

Buchstaber Genus and Formal Groups*

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Structure of the Talk

1. Overview of Results
2. Basics of the Theory of n -Valued Formal Groups
3. Main Theorem
4. Corollaries

Overview of Results

In [Buc90], a family of polynomials

$$\mathcal{B}_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz),$$

where $\mathbf{a} = (a_1, a_2, a_3)$ free parameters, was introduced.

It was shown that each polynomial defines a two-valued formal group \mathbb{G}_a , universal in the class of two-valued formal groups obtained by the modulus square construction (introduced in [BN71]) from the class of formal groups

$$\mathcal{F}(u, v) = \frac{u^2\lambda_1(v) - v^2\lambda_1(u)}{u\lambda_2(v) - v\lambda_2(u)},$$

where $\lambda_k(u) \in A[[u]]$, $k = 1, 2$.

The ring A has a rather complicated structure (see [BU15]), but

$$A \otimes \mathbb{Q} \cong \mathbb{Q}[b_1, b_2, b_3, b_4],$$
$$b_1 = \alpha, \quad b_2 = \wp(z), \quad b_3 = \wp'(z), \quad b_4 = g_2,$$

with $\deg b_i = -2i$, $i = 1, \dots, 4$, and $\wp(z) = \wp(z; g_2, g_3)$ is the Weierstrass function satisfying the equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

The class of groups $\mathcal{F}(u, v)$ contains the formal groups corresponding to the Todd genus, the signature, the Ochanine (also elliptic) genus [Och87], and the Krichever genus.

In [Buc90], it was proved that the class $\mathcal{F}(u, v)$ is a realization of the class of Krichever formal groups (see [Kri90]) with exponential

$$f_{\text{Kr}}(x) = \frac{e^{\alpha x}}{\Phi(x, z)}, \quad \text{where} \quad \Phi(x, z) = \Phi(x, z; g_2, g_3) = \frac{\sigma(z-x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}$$

is the Baker–Akhiezer function, and $\sigma(z) = \sigma(z; g_2, g_3)$, $\zeta(z) = \frac{d}{dz} \log \sigma(z)$ are Weierstrass functions.

In the present paper, a new class of formal groups

$$F_{Bc}(u, v) = \left[\left(\frac{u\sqrt{Q(v)} - v\sqrt{Q(u)}}{u^2 - v^2} \right)^2 + a_3 u^2 v^2 \right]^{-1/2},$$

where $Q(t) = 1 - a_1 t^2 + a_2 t^4 - a_3 t^6$ is introduced, whose modulus square is given by the family of polynomials $\mathcal{B}_a(z; x, y)$.

The formal group law $F_{Bc}(u, v)$ belongs to the family of formal group laws $\mathcal{F}(u, v)$ if and only if $a_3 = 0$.

In the case of $a_3 = 0$, we get the Ochanine genus. Therefore, for $a_3 \neq 0$, the corresponding genus is distinct from the elliptic genera of level n introduced by Hirzebruch [HBJ92, Appendix III, Section 1].

The formal group $F_{\text{Bc}}(u, v)$ corresponds to a Hirzebruch genus

$$\text{Bc} : \Omega_{\text{U}} \rightarrow \mathbb{Z} \left[\frac{1}{2}, a_1, a_2, a_3 \right], \quad (1)$$

which we call the Buchstaber genus, where $\deg a_i = -4i$, $i = 1, 2, 3$.

We construct two cohomology theories $\text{Bc}^*(-)[\Delta^{-1}]$ and Bc^* giving the Buchstaber genus (1) on the coefficient rings.

Consider the elliptic curve $\mathcal{E} = \{\eta^2 = \zeta^3 + a_1\zeta^2 + a_2\zeta + a_3\}$ in the chart $\{\zeta \neq 0\}$.

There is an involution on \mathcal{E} :

$$\sigma: (\zeta, \eta) \longmapsto (\zeta, -\eta), \quad \infty \longmapsto \infty$$

The coset construction (\mathcal{E}, σ) yields a *2-valued algebraic coset group on $\mathbb{C}P^1$* with identity ∞ , inverse $\text{inv}(x) = x$, and multiplication defined in the chart $\{\zeta \neq 0\}$ by the equation:

$$x * y = [z \mid D_a(z; x, y) = 0],$$

where

$$D_a(-z; -x, -y) = \text{disc}_t(t^3 + a_1t^2 + a_2t + a_3 - (t-x)(t-y)(t-z))$$

— a *generalized Kontsevich polynomial*, introduced by Buchstaber–Gaiur–Rubtsov in connection with the Picard–Fuchs equation [BGR26].

$$(xyz)^2 \mathcal{B}_a(1/z; 1/x, 1/y) = D_a(-z; -x, -y).$$

Theorem

(I) [BGR26] For each $(a_1, a_2, a_3) = \mathbf{a} \notin \text{disc}_t(t^3 + a_1 t^2 + a_2 t + a_3)$, the structure of the 2-valued algebraic involutive coset group $\mathbb{G}_{\mathbb{C}}(\mathcal{B}_{\mathbf{a}})$ with the multiplication

$$x * y = [z \mid \mathcal{B}_{\mathbf{a}}(z; x, y) = 0]$$

extends to the structure of a regular 2-valued algebraic group $\mathbb{G}_{\mathbb{C}P^1}(\mathcal{B}_{\mathbf{a}})$ with neutral element 0.

(II) [BK26] For $\mathbf{a} \in \text{disc}_t(t^3 + a_1 t^2 + a_2 t + a_3)$, the structure $\mathbb{G}_{\mathbb{C}}(\mathcal{B}_{\mathbf{a}})$ extends only to the structure of a 2-valued algebraic involutive coset monoid on $\mathbb{C}P^1$.

When $a_1 = a_2 = a_3 = 0$, we get the law

$$z^2 - 2(x + y)z + (x - y)^2 = 0$$

giving the 2-valued group $\mathbb{G}_2(\mathbb{C})$.

Definition

A *symmetric n -algebraic n -valued monoid (group)* on \mathbb{C} is an algebraic n -valued monoid (group) $\mathbb{G}_{\mathbb{C}}(f(z; x, y))$ with a (partially defined) multiplication

$$x * y = [z \mid f(z; x, y) = 0],$$

where the polynomial $f(z; (-1)^n x, (-1)^n y)$ is symmetric and has a degree in each variable not exceeding n , with zero 0 , and $\text{inv}(x) = x$.

Each such polynomial is of the form (our Ansatz)

$$P_2(z; x, y) = \sigma_1^2 - 4\sigma_2 + k_2\sigma_3 + k_8\sigma_3^2 + k_4\sigma_1\sigma_3 + k_6\sigma_2\sigma_3$$

modulo associativity conditions, σ_i is the i th elementary symmetric polynomial.

Theorem [BV19], [BK25]

The universal symmetric 2-algebraic 2-valued group $P_2(z; x, y)$ over rings without zero divisors is defined over the coefficient ring

$$\mathbb{Z}[k_2, k_4, k_6, k_8]/(4k_8 - k_4^2 + k_6k_2).$$

In particular, over a $\mathbb{Z}[\frac{1}{2}]$ -algebra K without zero divisors, the family of polynomials P_2 has the form:

$$P_2(z; x, y) = \sigma_1^2 - 4\sigma_2 + k_2\sigma_3 + \frac{k_4^2 - k_6k_2}{4}\sigma_3^2 + k_4\sigma_1\sigma_3 + k_6\sigma_2\sigma_3.$$

To get $\mathcal{B}_a(z; x, y)$, set

$$k_2 = -4a_1, \quad k_4 = -2a_2, \quad k_6 = -4a_3.$$

Theorem [BK25]

There exist exactly two classes of universal symmetric 3-algebraic 3-valued groups with fixed homogeneous components $P_3^h(z; -x, -y)$. In the case of the first class, $k_0 = -27$, and in the case of the second, $k_0 = 0$. In more detail:

(i) If $k_0 = -27$, we have for any $c \in \mathbb{C}$:

$$P_3(z; -x, -y) = \sigma_1^3 - 27\sigma_3 + 18c\sigma_1^2\sigma_3 - 54c\sigma_2\sigma_3 - 27c^2\sigma_2^2\sigma_3 + 81c^2\sigma_1\sigma_3^2.$$

(ii) If $k_0 = 0$, we have for any $\alpha \in \mathbb{C}$:

$$P_3(z; -x, -y) = (\sigma_1 + \alpha\sigma_3)^3.$$

Ansatz for $n = 3$:

$$\begin{aligned} & \sigma_1^3 + k_0\sigma_3 + k_6\sigma_3^2 + k_{12}\sigma_3^3 + k_2\sigma_1\sigma_3 + k_4\sigma_2\sigma_3 \\ & + \ell_6\sigma_1\sigma_2\sigma_3 + \ell_4\sigma_1^2\sigma_3 + k_8\sigma_2^2\sigma_3 + \ell_8\sigma_3^2\sigma_1 + k_{10}\sigma_3^2\sigma_2, \end{aligned}$$

Basics of the Theory of n -Valued Formal Groups

Substitution of Multivalued Formal Series

An *n-valued formal series* is an equation

$$\mathcal{F}^n + \sum_{i=1}^n (-1)^i \Theta_i(x) \mathcal{F}^{n-i} = 0,$$

Consider two multivalued formal series:

$$\mathcal{F}_1^n + \sum_{i=1}^n (-1)^i \Theta_{i,1}(x) \mathcal{F}_1^{n-i} = 0, \quad \mathcal{F}_2^m + \sum_{j=1}^m (-1)^j \Theta_{j,2}(x) \mathcal{F}_2^{m-j} = 0.$$

Let

$$\Theta_{i,1}(0) = 0, \quad 1 \leq i \leq n, \quad \Theta_{j,2}(0) = 0, \quad 1 \leq j \leq m.$$

Then *the operation of substituting* the *m-valued series* $\mathcal{F}_2(x)$ into the *n-valued series* $\mathcal{F}_1(x)$ is defined.

Result of Substitution

The result of the substitution is an *nm-valued formal series*

$$\mathcal{F}(x) = \mathcal{F}_1(\mathcal{F}_2(x)).$$

It is defined by the equation

$$\prod_{k=1}^m \left(\mathcal{F}^n + \sum_{i=1}^n (-1)^i \Theta_{i,1}(\mathcal{F}_2^{(k)}) \mathcal{F}^{n-i} \right) = 0.$$

Here

$$\mathcal{F}_2^{(1)}, \dots, \mathcal{F}_2^{(m)}$$

denote the formal branches (roots) of the m -valued series \mathcal{F}_2 .

One-dimensional n -valued Formal Group

A *commutative one-dimensional n -valued formal group* over a ring A is an n -valued formal series $\mathcal{F}(x, y)$ defined by the equation

$$\mathcal{F}^n + \sum_{i=1}^n (-1)^i \Theta_i(x, y) \mathcal{F}^{n-i} = 0, \quad \Theta_i(x, y) \in A[[x, y]]$$

The formal series $\mathcal{F}(x, y)$ must satisfy the following conditions:

(I) $\mathcal{F}(x, 0)$ is defined by the equation $(\mathcal{F} - x)^n = 0$.

(II) $\mathcal{F}(\mathcal{F}(x, y), z) = \mathcal{F}(x, \mathcal{F}(y, z))$.

(III) $\mathcal{F}(x, y) = \mathcal{F}(y, x)$.

Examples of Two-valued Formal Groups

Next, let $z(x, y) = \mathcal{F}(x, y)$.

1. Elementary formal groups:

$$z^2 - 2(x + y)z + (x - y)^2 = 0, \quad z^2 - 2(x + y)z + (x + y)^2 = 0.$$

2. The Buchstaber polynomial

$$\mathcal{B}_a(z; x, y) = (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + xz + a_1xyz),$$

introduced in [Buc90], defines a two-valued formal group

$$\mathcal{B}_a(z; x, y) = 0.$$

Types of Two-Valued Formal Groups

A two-valued formal group

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$$

over a ring R is called *of the first or the second type* if

$$\Theta_2(x, y) \equiv (x - y)^2 \quad \text{or} \quad \Theta_2(x, y) \equiv (x + y)^2 \pmod{\deg 3}.$$

Theorem [Buc75]

Over a ring R without zero divisors, any 2-valued formal group is either of the first or the second type.

Theorem 6.4(i), [Buc75]

Let $z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$ be an arbitrary two-valued formal group $\mathbb{G}(R)$ in formal power series over an arbitrary \mathbb{Q} -algebra R . Let $B(x)$ be its logarithm.

Then $B(x)$ satisfies the differential equation

$$\frac{1}{2}\phi_1(x)B'(x) + \frac{1}{8}\phi_2(x)B''(x) = 1$$

with the initial condition $B(0) = 0$, where

$$\phi_1(x) = \left. \frac{\partial \Theta_1(x, y)}{\partial y} \right|_{y=0}, \quad \phi_2(x) = \left. \frac{\partial \sigma(x, y)}{\partial y} \right|_{y=0}, \quad \sigma(x, y) = \Theta_1^2 - 4\Theta_2.$$

Suppose $\mathbb{G}(R)$ is of the first type. Then the series $B(x)$ defines a strong isomorphism of this two-valued formal group with the elementary two-valued formal group of first type.

Theorem 6.4, [Buc75]

Moreover, for the logarithm of $\mathbb{G}(R)$ we have:

$$B(x) = \left(\int_0^{\sqrt{x}} \frac{dt}{\sqrt{\phi(t^2)}} \right)^2, \quad \phi_2(x) = 8 \int_0^x \phi_1(t) dt,$$

where $\phi(t) = \phi_2(t)/(16t)$ and $\phi(0) = 1$.

Geometric 2-Valued Formal Group

How can we get an analogue of the formal group of complex cobordisms in the quaternionic case?

Let $\zeta_1 = \text{pr}_1^*(\zeta)$ and $\zeta_2 = \text{pr}_2^*(\zeta)$ be quaternionic line bundles, where ζ is the universal quaternionic line bundle over $\mathbb{H}P^\infty$

Problem: $\zeta_1 \otimes_{\mathbb{H}} \zeta_2$ and $\zeta_1 \otimes_{\mathbb{C}} \zeta_2$ are not quaternionic.

$$\begin{array}{ccc} (\eta_1 \oplus \bar{\eta}_1) \otimes_{\mathbb{C}} (\eta_2 \oplus \bar{\eta}_2) & \longrightarrow & \zeta_1 \otimes_{\mathbb{C}} \zeta_2 \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty & \xrightarrow{\iota \times \iota} & \mathbb{H}P^\infty \times \mathbb{H}P^\infty \end{array}$$

$$(\eta_1 \oplus \bar{\eta}_1) \otimes_{\mathbb{C}} (\eta_2 \oplus \bar{\eta}_2) \cong (\eta_1 \eta_2 \oplus \bar{\eta}_1 \bar{\eta}_2) \oplus (\eta_1 \bar{\eta}_2 \oplus \bar{\eta}_1 \eta_2).$$

Introduce classes $z_1, z_2 \in U^4(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ as Chern classes in U-theory:

$$z_1 := c_2^U(\eta_1 \eta_2 \oplus \bar{\eta}_1 \bar{\eta}_2) = c_1^U(\eta_1 \eta_2) c_1^U(\bar{\eta}_1 \bar{\eta}_2) = F_U(u, v) F_U(\bar{u}, \bar{v}),$$

$$z_2 := c_2^U(\eta_1 \bar{\eta}_2 \oplus \bar{\eta}_1 \eta_2) = c_1^U(\eta_1 \bar{\eta}_2) c_1^U(\bar{\eta}_1 \eta_2) = F_U(u, \bar{v}) F_U(\bar{u}, v),$$

where $u = c_1^U(\eta_1)$, $v = c_1^U(\eta_2)$, $\bar{u} = c_1^U(\bar{\eta}_1)$, $\bar{v} = c_1^U(\bar{\eta}_2)$.

Let

$$x = \iota^* p_1^{\text{Sp}}(\zeta_1) = u\bar{u} \quad \text{and} \quad y = \iota^* p_1^{\text{Sp}}(\zeta_2) = v\bar{v} \in \text{Sp}^4(\mathbb{C}P^\infty),$$

where $p_1^{\text{Sp}}(\zeta_j) \in \text{Sp}^4(\mathbb{H}P^\infty)$ denotes the Borel class in Sp-theory, and ι is the map $\mathbf{B}S^1 \rightarrow \mathbf{B}Sp(1)$. Define series:

$$\Theta_1 := p_1^{\text{Sp}}(\iota^*(\zeta_1 \otimes_{\mathbb{C}} \zeta_2)) = z_1 + z_2,$$

$$\Theta_2 := p_2^{\text{Sp}}(\iota^*(\zeta_1 \otimes_{\mathbb{C}} \zeta_2)) = z_1 z_2.$$

Modulus Square Construction

Let $F(u, v)$ be a formal group over a ring A .

Let us define the series:

$$\begin{aligned} z_1 &= F(u, v)F(\bar{u}, \bar{v}), & z_2 &= F(u, \bar{v})F(\bar{u}, v) \\ \Theta_1 &= z_1 + z_2, & \Theta_2 &= z_1 z_2. \end{aligned}$$

The series Θ_1 and Θ_2 belong to the ring $A[[x, y]]$.

Proposition [BN71]

A two-valued formal group \mathbb{G}_F is defined by the roots z_1, z_2 of an equation over $A[[x, y]]$:

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0.$$

Main Theorem

Theorem (K., 2026)

(I) Let

$$u := I(x) = \int_0^{\sqrt{x}} \frac{dt}{\sqrt{1 + a_1 t^2 + a_2 t^4 + a_3 t^6}},$$
$$g_2 = 4 \left(\frac{a_1^2}{3} - a_2 \right), \quad g_3 = 4 \left(\frac{a_1 a_2}{3} - \frac{2a_1^3}{27} - a_3 \right).$$

Then

$$x(u) = \frac{1}{\wp(u; g_2, g_3) - a_1/3}.$$

(If $g_2^3 - 27g_3^2 = 0$, then the function

$\wp(u; g_2, g_3) = -\frac{d^2}{du^2} \log \sigma(u; g_2, g_3)$ corresponds to a degeneration of the Weierstrass σ -function).

Theorem (K., 2026)

(III) The logarithm and exponential of the formal two-valued group \mathbb{G}_a are given by the following formal series over the algebra $\mathbb{Q}[a_1, a_2, a_3]$:

$$B(x) = I^2(x) = \left(\int_0^{\sqrt{x}} \frac{dt}{\sqrt{1 + a_1 t^2 + a_2 t^4 + a_3 t^6}} \right)^2,$$
$$B^{-1}(x) = \frac{1}{\wp(\sqrt{x}; g_2, g_3) - a_1/3}.$$

Theorem (K., 2026)

(III) Let $Q(t) := 1 - a_1 t^2 + a_2 t^4 - a_3 t^6$. Then the law $F_{Bc}(u, v)$, its exponential $f_{Bc}(u)$, and its logarithm $g_{Bc}(u)$ have the form:

$$F_{Bc}(u, v) = \left[\left(\frac{u\sqrt{Q(v)} - v\sqrt{Q(u)}}{u^2 - v^2} \right)^2 + a_3 u^2 v^2 \right]^{-1/2},$$

$$f_{Bc}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3) + \frac{a_1}{3}}},$$

$$g_{Bc}(u) = \int_0^u \frac{dt}{\sqrt{1 - a_1 t^2 + a_2 t^4 - a_3 t^6}}.$$

Theorem (K., 2026)

(IV) The formal group $F_{\text{Bc}}(u, v)$ is universal (over \mathbb{Q} -algebras without zero divisors) in the class of all single-valued formal groups with the condition

$$\bar{u} = -u,$$

for which the modulus square construction yields the two-valued formal group \mathbb{G}_a with the two-valued law $\mathcal{B}_a(z; x, y)$.

(V) The intersection of the classes $\mathcal{F}(u, v)$ and $F_{\text{Bc}}(u, v)$ coincides with the Ochanine genus. This is characterized by the conditions $\lambda_1(u) \equiv 1$ for $\mathcal{F}(u, v)$ and $a_3 = 0$ for $F_{\text{Bc}}(u, v)$.

Corollaries

Corollary (K., 2026)

The formal group $F_{Bc}(u, v)$ corresponds to a Hirzebruch genus

$$Bc : \Omega_U \rightarrow \mathbb{Z} \left[\frac{1}{2}, a_1, a_2, a_3 \right],$$

which we call the Buchstaber genus, where $\deg a_i = -4i$, $i = 1, 2, 3$.

In [BV24], it was established that the exponential of the universal formal group of complex cobordism is given by the series

$$f_U(u) = u + \sum_{n \geq 1} [\Theta_n] \frac{u^{n+1}}{(n+1)!},$$

where $[\Theta_n]$ denotes the cobordism class of a smooth theta divisor (of complex dimension n) on a general principally polarised abelian variety \mathcal{A}^{n+1} .

From the universality of the exponential f_U , we obtain the identity:

$$f_{\text{Bc}}(u) = u + \sum_{n \geq 1} \text{Bc}(\Theta_n) \frac{u^{n+1}}{(n+1)!}.$$

From the oddness of $f_{\text{Bc}}(u)$ it follows that

$$\text{Bc}(\Theta_{2n+1}) = 0 \quad \text{for all } n \geq 0.$$

We have:

$$\text{Bc}(\Theta_2) = -a_1, \quad \text{Bc}(\Theta_4) = a_1^2 + 12a_2,$$

$$\text{Bc}(\Theta_6) = -a_1^3 - 132a_1a_2 - 360a_3.$$

Observe that the first three nonzero values $\text{Bc}(\Theta_n)$ lie in the ring $\mathbb{Z}[a_1, a_2, a_3]$. It is not a coincidence.

Proposition (K., 2026)

The values $\text{Bc}(\Theta_n)$ belong to the ring $\mathbb{Z}[a_1, a_2, a_3]$.

Now we compute the values of the Buchstaber genus on $\mathbb{C}P^n$'s. Applying the genus to Mishchenko's logarithm for the formal group $F_U(u, v)$, we get:

$$g_{\text{Bc}}(u) = \sum_{n \geq 0} \text{Bc}(\mathbb{C}P^n) \frac{u^{n+1}}{n+1}.$$

Hence,

$$\text{Bc}(\mathbb{C}P^{2n+1}) = 0 \quad \text{for all } n \geq 0.$$

We obtain:

$$\text{Bc}(\mathbb{C}P^2) = \frac{a_1}{2}, \quad \text{Bc}(\mathbb{C}P^4) = \frac{3a_1^2 - 4a_2}{8}$$

$$\text{Bc}(\mathbb{C}P^6) = \frac{5a_1^3 - 12a_1a_2 + 8a_3}{16}.$$

There exists a class $K \in \Omega_U^{-12}$ on which the Krichever genus tensored by \mathbb{Q} vanishes, while the Buchstaber genus remains nonzero, even for $a_1 = a_2 = 0$. From the value $\text{Bc}(\mathbb{C}P^6)$, degree 12 is minimal for the existence of such a class.

Proposition (K., 2026)

(I) Let $K \in \Omega_U^{-12}$:

$$K = \Theta_6 + 9\Theta_1^6 - 15\Theta_1^4\Theta_2 - 3\Theta_1^3\Theta_3 - 13\Theta_1^2\Theta_2^2 + 3\Theta_1^2\Theta_4 \\ + 29\Theta_1\Theta_2\Theta_3 + 10\Theta_2^3 - 11\Theta_2\Theta_4 - 10\Theta_3^2.$$

Then $\text{Bc}(K) = -360a_3$, $\text{Kr}(K) = 0$ in $A \otimes \mathbb{Q}$.

(II) $\text{Ker}(\text{Kr} : \Omega_U^{-12} \otimes \mathbb{Q} \rightarrow \mathbb{Q}[\alpha, \wp(z), \wp'(z), g_2])$ is generated by K and

$$L = \Theta_5\Theta_1 - 3\Theta_4\Theta_1^2 - 11\Theta_3\Theta_2\Theta_1 + 12\Theta_3\Theta_1^3 + 22\Theta_2^2\Theta_1^2 - 30\Theta_2\Theta_1^4 + 9\Theta_1^6.$$

Recall that the Witten genus [Wit88] corresponds to a formal group with the exponential $f_{\text{Wt}}(u) = \sigma(u; g_2, g_3)$ given by the Weierstrass σ -function.

Set $a_1 = 0$ and compare the Buchstaber $f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}$ and Witten $f_{\text{Wt}}(u) = \sigma(u; g_2, g_3)$ exponentials:

$$f_{\text{Bc}}(u) = u - 3g_2 \frac{u^5}{5!} + 90g_3 \frac{u^7}{7!} + 189g_2^2 \frac{u^9}{9!} - 43740g_2g_3 \frac{u^{11}}{11!} + (-68607g_2^3 + 2673000g_3^2) \frac{u^{13}}{13!} + O(u^{15})$$

$$f_{\text{Wt}}(u) = u - \tilde{g}_2 \frac{u^5}{5!} - \tilde{g}_3 \frac{u^7}{7!} - 9\tilde{g}_2^2 \frac{u^9}{9!} - 6\tilde{g}_2\tilde{g}_3 \frac{u^{11}}{11!} + (69\tilde{g}_2^3 - 6\tilde{g}_3^2) \frac{u^{13}}{13!} + O(u^{15}),$$

where $\tilde{g}_2 = \frac{g_2}{2}$ and $\tilde{g}_3 = 6g_3$.

Recall that a formal power series $u + \sum_{n \geq 1} c_n \frac{u^{n+1}}{(n+1)!}$, where $c_n \in R$, is called a *Hurwitz series* over a ring R .

It is known (see, e.g., [BL05]) that $f_{\text{Wt}}(z)$ is a Hurwitz series over the ring $\mathbb{Z}\left[\frac{g_2}{2}, 2g_3\right]$.

In [Bun17], Bunkova conjectured that $f_{\text{Wt}}(z) = \sigma(z; g_2, g_3)$ is a Hurwitz series over the ring $\mathbb{Z}\left[\frac{g_2}{2}, 6g_3\right]$.

Proposition (K., 2026)

For $a_1 = 0$, the exponential $f_{\text{Bc}}(u) = \frac{1}{\sqrt{\wp(u; g_2, -g_3)}}$ of the Buchstaber genus is a Hurwitz series over the ring $\mathbb{Z}[g_2, g_3]$.

Theory $\text{Bc}^*(-)[\Delta^{-1}]$

Theorem (K., 2026)

There exists a complex oriented multiplicative commutative cohomology theory

$$\text{Bc}^*[\Delta^{-1}]$$

with the coefficient ring

$$\mathbb{Z} \left[\frac{1}{2}, a_1, a_2, a_3, \Delta^{-1} \right],$$

$\Delta = \text{disc}_t(t^3 - a_1 t^2 + a_2 t - a_3)$ such that on the natural multiplicative transformation $U^*(-) \rightarrow \text{Bc}^*[\Delta^{-1}]$ induces the localized Buchstaber genus map

$$\Omega_U \rightarrow \text{Bc}^*(\text{pt})[\Delta^{-1}] \cong \mathbb{Z} \left[\frac{1}{2}, a_1, a_2, a_3, \Delta^{-1} \right].$$

Theorem (K., 2026)

There exists a complex oriented multiplicative commutative cohomology theory $Bc^(-)$ such that the map $U^*(-) \rightarrow Bc^*$ give the Buchstaber genus map*

$$\Omega_U \rightarrow \mathbb{Z} \left[\frac{1}{2}, a_1, a_2, a_3 \right].$$

The previous two theorems are proved by the same technique as used in the paper [LRS95].

Rigidity of Equivariant Odd Genera

Proposition (K., 2026)

Let m_0, \dots, m_n be pairwise distinct vector in \mathbb{Z}^k (where $k \geq 1$) such that

$$m_{n-i} + m_i = m$$

for any $0 \leq i \leq n$ and some fixed $m \in \mathbb{Z}^k$.

For each integer $n \geq 0$ consider a manifold $\mathbb{C}P^n$ with a torus action \mathbb{T}^k :

$$t \cdot [z_0 : \dots : z_{2n+1}] = [t^{m_0} z_0 : t^{m_1} z_1 \dots : t^{m_n} z_n],$$

where $t^{m_i} = t_1^{m_{i_1}} \dots t_k^{m_{i_k}}$. Let n be odd.

Then any odd \mathbb{T}^k -equivariant genus $\mathcal{G}^{\mathbb{T}^k}$ vanishes and, hence, it is \mathbb{T}^k -rigid on the manifold $\mathbb{C}P^n$.

Example: Action on $\mathbb{C}\mathbb{P}^3$

Consider $\mathbb{C}\mathbb{P}^3$ with an effective circle action:

$$t \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : tz_1 : t^2 z_2 : t^4 z_3]$$

The fixed points are of the form $z_i = 1, z_j = 0$ for all $i = 0, \dots, 3, j \neq i$.

At these fixed points, the weights of the tangent representation of the torus action are:

$$(1, 2, 4), \quad (-1, 1, 3), \quad (-2, -1, 2), \quad (-4, -3, -2).$$

From the Buchstaber–Panov–Ray formula:

$$\begin{aligned} \mathrm{Bc}^{\mathbb{S}^1}(\mathbb{C}\mathbb{P}^3)(f(u)) &= \frac{1}{f(u)f(2u)f(4u)} + \frac{1}{f(-u)f(u)f(3u)} \\ &+ \frac{1}{f(-2u)f(-u)f(2u)} + \frac{1}{f(-4u)f(-3u)f(-2u)}, \end{aligned}$$

where $f = f_{\mathrm{Bc}}$.

Example: Action on $\mathbb{C}P^3$

From formulas (41) and

$$f'(u)^2 = 1 - a_1 f(u)^2 + a_2 f(u)^4 - a_3 f(u)^6$$

a direct calculation implies that:

$$\mathrm{Bc}^{\mathbb{S}^1}(\mathbb{C}P^3) = \frac{15}{2} a_3 u^2 + O(u^4). \quad (2)$$

It follows from (2) that $\mathrm{Bc}^{\mathbb{S}^1}(\mathbb{C}P^3) = 0$ if and only if Bc coincides with the Krichever genus.

This is consistent with the classical result of Hirzebruch [HBJ92, Section 4.6, Proposition].

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Thank you
for your attention!