Hyperbolic links associated to Hamiltonian subgraphs in simple 3-polytopes

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- In

A.D. Mednykh,
Three-dimensional hyperelliptic manifolds.
Ann. Global. Anal. Geom., 8:1 (1990), 13–19.

A.Yu. Vesnin, A.D. Mednykh,
Three-dimensional hyperelliptic manifolds and Hamiltonian graphs.
Siberian Math. J., 40:4 (1999), 628–643.

  A.D. Mednykh and A.Yu. Vesnin introduced a construction that

  - for a compact right-angled polytope P in geometry $\mathbb{L}^3$, $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{L}^2 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}$ and
  - a Hamiltonian cycle, Hamiltonian theta-subgraph or Hamiltonian K_4-subgraph Γ in the 1-skeleton of P

  builds a geometric 3-manifold $N(P, \Gamma)$ with an involution τ such that $N(P, \Gamma)/\langle\tau\rangle \simeq S^3$.

- The brach set of the corresponding 2-sheeted branched covering $N(P, \Gamma) \to S^3$ is a link $C_\Gamma \subset S^3$ consisting of trivially embedded circles.

- This construction reformulated in the language of toric topology works for such a Hamiltonian subgraph Γ in any simple 3-polytope P and gives a topological 3-manifold $N(P,\Gamma)$.
- We give a criterion when $S^3 \setminus C_\Gamma$ has a complete hyperbolic structure of finite volume and generalize this criterion to similar links in 3-manifolds different from S^3.
- A a corollary we prove that hyperbolic links C_Γ are parametrized by nonselfcrossing Eulerian cycles, theta-subgraphs and K_4-subgraphs in hyperbolic right-angled 3-polytopes of finite volume with 0, 2 or 4 finite vertices.

Introduction

- The link corresponding to a Hamiltonian cycle in a simple 3-polytope always contains the Hopf link consisting of two trivially embedded circles linked once.
- A theta-subgraph in the cube I^3 corresponds to the Borromean rings.
- We give a criterion when the link C_Γ consists of mutually unlinked circles and prove that if such a link is nontrivial, then it contains the Borromean rings.

Introduction

- This is a partial answer to the question posed by Victor Buchstaber

Using technique of toric topology to build a rich family of Brunnian links, that is nontrivial links that become a set of trivial unlinked circles if any one component is removed.

- The question is motivated by the notion of a Efimov state in quantum mechanics.
- This is a bound state of three bosons such that the two-particle attraction is too weak to allow two bosons to form a pair. If one of the particles is removed, the remaining two fall apart.

The only Brunnian link C_Γ is the Borromean rings, but there are many nontrivial links C_Γ consisting of mutually unlinked circles.

Manifolds defined by vector-colorings of polytopes

- A vector-coloring of rank r of a simple n-polytope P is a mapping Λ from the set of its facets $F_1, \dots, F_m$ to $\mathbb{Z}_2^r$, $F_i \to \Lambda_i$, such that $\langle \Lambda_1, \dots, \Lambda_m \rangle = \mathbb{Z}_2^r$.
- It corresponds to the space

$$N(P, \Lambda) = P \times \mathbb{Z}_2^r / \sim, \text{ where } (p, a) \sim (q, b)$$

if and only if $p = q$ and $a - b \in \langle \Lambda_i : p \in F_i \rangle$.

- This space has an action of $\mathbb{Z}_2^r$.

Free actions

- The action of $H(\Lambda)$ is free if and only if for any vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$ the images $\Lambda(F_{i_1}), \dots, \Lambda(F_{i_n})$ are linearly independent.
- We call such vector-colorings linearly independent.
- In this case $N(P, \Lambda)$ is a smooth manifold.
- $N(P, \Lambda)$ is a topological manifold if and only if for any vertex different nonzero vectors Λ_j are linearly independent.
- The boundary is glued of copies of facets F_i with $\Lambda_i = 0$.
- $N(P, \Lambda)$ is orientable iff there is $c \in (\mathbb{Z}_2^r)^*$ such that $\langle c, \Lambda_i \rangle = 1$ for any nonzero Λ_i.

- The following construction was invented in the works by A.Yu. Vesnin and A.D. Mednykh (starting from 1985).
- Given a right-angled polytope P of finite volume in some geometry X, where $X = \mathbb{S}^n, \mathbb{R}^n, \mathbb{L}^n$, or a product of such spaces, consider the right-angled Coxeter group

$$G(P) = \langle \rho_1, \dots, \rho_m | \rho_i^2 = 1, \rho_i\rho_j = \rho_j\rho_i \text{ if } F_i \cap F_j \neq \varnothing \rangle$$

  It is isomorphic to a subgroup of isometries of X generated by reflexions ρ_i in hyperplanes corresponding to facets F_i.
- An epimorphism $\varphi_\Lambda \colon G(P) \to \mathbb{Z}_2^r$, $\rho_i \to \Lambda_i$ such that $\Lambda_{i_1}, \dots, \Lambda_{i_k}$ are linearly independent, whenever $F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing$, gives a subgroup $G(P, \Lambda) = \operatorname{Ker} \varphi_\Lambda$ which acts freely on X.

Geometric manifolds

If P is compact, then $X/G(P,\Lambda)$ is a geometric manifold homeomorphic to $N(P,\Lambda)$.

If $P \subset L^n$ has finite volume, then $X/G(P,\Lambda)$ is a geometric manifold homeomorphic to $\operatorname{int} N(\widehat{P},\widehat{\Lambda})$, where $\widehat{P}$ is obtained from P by cutting off ideal vertices, and $\widehat{\Lambda}(F_i) = 0$ for each cubical facet F_i corresponding to a cut vertex.

Factor-spaces

Proposition

Let Λ be a vector-coloring of rank r of a simple polytope P. For a subgroup $H \subset \mathbb{Z}_2^r$ we have $N(P,\Lambda)/H \simeq N(P,\Lambda_H)$, where Λ_H is a vector coloring of rank $r-1$ obtained as the composition $\pi \circ \Lambda$, where π is the projection $\mathbb{Z}_2^r \to \mathbb{Z}_2^r/H$.

- Let Λ be a linearly independent vector-coloring of rank r of a simple 3-polytopes P, and $\tau \in \mathbb{Z}_2^r \setminus \{0\}$;
- If the orbit space $N(P,\Lambda)/\langle\tau\rangle = N(P,\Lambda_\tau)$ is a closed manifold, then the mapping $N(P,\Lambda) \to N(P,\Lambda)/\langle\tau\rangle$ is a 2-sheeted branched covering. That is, locally it is either a 2-sheeted covering, or it is modeled by the mapping

$$(z,t) \in \mathbb{C} \times \mathbb{R} \to (z^2,t) \in \mathbb{C} \times \mathbb{R}$$

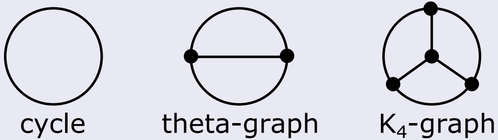
- The branch set is the link $C_\tau \subset N(P,\Lambda)/\langle\tau\rangle$ consisting of circles corresponding to edges $E_{i,j} = F_i \cap F_j$ of P such that $\Lambda_i + \Lambda_j = \tau$.
- Denote this set of edges M_τ. It is a matching, that is any two edges are disjoint.
- The edge $E_{i,j}$ connecting the vertices $F_i \cap F_j \cap F_k$ and $F_i \cap F_j \cap F_l$ corresponds to $2^{r-2-c_{i,j}}$ circles, where $c_{i,j} = 2$, if Λ_i, Λ_j, Λ_k and Λ_l are linearly independent, and $c_{i,j} = 1$, otherwise.
- Each circle is glued of $2^{c_{i,j}}$ copes of $E_{i,j}$.
- If $M_\tau = \varnothing$, then τ acts freely, the mapping $N(P,\Lambda) \to N(P,\Lambda)/\langle\tau\rangle$ is a 2-sheeted covering, and $C_\tau = \varnothing$.

Definition

- A hyperelliptic manifold M^n is a topological manifold M^n with an involution τ such that the orbit space $M^n/\langle\tau\rangle$ is homeomorphic to S^n.
- the involution τ is called hyperelliptic.

Theorem (E., 24)

Let Λ be a linearly independent vector-coloring of rank r of a simple 3-polytopes P, and $\tau \in \mathbb{Z}_2^r \setminus \{0\}$. Then $N(P,\Lambda)/\langle\tau\rangle \simeq S^3$ if and only if $\partial P \setminus M_\tau$ has one of the forms



and Λ_τ sends faces of this graph to a basis.

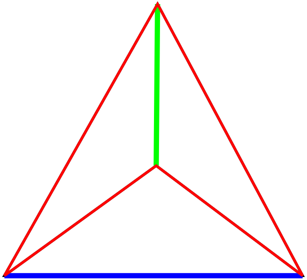
Vector-coloring induced by a Hamiltonian subgraph

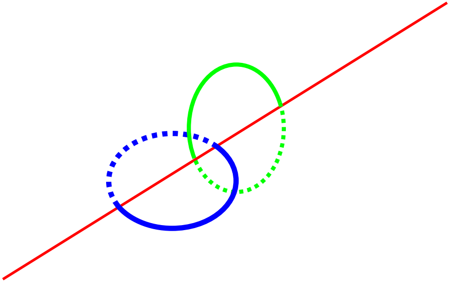
- The subgraph Γ in the 1-skeleton of P of the above form is either a Hamiltonian cycle, or a Hamiltonian theta-subgraph, or a Hamiltonian K_4-subgraph.
- Hamiltonian means that it contains all the vertices of P.
- Λ and τ can be recovered from Γ as follows
  - For each face of Γ the adjacency graph of facets of P lying in this face is a tree and the facets can be colored in black and white colors in such a way that adjacent facets have different colors.
  - Let $e_1, \dots, e_{r-1}, \tau$ be a basis in $\mathbb{Z}_2^r$, where $(r-1)$ is the number of faces in Γ.
  - Assume that F_i lies in j-th face of Γ. Set $\Lambda_\Gamma(F_i) = e_j$, if F_j is black, and $\Lambda_\Gamma(F_i) = e_j + \tau$, if F_j is white.
  - Then $\Lambda = C\Lambda_\Gamma$ for $C \in Gl_r(\mathbb{Z}_2)$.

If Γ is a Hamiltonian cycle, Hamiltonian theta-subgraph or a Hamiltonian K_4-subgraph in a 1-skeleton of a compact right-angled polytope P in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{L}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{L}^2 \times \mathbb{R}$, then $N(P, \Lambda_\Gamma)$ is exactly a geometric hyperelliptic manifold built by A.D. Mednykh and A.Yu.Vesnin.

Example: Hopf link

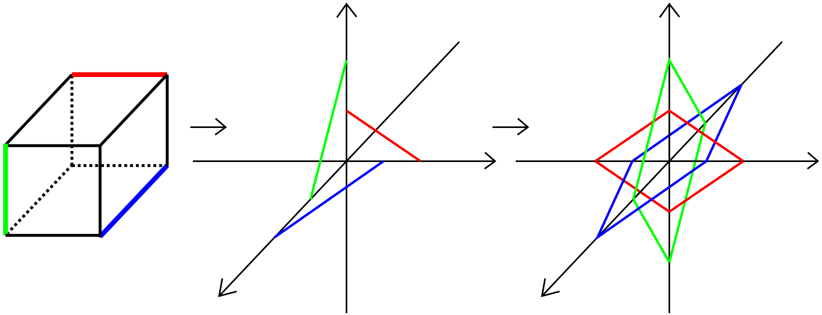
The simplest example is given by a Hamiltonian cycle in the simplex. We have $\mathrm{N}(\Delta^3, \Lambda_\Gamma) \simeq \mathbb{R}\mathrm{P}^3$ with spherical structure, and C_Γ is the Hopf link.





Example: Borromean rings

Theta-subgraph in the cube gives a Euclidean manifold $N(I^3, \Lambda_\Gamma)$, and C_Γ is the Borromean rings.



This example goes back to the Thurston's book The Geometry and Topology of 3-manifolds.

Definition

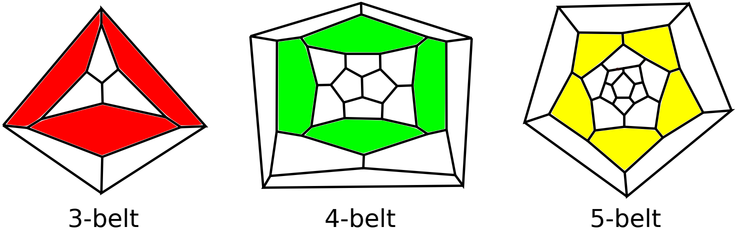
A link C in a closed 3-manifold M is a collection of embedded circles, which is a topological submanifold, that is for any point in M there is a homeomorphism φ of its neighbourhood U to an open set $V \subset \mathbb{R}^3$ such that $\varphi(C \cap U) = V \cap \{y = z = 0\}$.

Definition

We call a link C in a 3-manifold M hyperbolic, if its complement $M \setminus C$ admits a complete hyperbolic metric of finite volume.

We will give a criterion when the link C_τ is hyperbolic.

A k-belt ($k \geqslant 3$) of a 3-polytope is a cyclic sequence of k faces such that faces are adjacent if and only if they follow each other and no three faces have a common vertex.



Proposition

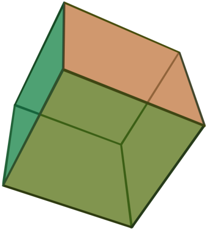
Any simple 3-polytope $P \neq \Delta^3$ has a 3-, 4-, or 5-belt.

Flag polytopes

Definition

A simple 3-polytope P is flag, if $P \neq \Delta^3$ and P has no 3-belts.

A flag 3-polytope with the smallest number of facets is the cube.



A closed manifold $N(P, \Lambda)$ corresponding to a linearly independent coloring Λ of a simple 3-polytope is aspherical (i.e. $\pi_i(N(P, \Lambda)) = 0$ for $i \geqslant 2$) iff P is flag.

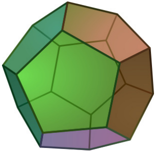
Pogorelov polytopes

A simple 3-polytope is Pogorelov, if it is flag and has no 4-belts.

Theorem (A.V. Pogorelov, 1967, E.M. Andreev, 1970)

A combinatorial polytope P can be realized in $\mathbb{L}^3$ as compact right-angled polytope iff P is Pogorelov.

A Pogorelov polytope with minimal m is the dodecahedron.



A closed manifold $N(P, \Lambda)$ corresponding to a linearly independent coloring Λ of a simple 3-polytope is hyperbolic iff P is Pogorelov.

Almost Pogorelov Polytopes

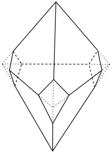
A simple 3-polytope is almost Pogorelov if it is flag and any 4-belt surrounds a face.

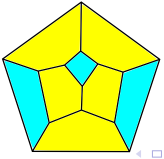
An almost Pogorelov polytope has adjacent quadrangles iff P is I^3 or the 5-prism.

Andreev's theorem implies (M.W. Davis – B. Okun, 2001)

Cutting off 4-valent (=ideal) vertices gives a bijection between right-angled polytopes of finite volume in $\mathbb{L}^3$ and almost Pogorelov polytopes without adjacent quadrangles and produces all the quadrangles.

Al. Pog. polytope with minimal m is the Stasheff polytope.





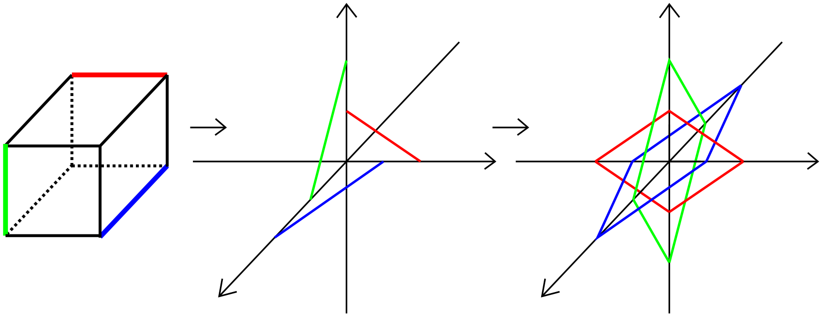
- Denote by $G(P)$ the 1-skeleton of P. For a matching $M \subset G(P)$ let $G(P)/M$ be the plane graph obtained by contraction of all edges of M.
- A belt has k edges in M if k edges of intersection of its successive facets lie in M.

Theorem (E., 26)

Let Λ be a linearly independent vector-coloring Λ of rank r of a simple 3-polytope P and $\tau \in \mathbb{Z}_2^r \setminus \{0\}$ be an involution such that $N(P,\Lambda)/\langle\tau\rangle$ is a closed topological manifold and $M_\tau \neq \varnothing$. Then the following conditions are equivalent.

1. P is flag and any its 4-belt has at least one edge in M_τ.
2. $G(P)/M_\tau$ is a graph of a right-angled hyperbolic polytope of finite volume.
3. The link C_τ is hyperbolic.

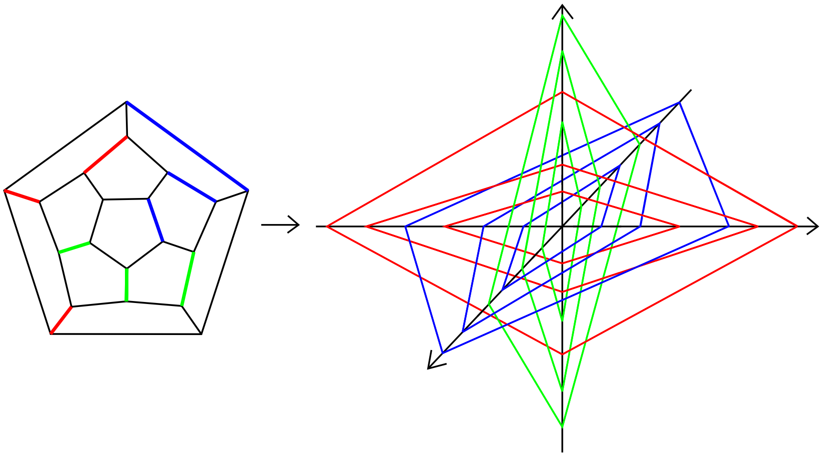
Example: Borromean rings



- In the case of the above theta-graph in the cube $G(P)/M_\Gamma$ is the right-angled 3-bipyramid. In particular, C_Γ is hyperbolic.
- This example also arises in recent works by B. Martelly.
- Recently A.Yu. Vesnin, A.A. Egorov proved that the 3-bipyramid has the smallest volume among all hyperbolic right-angled 3-polytopes of finite volume.

Corollary

For any Hamiltonian subgraph Γ in a Pogorelov polytope P the link C_Γ is hyperbolic.



Denote by $\mathcal{R}$ the family of right-angled hyperbolic 3-polytopes of finite volume.

Theorem (E., 2026)

Let M be a matching in a simple 3-polytope P. If $P = \Delta^3$ or $\Delta^2 \times I$, then $G(P)/M$ is not a graph of a polytope in $\mathcal{R}$. If $P \neq \Delta^3, \Delta^2 \times I$, then $G(P)/M$ is a graph of a polytope in $\mathcal{R}$ if and only if the following two conditions hold:

1. any 3-belt of P has at least two edges in M;
2. any 4-belt of P has at least one edge in M;

Idea of the proof

- If $G(P)/M_\tau$ is a graph of a polytope Q in $\mathcal{R}$, then $N(P,\Lambda_\tau) \setminus C_\tau$ is glued of copies of Q according to its vector-coloring induced from Λ_τ.
- If $N(P,\Lambda_\tau) \setminus C_\tau$ is hyperbolic, then is is covered by a hyperbolic manifold X, which is a subset in $\mathbb{R}Z_{P_M}$, where P_M is obtained from P by cutting off M.
- If P has a 3-belt with no edges in M_τ, then there is an essential sphere in X.
- If P has a 3-belt with one edge in M_τ, then two elements in $\pi_1(X)$ corresponding to circles in different boundary tori are conjugated.
- If P has a 4-belt with no edges in M_τ, then there is an incompressible torus in X, which is not boundary parallel.

Parametrisation of hyperbolic links C_Γ

- A cycle in a graph G is called Eulerian if it passes each edge of G once (it may pass one vertex many times).
- An Eulerian theta-subgraph in a graph G consists of three paths connecting two vertices. Each edge of G belongs to exactly one path and is traversed exactly once. (Similarly for K_4).
- An Eulerian subgraph in a plane graph is nonselfcrossing if each path traverses any its interior vertex (that is, different from its ends) by edges successive in the cyclic order around this vertex (it may visit a vertex more than once).

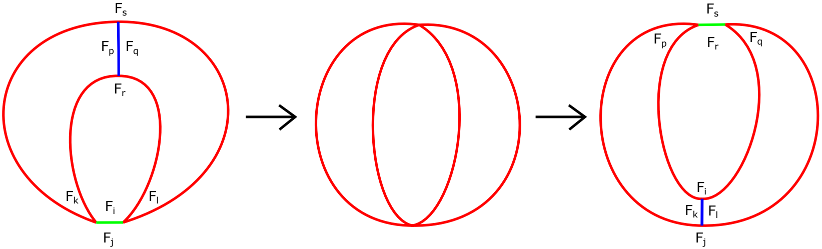
Theorem (E., 26)

Hyperbolic links C_Γ bijectively correspond to nonselfcrossing Eulerian subgraphs in right-angled hyperbolic 3-polytopes of finite volume with 0, 2, or 4 finite vertices and all the other vertices lying at infinity.

Links corresponding to Eulerian cycles

Results by A. Kotzig (1968) imply that

- Any ideal right-angled 3-polytope Q has a nonselfcrossing Eulerian cycle.
- Any two such cycles can be connected by a sequence of elementary transformations.

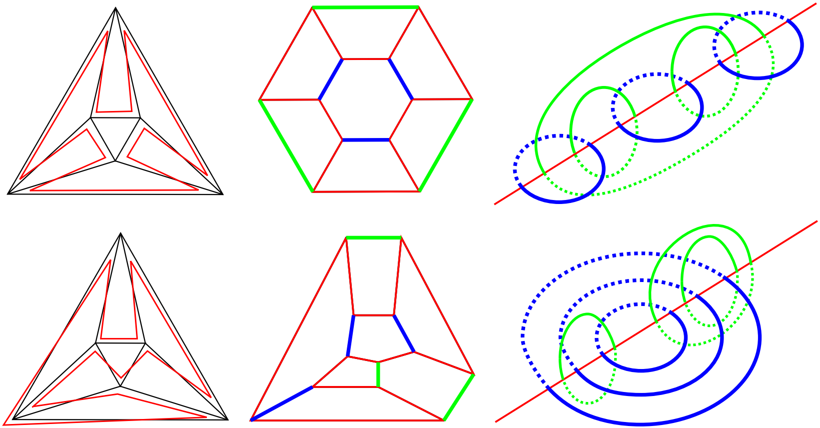


Links corresponding to Eulerian cycles

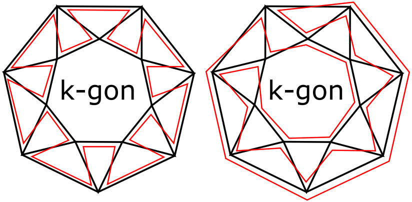
- Any nonselfcrossing Eulerian cycle γ in the ideal right-angled 3-polytope Q corresponds to a simple 3-polytope P_γ with a Hamiltonian cycle Γ_γ.
- Complement to the link C_Γ is glued of 4 copies of Q according to its checkerboard coloring.

Example: Ideal octahedron

The ideal hyperbolic octahedron has exactly 2 combinatorially different nonselfcrossing Eulerian cycles.



Each k-antiprism with k $\geqslant$ 4 has at least 2 combinatorially different nonselfcrossing Eulerian cycles.



The left cycle corresponds to the hyperbolic 2k-link chain from Thurston's book.

Links consisting of mutually unlinked circles

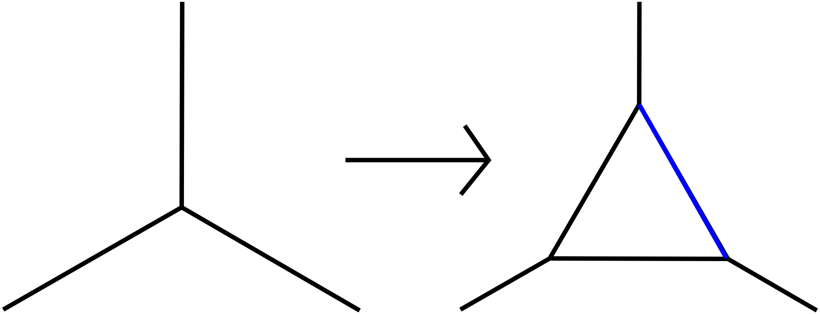
Any circle in the link C_Γ corresponding to a Hamiltonian cycle Γ forms a Hopf link with at least one other circle.

Theorem (E., 25)

Let Γ be a Hamiltonian theta-subgraph in a simple 3-polytope P. Then

1. the link C_Γ consists of mutually unlinked circles if and only if each edge of M_Γ connects vertices of different paths of Γ;
2. if C_Γ consists of mutually unlinked circles and is nontrivial, then it contains a triple of Borromean rings;
3. the link C_Γ is trivial if and only if (P, Γ) is obtained from (Δ^3, Γ_0), by a sequence of operations of cutting off a vertex.

Cutting off a vertex



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Links consisting of mutually unlinked circles

Theorem

Let Γ be a Hamiltonian K_4-subgraph in a simple 3-polytope P. Then

1. the link C_Γ consists of mutually unlinked circles if and only if M_Γ splits into matchings $M_\Gamma(v_i)$ corresponding to vertices of K_4, such that each matching consists of edges connecting the vertices on different paths of K_4 containing v_i and for any two edges $E_1 \in M_\Gamma(v_i)$ and $E_2 \in M_\Gamma(v_j)$, $i \neq j$ the triangles $v_i * E_1$ and $v_j * E_2$ do not intersect;
2. if C_Γ consists of mutually unlinked circles and is nontrivial, then it contains a triple of Borromean rings;
3. the link C_Γ is trivial if and only if (P, Γ) is obtained from $(\Delta^3, G(\Delta^3))$ by a sequence of operations of cutting off a vertex.

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