Postnikov theory of Grassmannians and its applications

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Postnikov Theory

Definition

The totally non-negative Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$ is the subset of all points of the complex Grassmannian $\mathrm{Gr}(k, n)$, whose Plücker coordinates $\Delta_I, \ I \subset \binom{n}{k}$ have the same sign or vanish.

The main result of Postnikov theory is that each point of $\mathrm{Gr}_{\geq 0}(k, n)$ can be parametrized by special weighted planar graphs – models. There are three equivalent ways to introduce them, namely Lam's approach, Postnikov's approach, and the formalism of flow models.

Lam Models: Definition

Lam, «Totally nonnegative Grassmannian and Grassmann polytopes»

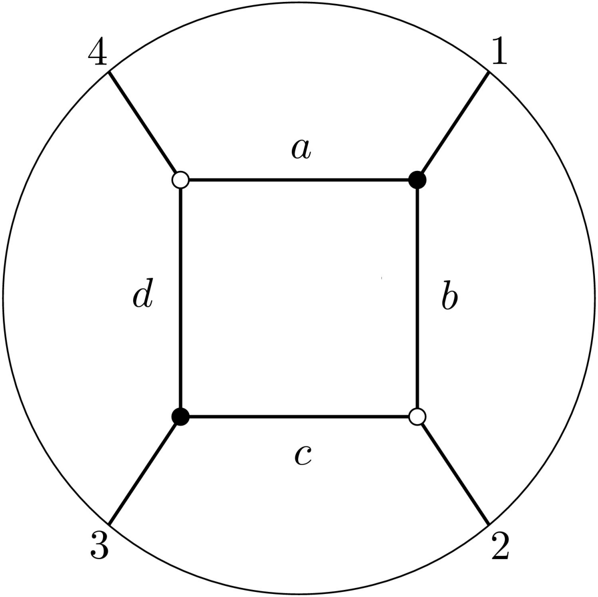
Definition

A *Lam model* $N(G, w)$ is a planar graph embedded into a disk and equipped with a function $w : E \to \mathbb{R}_{>0}$ satisfying the following conditions:

- The nodes lying on the boundary of the disk are numbered clockwise;
- The degree of each boundary node is one;
- All nodes are colored either black or white, and each edge connects nodes of different colors.

We do not indicate the colors of boundary nodes. Edge weights equal to 1 are also omitted.

Example: Lam Model



Lam Models: Dimers

Definition

A subset $\Pi \subset E$ is called a **dimer** (almost perfect matching) if it satisfies the following conditions:

- Edges in Π share no common vertices;
- All internal nodes of G are covered by edges in Π.

The weight $wt(\Pi)$ is the product of the weights of all edges in Π.

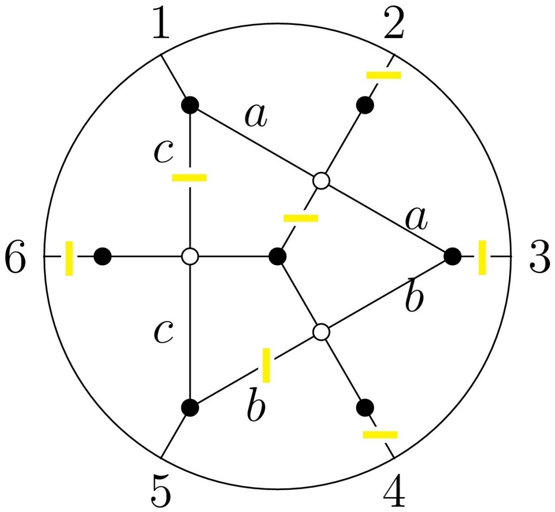
Theorem

For each dimer Π of a model $N(G, w)$, associate the Postnikov number $k(G)$ by the formula:

$$k(G) = d_1(\Pi) + d_2(\Pi),$$

where $d_1(\Pi)$ is the number of black boundary nodes covered by Π, and $d_2(\Pi)$ is the number of white boundary nodes not covered by Π. The Postnikov number does not depend on the choice of dimer Π.

Example: Dimer



Postnikov Number

Theorem

Consider a Lam model $N(G, w)$. Its Postnikov number can be calculated as follows:

$$k(G) = \frac{1}{2}\left(n + \sum_{v \in black} (\deg(v) - 2) + \sum_{v \in white} (2 - \deg(v))\right),$$

where summations are over black or white inner nodes of G.

Lam Models: Grassmannians

Definition

Consider a Lam model $N(G, w)$ with n boundary nodes. For each subset I of boundary nodes with $|I| = k(G)$, define the **dimer partition function** Δ_I by

$$\Delta_I = \sum_{\partial \Pi = I} wt(\Pi),$$

where the sum is taken over all dimers Π such that for each such dimer, the black (resp. white) boundary nodes in I are covered (resp. not covered).

Theorem

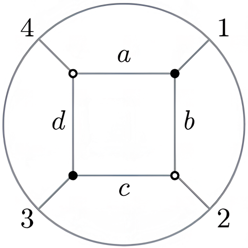
The set of partition functions Δ_I determines the Plücker coordinates of a point $X(N) \in \mathrm{Gr}_{\geq 0}(k(G), n)$; that is, all Δ_I are nonnegative and satisfy the Plücker relations.

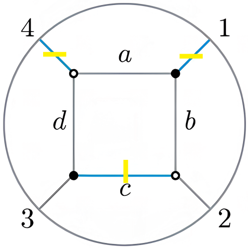
Lam Models: Example

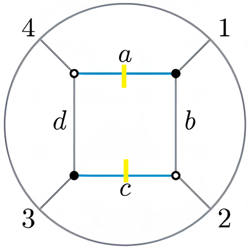
$$\Delta_{12} = a \quad \Delta_{13} = ac + bd \quad \Delta_{14} = b$$

$$\Delta_{23} = d \quad \Delta_{24} = 1 \quad \Delta_{34} = c$$

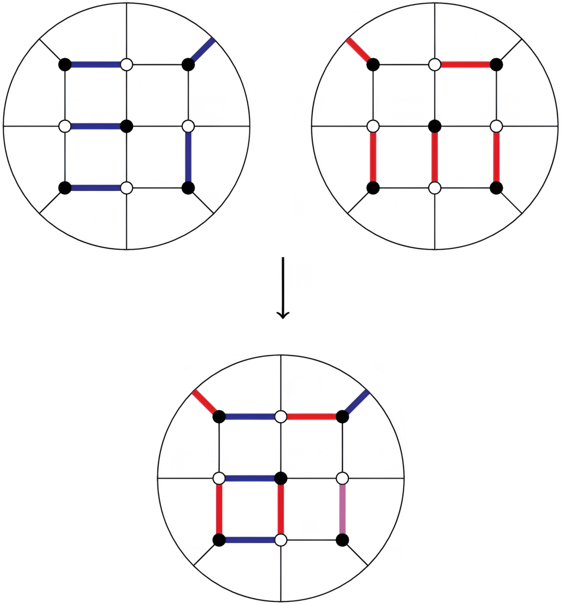
$$\Delta_{13}\Delta_{24} = \Delta_{14}\Delta_{23} + \Delta_{34}\Delta_{12}$$







Double Dimers



Direct Problem

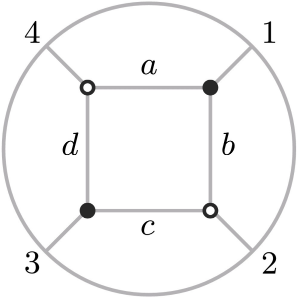
Problem

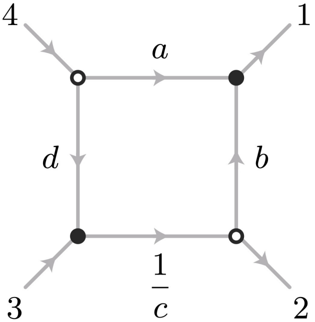
For a given Lam model $N(G, w)$ with n boundary nodes, it is required to construct a matrix that represents its corresponding point of $\mathrm{Gr}_{\geq 0}(k(G), n)$.

Theorem

Consider a Lam model $N(G, w)$, and let $O(I)$ be a perfect orientation of the edge set of G with source set I. Consider the Postnikov model $NP(G, w', O(I))$, where the weights w' are obtained by taking the reciprocals of the weights of edges going from black to white vertices (the weights of the other edges remain unchanged). Then the points in $Gr_{\geq 0}(k(G), n)$ associated with the models $N(G, \omega)$ and $NP(G, \omega', O(I))$ coincide.

Direct Problem: Example





$$A = \begin{pmatrix} \frac{b}{c} & \frac{1}{c} & 1 & 0 \\ -a - \frac{bd}{c} & -\frac{d}{c} & 0 & 1 \end{pmatrix}$$

Postnikov Models

Definition

A Postnikov model $NP(\Gamma, O, w)$ is a Lam model $NP(\Gamma, w)$ together with a **perfect orientation** O, i.e. each black interior vertex is the source of a unique edge and each white interior vertex is the target of a unique edge.

Definition

Consider a Postnikov model $NP(\Gamma, O, w)$ with n boundary vertices, and suppose that the perfect orientation O has k sources and $n-k$ sinks. Assign indices to all sources in the clockwise direction along the boundary circle: $I = \{i_1, \ldots, i_k\}$. Define a boundary measurement matrix $M(P) = (m_{i_r,j}) \in \mathrm{Mat}_{k\times(n-k)}$ as follows:

$$m_{i_r,j} = \sum_{p: i_r \to j} (-1)^{\mathrm{wind}(p)} \, \mathrm{wt}(p),$$

here $\mathrm{wt}(p)$ is the product of the weights of all edges along the path p from i_r to j; $\mathrm{wind}(p)$ is the winding index of the path p.

Postnikov Models

Theorem

Consider a Postnikov model $NP(\Gamma, \omega, O)$ with k sources and $n-k$ sinks, and define its $k \times n$ extended boundary measurement matrix A in the following way:

- *the submatrix formed by the columns with numbers equal to the indices of sources is the identity matrix;*
- *the other matrix elements are defined by $a_{r,j} = (-1)^s M(P)_{i_r,j}$, where s is the number of elements in I lying strictly between the source i_r and the sink j.*

The matrix A defines a point in $Gr_{\geq 0}(k,n)$. Each point in $Gr_{\geq 0}(k,n)$ corresponds to the extended boundary measurement matrix of an appropriate Postnikov model.

Inverse Problems

Theorem

For any point $X \in \mathrm{Gr}_{\geq 0}(k, n)$*, there exists a unique (up to Postnikov transformations) Lam model* $N(G, w)$ *whose partition functions* Δ_I *coincide with the Plücker coordinates of* X*.*

Problem

Consider a point $X \in \mathrm{Gr}_{\geq 0}(k, n)$ *and its associated Lam model* $N(G, w)$ *on a given graph* G*. Weights* w *must be recovered from Plücker coordinates of* X*.*

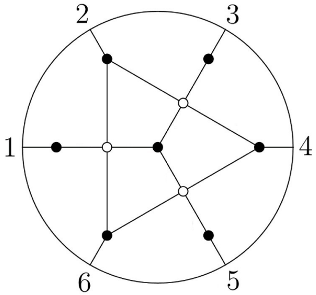
Problem

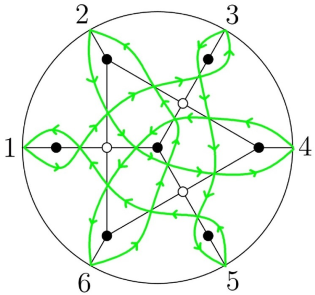
Consider a point $X \in \mathrm{Gr}_{\geq 0}(k, n)$*. Its associated Lam model* $N(G, w)$ *must be recovered from Plücker coordinates of* X*.*

Lam Models: Graph Reconstruction

Theorem

A Lam model is uniquely defined (up to Postnikov transformations) by the strand permutation g coming from its oriented medial graph.

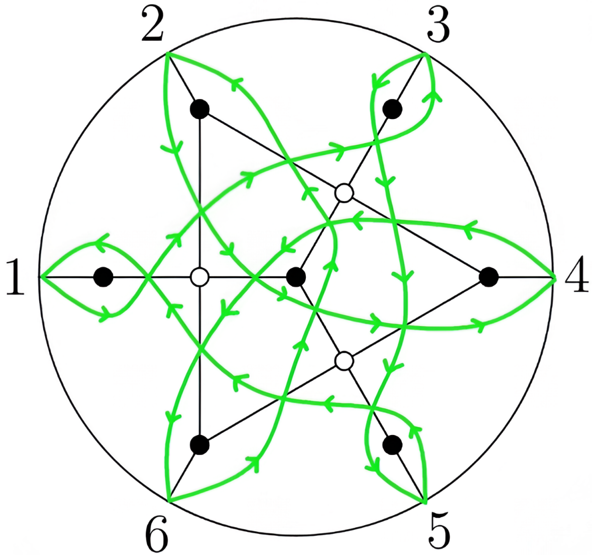




For the depicted Lam model, the strand permutation in window notation is

$$g = [345612]$$

Strand Permutation: Example



Oriented Medial Graph

Consider a Lam model $N(G, w)$. Then, its **oriented medial graph** G_M is defined as follows:

- Boundary nodes of G_M coincide with boundary nodes of $N(G, w)$;
- Inner nodes of G_M are the midpoints of the edges connecting inner nodes of G.

Two nodes of G_M are connected by an edge if the edges of the original graph G are adjacent. Edges of G_M are oriented as follows:

- Clockwise around each inner white node of G;
- Counterclockwise around each inner black node of G.

Since all inner nodes of G_M have a degree of four, we can correctly define the **strands** of G_M as the oriented paths that always go straight through any degree four node.

Lam Models: Graph Reconstruction

Theorem

Let us consider a matrix A representing a point in $Gr_{\geq 0}(k, n)$ and denote its columns by A_i. Define the column permutation $g(X)$ as follows: $g(X)(i) = j$ if j is the minimal index such that

$$A_i \in \mathrm{span}(A_{i+1}, \dots, A_j),$$

where indices are taken modulo n. Then $g(A)$ coincides with the strand permutation $g(N)$ of the Lam model associated with A.

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}.$$

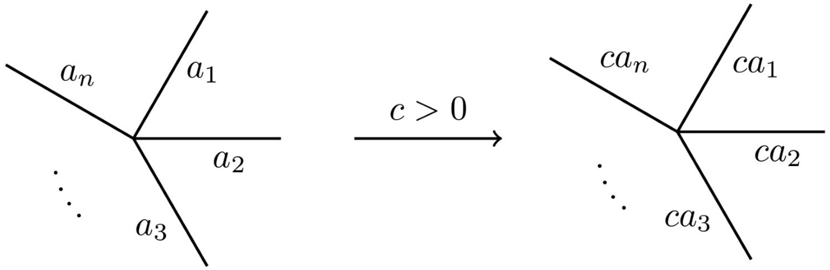
Then the following holds

$$g(A) = [4\,6\,5\,7\,8\,2\,1\,3].$$

Lam Models: Weights Recovering

Theorem

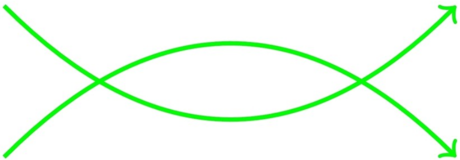
Consider a minimal Lam model $N(G, w)$ on a given graph G associated with a point $X \in Gr_{\geq 0}(k, n)$. Then, a weight function w is uniquely recovered by the Plücker coordinates of X, up to the gauge transformations.



Minimal Lam Models

A Lam model $N(G, w)$ is called **minimal** (reduced) if it satisfies the following condition:

- Each strand does not intersect itself;
- There are no loops apart from loops attached to isolated boundary nodes;
- Strands do not form oriented lenses.

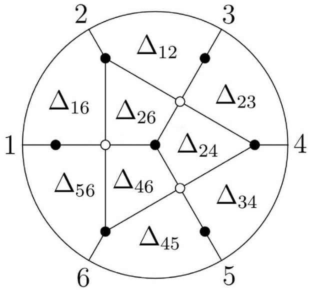


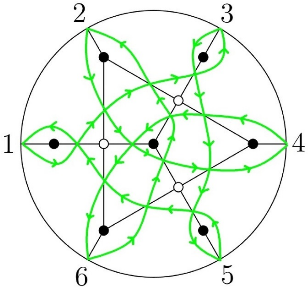
Scott Rule

Scott, «Grassmannians and cluster algebras»

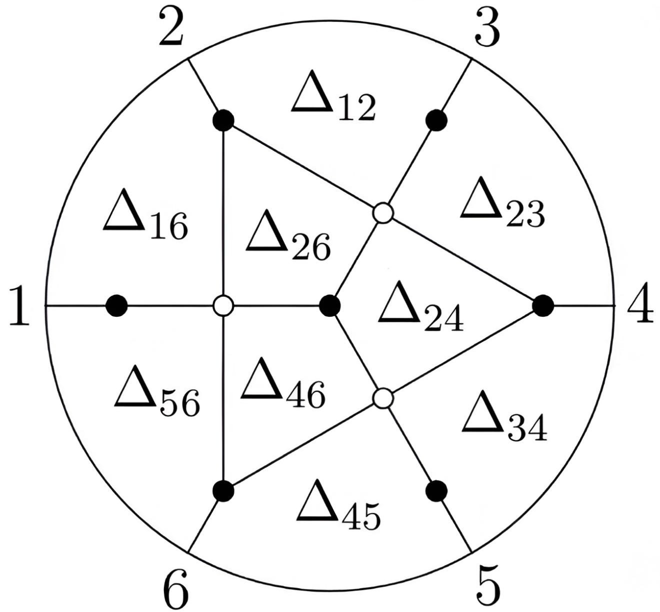
Theorem

Consider a minimal Lam model $N(G, w)$ with n boundary nodes. Let us label each of its faces F by $I(F) \subset [n]$ according to the following ***Scott rule****: if a face F lies on the left-hand side of the oriented strand from a source numbered i, we have that $i \in I(F)$. This labeling rule ensures that each face is labeled by $k(G)$ distinct indices.*





Scott Rule: Example



Twist

Muller, Speyer, «The twist for positroid varieties»

Definition

Consider a matrix $A \in \mathrm{Mat}_{k \times n}(\mathbb{R})$ and denote by A_i, $i \in \{1, \ldots, m\}$ its columns. We define the **twist** $\tau(A) \in \mathrm{Mat}_{k \times n}(\mathbb{R})$ as follows:

$$\begin{cases} (\tau(A)_i, A_i) = 1, & \text{if } i = j \\ (\tau(A)_i, A_j) = 0, & \text{if } A_j \notin \mathrm{span}(A_i, A_{i+1}, \ldots, A_{j-1}), \end{cases}$$

where $\tau(A)_i$, $i \in \{1, \ldots, m\}$ are the columns of $\tau(A)$; $(\cdot, \cdot)$ is the standard scalar product; and the index operation $+$ is taken modulo n.

Twist: Example

Example

Consider a matrix A (all its 2×2 minors are positive) of the form:

$$A = \begin{pmatrix} \frac{ca+cb}{a+b+c} & 1 & \frac{ca}{a+b+c} & 0 & \frac{-cb}{a+b+c} & -1 \\ \frac{ca}{a+b+c} & 1 & \frac{ca+ab}{a+b+c} & 1 & \frac{ab}{a+b+c} & 0 \end{pmatrix}$$

Then, the following holds:

$$\tau(A) = \begin{pmatrix} \frac{a+b+c}{bc} & \frac{c+b}{b} & \frac{a+b+c}{ac} & \frac{a}{c} & 0 & -1 \\ -\frac{a+b+c}{bc} & -\frac{c}{b} & 0 & 1 & \frac{a+b+c}{ab} & \frac{a+b}{a} \end{pmatrix}$$

Reconstruction Algorithm

Consider a minimal Lam model $N(G, w)$ on a given graph G associated with a point $X \in \mathrm{Gr}_{\geq 0}(k, n)$ defined by a matrix A. Then, a weight function ω can be found as follows:

- The first step is to label each face F of Γ by $I(F)$ according to the Scott rule;
- The second step is to find the twist $\tau(A)$ of A.

Finally, we calculate the weight function w according to the rule:

Reconstruction Algorithm

- If an edge e connects two inner nodes, then

$$w(e) = \frac{1}{\Delta_{I(F_1)}(\tau(A))\Delta_{I(F_2)}(\tau(A))},$$

  where the edge e is shared by faces F_1 and F_2;

- If an edge e connects an inner node with a boundary node, then

$$w(e) = \frac{1}{\Delta_{I(F)}(\tau(A))},$$

  where F is:

  - The closest counterclockwise face that shares the edge e, if e is connected to a white boundary node;
  - The closest clockwise face that shares the edge e, if e is connected to a black boundary node.

Duality

Theorem

For each face F of $N(G, w)$ we define its weight as follows:

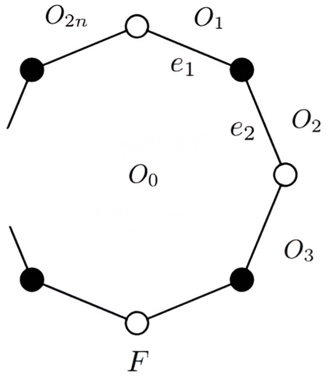
$$\omega(F) = \frac{\prod\limits_{e \in \Pi, e = wb} w(e)}{\prod\limits_{e \in \Pi, e = bw} w(e)},$$

where the numerator is the product over all from white to black (clockwise) edges of F and the denominator is the product over all from black to white (clockwise) edges of F. The coordinate $\omega(F)$ doesn't depend on the gauge transformations.

Theorem

Twist provides the duality between $\{\omega(F)\}$ coordinates of X and $\{\Delta_{I(F)}\}$ coordinates of $\tau(X)$.

Duality: Example



Theorem

The following holds $\omega(F) = \dfrac{O_1 \dots O_{2n}}{O_1 \dots O_{2n-1}}$, *here* $O_i = \Delta_{I(F_i)}(\tau(A))$.

Therefore $w(e_i) = \dfrac{1}{O_0 O_i}$.

Electrical Networks

Curtis, Morrow, «Inverse problems for electrical networks»

Definition

An **electrical network** $\mathcal{E}(\Gamma, \omega)$ is embedded into a disk connected weighted planar graph $\Gamma(V, E)$ that satisfies the following conditions:

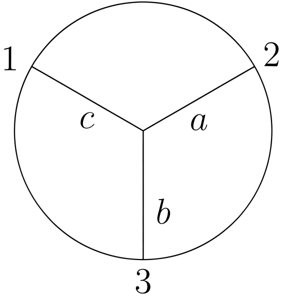
- All nodes are divided into the set of inner nodes and the set of boundary nodes V_B, and each boundary node lies on the boundary circle;
- Boundary nodes are enumerated clockwise from 1 to $|V_B| = n$. Inner nodes are enumerated arbitrarily from $n + 1$;
- Every edge is equipped with a positive weight ω_{ij}, which denotes the conductivity of this edge.

Electrical Networks

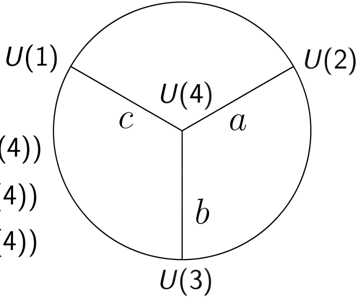
Let us apply voltages $\mathbf{U} : V_B \to \mathbb{R}$ to its boundary nodes V_B. These boundary voltages give rise to the unique harmonic extension $U : V \to \mathbb{R}$ on all vertices, which might be restored by the Kirchhoff and Ohm laws:

$$\sum_{j \in V} \omega_{ij}\big(U(i) - U(j)\big) = 0, \ \forall i \in V_I.$$

$$c(U(4) - U(1)) + a(U(4) - U(2)) + b(U(4) - U(3)) = 0$$



$$I_1 = c(U(1) - U(4))$$
$$I_2 = a(U(2) - U(4))$$
$$I_3 = b(U(3) - U(4))$$



Response Matrix

Definition

Voltages $\mathbf{U}$ induce boundary currents $\mathbf{I} = \{I_1, \ldots, I_n\}$ flowing through boundary nodes:

$$I_k := \sum_{j \in V} \omega_{ij} \big(U(k) - U(j) \big), \ k \in \{1, \ldots, n\}.$$

Theorem

There is a matrix $M(\mathcal{E}) \in Mat_{n \times n}(\mathbb{R})$, which relates $\boldsymbol{U}$ and $\boldsymbol{I}$ to each other as follows:

$$M(\mathcal{E})\boldsymbol{U} = \boldsymbol{I}.$$

*This matrix is called the **response matrix** of the network $\mathcal{E}$.*

Discrete Calderón Problem

Problem

Consider an electrical network $\mathcal{E}(\Gamma, \omega)$ on a given graph Γ. Conductivities ω must be recovered by a known response matrix $M(\mathcal{E})$.

Problem

Consider an electrical network $\mathcal{E}$ with n boundary nodes on an unknown graph Γ. A graph Γ must be recovered by a known response matrix $M(\mathcal{E})$.

Discrete Calderón Problem

Theorem

Conductivities ω of any minimal electrical network $\mathcal{E}$ can be uniquely recovered by its response matrix $M(\mathcal{E})$.

- Recursive bridge-spike decompositions;
- Special standard or layer networks technique;
- Cluster algebra technique for non-negative Grassmannians.

Theorem

Each minimal electrical network $\mathcal{E}$ can be uniquely recovered by its response matrix $M(\mathcal{E})$ up to the star-triangle transformation.

- Recursive rank pattern calculation;
- Rank pattern calculation via positroids.

From Electrical Networks To Non-Negative Grassmannians

Bychkov, Gorbounov, Kazakov, Talalaev, «Electrical networks, Lagrangian Grassmannians and symplectic groups»

Theorem

Consider an electrical network $\mathcal{E}(\Gamma, \omega)$ with n boundary nodes. Using its response matrix $M(\mathcal{E}) = (x_{ij})$, let us define a point of the Grassmannian $\mathrm{Gr}(n-1, 2n)$as the row space of the matrix:

$$\Omega_n(\mathcal{E}) = \begin{pmatrix} x_{11} & 1 & -x_{12} & 0 & x_{13} & 0 & \cdots & (-1)^n \\ -x_{21} & 1 & x_{22} & 1 & -x_{23} & 0 & \cdots & 0 \\ x_{31} & 0 & -x_{32} & 1 & x_{33} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

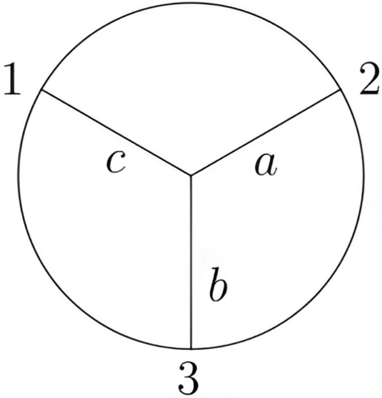
Then $\Omega_n(\mathcal{E})$ defines a non-negative point in $\mathrm{Gr}(n-1, 2n)$ i.e. the dimension of the row space of $\Omega_n(\mathcal{E})$ is equal to $n-1$ and each $n-1 \times n-1$ minor (each Plücker coordinate) of the matrix $\Omega_n(\mathcal{E})$ is non-negative.

Example

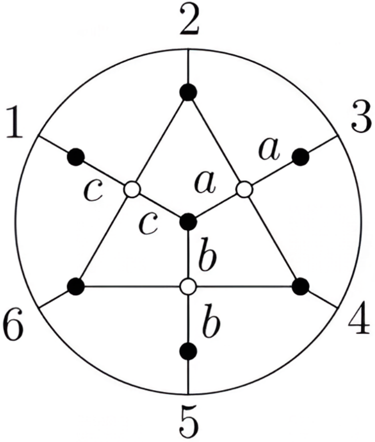
Example

$$\Omega_4(\mathcal{E}) = \begin{pmatrix} x_{11} & 1 & -x_{12} & 0 & x_{13} & 0 & -x_{14} & 1 \\ -x_{21} & 1 & x_{22} & 1 & -x_{23} & 0 & x_{24} & 0 \\ x_{31} & 0 & -x_{32} & 1 & x_{33} & 1 & -x_{34} & 0 \\ -x_{41} & 0 & x_{42} & 0 & -x_{43} & 1 & x_{44} & 1 \end{pmatrix}$$

Temperley Trick: Example

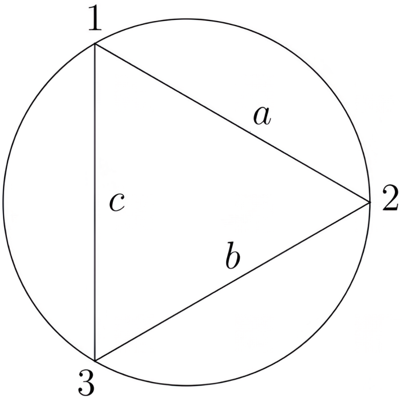


Γ



$N(\Gamma)$

Temperley Trick: Example



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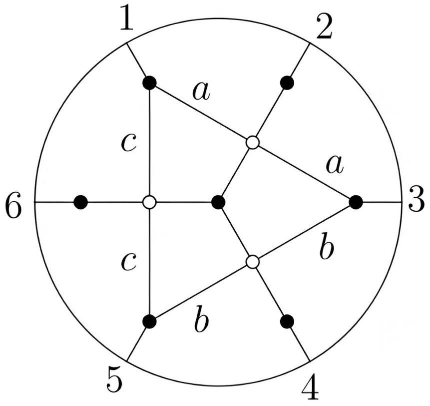


$N(\Gamma)$

Temperley Trick: Main Result

Lam, «Electroid varieties and a compactification of the space of electrical networks»; Bychkov, Gorbounov, Kazakov, Talalaev, «Electrical networks, Lagrangian Grassmannians and symplectic groups»

Theorem

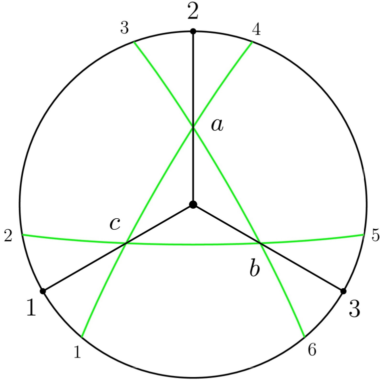
Consider an electrical network $\mathcal{E}(\Gamma, \omega)$ and its associated Lam model $N(\Gamma)$. Then $N(\Gamma)$ and $\Omega_n(\mathcal{E})$ define the same point in $\mathrm{Gr}_{\geq 0}(n-1, 2n)$.

Topology Reconstruction

Theorem

Any two minimal electrical networks that share the same resistance matrix can be converted into each other only by the star-triangle transformations. As a consequence, the topology of a minimal electrical network $\mathcal{E}$ is recovered, up to the star-triangle transformations, by its ***strand permutation*** $\tau(\mathcal{E})$.

Strand Permutation: Example



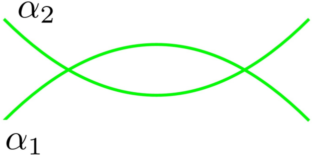
For the star-shaped network, its strand permutation is

$$\tau(\mathcal{E}) = (14)(25)(36)$$

Minimal Networks

A **medial graph** of an electrical network $\mathcal{E}(\Gamma, \omega)$ is a graph Γ_M whose internal vertices are the midpoints of the edges of Γ and two internal vertices are connected by an edge if the edges of the original graph Γ are adjacent. The boundary vertices of Γ_M are defined as the intersection of the natural extensions of the edges of Γ_M with the boundary circle. Since the interior vertices of the medial graph have degree four, we can define the **strands** of the medial graph as the paths that always go straight through any degree four vertex.

An electrical network is called **minimal** if the strands of its medial graph do not have self-intersections; each strand does not form a closed loop; any two strands intersect at most once, i.e. the medial graph has no lenses.



Topology Recovering

Definition

Denote by A_i the columns of the matrix $\Omega_n(\mathcal{E})$ and define the column permutation $g(\mathcal{E})$ as follows: $g(\mathcal{E})(i) = j$, if j is the minimal number such that $A_i \in \mathrm{span}(A_{i+1}, \ldots, A_j)$, where the indexes are taking modulo $2n$.

Theorem

The following holds

$$g(\mathcal{E}) + 1 = \tau(\mathcal{E}).$$

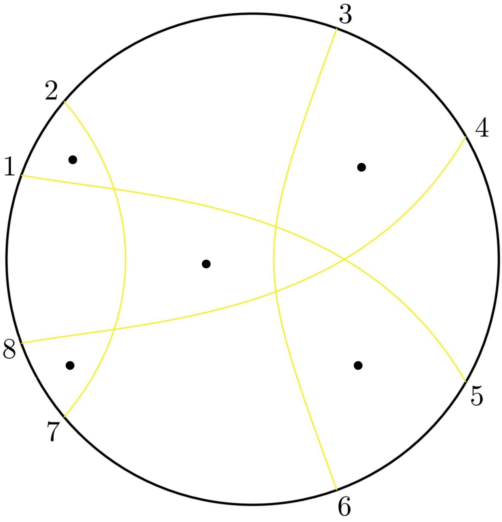
Topology Recovering: Example

Curtis, Morrow, «Inverse problems for electrical networks»; Lam, «Electroid varieties and a compactification of the space of electrical networks»; Gorbounov, Kazakov, «Electrical networks and data analysis in phylogenetics».

$$\Omega_4(\mathcal{E}) = \begin{pmatrix} 1 & 3 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}.$$

Then the following holds $g(\mathcal{E}) = [4\,6\,5\,7\,8\,2\,1\,3]$ in the one window notation, therefore the strand permutation is $\tau(e) = (15)(27)(36)(48)$.

Topology Recovering: Example



Topology Recovering: Example