Toric topology
of Milnor manifolds $H_{p,q}$

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Conference
Algebraic topology, Group actions and Combinatorics
May 12, 2026

Milnor Manifolds manifolds $H_{p,q}$

Definition

The Milnor manifold $H_{p,q}$ $(p \leqslant q)$ is defined as the complex submanifold in $\mathbb{C}P^{p-1} \times \mathbb{C}P^{q-1}$ given by the equation

$$z_1 w_1 + z_2 w_2 + \cdots + z_p w_p = 0.$$

- This is a hypersurface of bidegree $(1, 1)$ in $\mathbb{C}P^{p-1} \times \mathbb{C}P^{q-1}$, or a hyperplane section of the Segre embedding

$$\mathbb{C}P^{p-1} \times \mathbb{C}P^{q-1} \to \mathbb{C}P^{pq-1},$$

$$[z, w] = ([z_1 : \cdots : z_p], [w_1 : \cdots : w_q]) \mapsto [z_1 w_1 : \cdots : z_i w_j : \cdots : z_p w_q].$$

- The Milnor manifold $H_{p,q}$ can be identified with the set of pairs (L_1, L_{q-1}), where L_1 is a line in $\mathbb{C}^p$, L_{q-1} is a $(q-1)$-dimensional subspace in $\mathbb{C}^q$, and $L_1 \subset L_{q-1}$. In particular, $H_{q,q}$ is the partial flag manifold $Fl_{1,q-1}(\mathbb{C}^q)$.

Torus action and Moment map

- The torus $(\mathbb{C}^*)^q$ acts on $\mathbb{C}P^{p-1} \times \mathbb{C}P^{q-1}$ and $H_{p,q}$ as follows:

$$t \cdot [z, w] = ([t_1 z_1 : t_2 z_2 : \cdots : t_p z_p], [t_1^{-1} w_1 : t_2^{-1} w_2 : \cdots : t_q^{-1} w_q]),$$

- The complexity of the action is $d = p - 2$.
- $H_{p,q}$ symplectic manifold with Hamiltonian action of compact torus T^q.
- $\mu : H_{p,q} \to \mathbb{R}^q$ is moment map defined as follows:

$$\mu([z, w]) = -\frac{1}{\sum_{i=1}^{p} |z_i|^2} \left(\sum_{i=1}^{p} |z_i|^2 \Lambda_i \right) + \frac{1}{\sum_{i=1}^{q} |w_i|^2} \left(\sum_{i=1}^{q} |w_i|^2 \Lambda_i \right).$$

Moment Polytopes

- $\mu(H_{p,q}) = -\Delta^{p-1} + \Delta^{q-1} = \mathrm{conv}(-e_i + e_j : i \in [p], j \in [q]$.
- The polytope $\mu(H_{p-1,q})$ is the convex hull of a subset of the vertices of the polytope $\mu(H_{p,q})$.

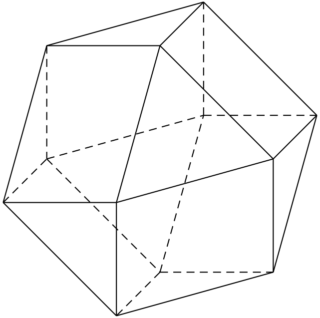

Figure 1: Cuboctahedron

Stratification of $H_{p,q}$

Definition

Fix index sets $I \subset [p]$ and $J \subset [q]$. Define the stratum

$$W_{I,J} := \left\{ [z,w] \in H_{p,q} \;\middle|\; \begin{array}{ll} z_i \neq 0 \text{ if } i \in I, & z_i = 0 \text{ if } i \notin I, \\ w_j \neq 0 \text{ if } j \in J, & w_j = 0 \text{ if } j \notin J \end{array} \right\}.$$

- The strata are T^q-invariant subsets.
- The strata are pairwise disjoint and cover the Milnor manifold.

Definition

A stratum $W_{p,q} = W_{I,J}$ is called *principal* if $I = [p]$ and $J = [q]$.

- The principal stratum $W_{p,q}$ is an open everywhere dense subset of $H_{p,q}$, which means that any other stratum $W_{I,J}$ lies on the boundary of $W_{p,q}$.

Admissible Polytopes

Definition

A polytope Q is called *admissible* if it $Q = \overline{\mu(W_{I,J})}$ for some non-empty stratum $W_{I,J}$.

- Every admissible polytope is of the form $P_{I,J} := \mathrm{conv}(-e_i + e_j : i \in I, j \in J)$.
- The admissible polytope of the principal stratum is $\mu(H_{p,q})$.

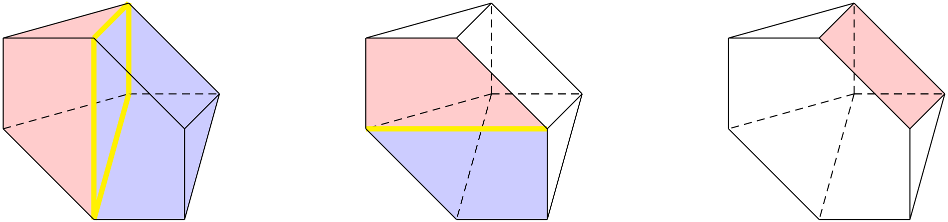

Figure 2: polyhedral subdivisions of the faces of the polytope $\mu(H_{3,4})$.

Parameter Spaces

Definition

The parameter space of $W_{I,J}$ is the orbit space $F_{I,J} = W_{I,J}/(\mathbb{C}^*)^q$.

- For main stratum $W_{p,q}$

$$F_{p,q} \cong \mathbb{C}P^{p-2} \setminus (\cup_{k=1}^{p} H_k) = \{[x_1 : x_2 : \cdots : x_{p-1}] \in \mathbb{C}P^{p-2} | x_i \neq 0, \sum_{k=1}^{p-1} x_k \neq 0\}.$$

- For strata with $I \cap J \neq \emptyset$

$$F_{I,J} \cong \mathbb{C}P^{p-2-|[p]\setminus(I\cap J)|} \setminus (\cup_{k\in I\cap J} H_k),$$

- For strata with $I \cap J = \emptyset$

$$F_{I,J} = \{pt\}.$$

Orbit spaces $W_{I,J}/T^q$

Proposition

The map $\widehat{\mu}_{I,J}\colon W_{I,J}/T^q \to \mathring{P}_{I,J}$ is the trivial bundle, and the space $W_{I,J}/T^q$ is homeomorphic to $\mathring{P}_{I,J} \times F_{I,J}$ via the fiberwise homeomorphism

$$\begin{tikzcd}
W_{I,J}/T^q \arrow[r] \arrow[d, "\widehat{\mu}"] & \mathring{P}_{I,J} \times F_{I,J} \arrow[d] \\
\mathring{P}_{I,J} \arrow[r, equal] & \mathring{P}_{I,J},
\end{tikzcd}$$

which is given by the pair of maps $(\widehat{\mu}_{I,J}, p)$, where $p\colon W_{I,J}/T^q \to F_{I,J}$ is the quotient map.

Symplectic Reduction

Let $x \in \operatorname{int}(-\Delta^{p-1} + \Delta^{q-1})$. Set $\mathfrak{S}(x) = \{(I, J) \mid x \in \mathring{P}_{I,J}\}$. Then

$$\hat{\mu}^{-1}(x) = \bigcup_{(I,J)\in\mathfrak{S}(x)} \{y \in W_{I,J}/T^q : \hat{\mu}(y) = x\}$$

Then the symplectic quotient of $H_{p,q}$ corresponding to the regular value x:

$$H_{p,q}/\!\!/_x T^q = \bigcup_{(I,J)\in\mathfrak{S}(x)} F_{I,J},$$

Theorem

The symplectic quotient of the Milnor manifold $H_{p,q}$ does not depend on the choice of a regular value and is homeomorphic to the projective space $\mathbb{C}P^{p-2}$

$$H_{p,q}/\!\!/ T^q \cong \mathbb{C}P^{p-2}.$$

Orbit Space $H_{p,q}/T^q$

We construct a continuous projection

$$h\colon (-\Delta_{p-1} + \Delta_{q-1}) \times \mathbb{C}P^{p-2} \to H_{p,q}/T^q.$$

And we obtain the following theorem:

Theorem

The orbit space $H_{p,q}$ is homeomorphic to the quotient space

$$(-\Delta^{p-1} + \Delta^{q-1}) \times \mathbb{C}P^{p-2}/\sim,$$

where $(x, c) \sim (y, c')$ if $x = y \in \partial(-\Delta^{q-1} + \Delta^{q-1})$.

Orbit Space $H_{p,q}/T^q$

Corollary

Consider $H_{q,q} \cong Fl_{1,q-1}(\mathbb{C}^q)$. If we identify the polytope with the disk D^{q-1}, and the disk with the cone $C(S^{q-2})$, we obtain

$$Fl_{1,q-1}(\mathbb{C}^q)/T^q \cong \frac{C(S^{q-2}) \times \mathbb{C}P^{q-2}}{\sim} \cong S^{q-2} * \mathbb{C}P^{q-2}$$

In particular $Fl(\mathbb{C}^3)/T^3$ is homeomorphic to

$$S^1 * S^2 \cong S^4$$

Orbit Space $H_{p,q}/T^q$

Corollary

The orbit space $H_{p,q}/T^q$, where $p \neq q$, is homeomorphic to

$$D^{q-2} \times C(\mathbb{C}P^{p-2}).$$

Indeed

$$\frac{(-\Delta^{q-1} + \Delta^{q-1}) \times \mathbb{C}P^{q-2}}{\sim} \cong \frac{D^{q-2} \times [0,1] \times \mathbb{C}P^{p-2}}{\sim},$$

where the latter equivalence relation is defined as follows

$$(d, 0, c) \sim (d', 0, c'), \text{ if } d = d'.$$

Then

$$\frac{D^{q-2} \times [0,1] \times \mathbb{C}P^{p-2}}{\sim} \cong D^{q-2} \times C(\mathbb{C}P^{p-2}).$$