it is the algebra of functions on the homogeneous space $SL(n)/SL(n)^{\mathbf{a}}$

Conjecture

There is a cluster algebra structure on the algebra $\mathbb{C}[SL(n)]^{SL_n^{\mathbf{a}}}$ which deforms the standard cluster algebra structure on $\mathbb{C}[SL_n]^U$.

We have sorted out completely the example $n = 4$. The deformation exists and is highly non-trivial-the mutations and the Poisson structure are both get deformed! The invariants are generated by the deformed minors as expected and the Poisson bracket controls the mutation graph structure in more involved way.

$$P_i = x_{i1} - a_1\, x_{i2} + a_1 a_2\, x_{i3} - a_1 a_2 a_3\, x_{i4}.$$

$$P_{ij} = \ \Delta_{ij}^{12} - a_2\, \Delta_{ij}^{13} + a_2 a_3\, \Delta_{ij}^{14} + a_1 a_2\, \Delta_{ij}^{23} - a_1 a_2 a_3\, \Delta_{ij}^{24}$$
$$+ \ a_1 a_2^2 a_3\, \Delta_{ij}^{34}.$$

$$\{P_i, P_j\} = \frac{1}{2}(P_i P_j + a_i P_{ij})$$

$$P_2 P_{13} = P_{12} P_3 + P_1 P_{23} - a_1 a_2 P_{123}$$