Deligne's conjecture for deformation of tensor categories

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Referrences

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Hochschild complex

Definition

Let k be a field. The (normalised) Hochschild complex $CH(A)$ of an assocative k-algebra A is a sequence of vector subpaces

$$CH^n(A) \subset Vect_k(A \otimes \ldots \otimes A, A)$$

of functions which vanish on tensors containing unit of A, equipped with the dfferential

$$d_n : CH^n(A) \to CH^{n+1}(A)$$

$$d(f)(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) +$$

$$+ \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) - (-1)^{n+1} f(x_1, \ldots, x_{n+1}) x_{n+1}$$

Cup product and brackets

The cup product

$$-\cup - : CH^p(A)\otimes CH^q(A)\to CH^{p+q}(A)$$

$$(f\cup g)(x_1,\dots,x_{p+q})=(-1)^{(p-1)(q-1)}f(x_1,\dots,x_p)\cdot g(x_{p+1},\dots,x_{p+q})$$

The bracket

$$\{-,-\} : CH^p(A)\otimes CH^q(A)\to CH^{p+q-1}(A)$$

$$\{f,g\}=f\circ g-(-1)^{(p-1)(q-1)}g\circ f$$

where

$$f\circ g(x_1,\dots,x_{p+q-1})=$$

$$\sum_{i=1}^{p}(-1)^{(q-1)i}f(x_1,\dots,x_i,g(x_{i+1},\dots,x_{i+q}),x_{i+q+1},\dots,x_{p+q-1})$$

Deligne's conjecture

Theorem (M.Gershtenhaber 1963)

The cup product and bracket on Hochschild complex of A induce a Gershenhaber algebra structure on Hochschild cohomology $HH^*(A) = H^*(CH(A))$.

Theorem (F.Cohen 1976)

The operad for Gershtenhaber algebras is isomorphic to the homology operad of the little 2-disks operad with coefficient in k.

Conjecture (Deligne conjecture 1993)

The cup product and bracket on Hochschild complex of an associative algebra can be lifted to a full e_2*-algebra structure on* $CH(A)$ *(that is an algebra of a chain version of the little 2-disks operad)*

Tensor categories

Definition

A tensor category is a monoidal category C enriched over symmetric monoidal category $(Vect, \otimes, k)$.

Example

- The category $Vect$;
- For any bialgebra H the category of modules Mod_H is a tensor category.

Tensor functors

Definition

A tensor functor is a *Vect*-enriched (strong) monoidal functor. That is it is a functor $F: C \to D$ such that for each $a, b \in C$ $C(a,b) \to D(F(a), F(b))$ is k-linear and additionally equipped with natural isomorphisms

$$F(a) \otimes_D F(b) \to F(a \otimes_C b)\ ,\ 1_D \to F(1_C)$$

satisfying usual coherence conditions.

Example

- The identity functor $Id : C \to C$;
- The forgetful functor $U : Mod_H \to Vect$.

Endomorphism spaces

For a tensor functor $F : C \to D$ define its n-th power by

$$F^{\otimes^n} : C \otimes \dots \otimes C \to D,$$

$$F^{\otimes^n}(X_1, \dots, X_n) = F(X_1 \otimes (X_2 \otimes (\dots (X_{n-1} \otimes X_n) \dots))$$

For $n=0$ define $F^{\otimes^0} : \mathbf{k} \to D$, $F^{\otimes^0}(\star) = 1_D$, where $\mathbf{k}$ is the *Vect*-enriched category with one object $\star$ and unique hom-space k.

Let $E^n(F) = End(F^{\otimes^n})$, be the vector space of linear endomorphisms of the functor $F^{\otimes^n}$. It is naturally an associative k-algebra, that is a monoid in *Vect*.

Cosimplicial structure on $E^*(F)$

For an endomorphism $a \in End(F^{\otimes n})$ the coface maps are defined by:

$$\partial_i(a)_{X_1,\dots,X_{n+1}} = \begin{cases} \phi(1_{F(X_1)} \otimes a_{X_2,\dots,X_{n+1}})\phi^{-1} & , \quad i=0 \\ F(\alpha_i)^{-1}(a_{X_1,\dots,X_i \otimes X_{i+1},\dots,X_{n+1}})F(\alpha_i) & , \quad 1 \le i \le n \\ \psi(a_{X_1,\dots,X_n} \otimes 1_{F(X_{n+1})})\psi^{-1} & , \quad i=n+1 \end{cases}$$

Here ϕ, ψ and α are appropriate coherence constraints. There are formulas for codegeneracies which use tensor units.

Proposition

The cosimplicial vector space $E^*(F)$ is a cosimplicial monoid in Vect. It is called cosimplicial deformation complex of F.

Davydov-Yetter complex

Recall that for a cosimplicial vector space V^* its (normalised) total complex $Tot(V^*)$ is the chain complex $K^n, n \geq 0$, where $K^n \subset V^n$ is the intersection of kernels of all codegeneracies equipped with the differential $\partial(a) = \sum_{i=0}^{n+1}(-1)^n\partial_i(a)$.

Definition

The Davydov-Yetter deformation complex $\mathcal{D}(F)$ of a tensor functor $F: C \to D$ is the total complex of the cosimplicial deformation complex $\mathcal{D}(F) = Tot(E^*(F)) = Tot(End(F^{\otimes^*}))$.

Definition

The Davydov-Yetter deformation complex $\mathcal{D}(C)$ of a tensor category C is the Davydov-Yetter deformation complex of the identity functor $Id: C \to C$.

Davydov-Yetter cohomology

Example

Let G be a finite group, k be an algebraically closed field (of any characteristic), and $C{=}Vec_G$ be the tensor category of G-graded finite dimensional vector spaces over k. Then $H^*(\mathcal{D}(C)){=}HH^*(k[G],k)$, the Hochschild cohomology of the group algebra $k[G]$ with trivial coefficients, that is $H^*(G,k)$ the group cohomology of G with coefficients in k.

Example

The deformation complex $E^*(U)$ where $U : Mod_H \to Vect$ is isomorphic to the bar complex of the bialgebra H and $\mathcal{D}(U) = C^*(H)$ is the co-Hochschild complex of H.

Theorem (Davydov 1995)

In general

- $H^2(\mathcal{D}(F))$ *parametrizes additively trivial first order deformations of* F *as tensor functor modulo equivalence,*
- $H^3(\mathcal{D}(F))$ *is the obstruction space for such deformations.*
- *Similarly for a tensor category* C *the space* $H^3(\mathcal{D}(C))$ *parametrizes additively trivial first order deformations of* C *as tensor category and* $H^4(\mathcal{D}(C))$ *is the space of obstructions for deformations.*

Natural operations on deformation complex

The exists a $\cup$-product on the deformation complex of a tensor functor $\mathcal{D}(F)$:

$$- \cup - : \mathcal{D}^p(F) \otimes \mathcal{D}^q(F) \to \mathcal{D}^{p+q}(F),$$

$$(a \cup b)_{X_1,\ldots,X_p,X_{p+1},\ldots,X_{p+q}} =$$

$$= (-1)^{(p-1)(q-1)} \phi(a_{X_1,\ldots,X_p} \otimes b_{X_{p+1},\ldots,X_{p+q}})\phi^{-1} \ .$$

Here ϕ is the unique composition of coherence isomorphisms.

Natural operations on deformation complex

One can also defined a bracket operation of degree -1. For example, for $a, b \in \mathcal{D}^1(F) \subset End(F)$ the bracket coincides with the commutator in algebra of endomorphisms:

$$\{a, b\}=ab - ba \ .$$

For $a \in \mathcal{D}^1(F), b \in \mathcal{D}^2(F)$

$$\{a, b\} = a(\partial_0(b) + \partial_2(b)) - \partial_1(b)a.$$

Natural operations on deformation complex

More generally, for $a \in \mathcal{D}^p(F), b \in \mathcal{D}^q(F)$ we have an operation

$$\{a, b\} = \sum_{i=1}^{p} (-1)^{i(q-1)} a \circ_i b - (-1)^{(p-1)(q-1)} \sum_{i=1}^{q} (-1)^{i(p-1)} b \circ_i a,$$

where $a \circ_i b = \tau^i_{q,p}(a) \pi^i_{q,p}(b)$ and $\tau^i_{q,p}(a)$ and $\pi^i_{q,p}(b)$ are certain cosimplicial operations which we will consider shortly.

Remark. This operation as well as cup product were defined by A.Davydov and (independently) D.Yetter. A proof that it is, indeed a bracket operation was missing until 2020. It is instructive to compare these formulas with classical Gershtenhaber formulas for Hochschild complex of an associative algebra.

Davydov-Deligne's conjecture (1996)

Conjecture

1. *The operations* $- \cup -$ *and* $\{-,-\}$ *on the deformation complex of a tensor functor* $\mathcal{D}(F)$ *can be lifted to a* E_2*-algebra structure on* $\mathcal{D}(F)$.
2. *For* $F{=}Id : C \to C$ *the bracket* $\{-,-\}$ *is homotopically trivial and there exists a secondary bracket* $\{-,-\}_2$ *of degree* -2 *on* $\mathcal{D}(C)$. *This secondary bracket together with the cup product is a part of a* E_3*-algebra structure on the deformation complex* $\mathcal{D}(C)$.

Cosimplicial monoids

Let Δ be the category of nonempty finite ordinals.

Definition

A cosimplicial monoid M^* in a monoidal category $(V, \otimes, I)$ is a functor

$$M : \Delta \to Mon(V).$$

A commutative cosimplicial monoid M^* in a symmetric monoidal category $(V, \otimes, I)$ is a functor

$$M : \Delta \to CMon(V).$$

Equally, a (commutative) cosimplicial monoid in V is a (commutative) monoid in the (symmetric) monoidal category of functors $\Delta \to V$.

Cosimplicial monoids

Example

- Plenty of examples of cosimplicial monoids constructed from DG-algebras using monoidal Dold-Kan correspondence.
- The cochain complex of a simplicial set with coefficients in a commutative ring R is a commutative cosimplicial monoid in the category Mod_R.
- Cosimplicial deformation complexes of a tensor functor or of a tensor category are cosimplicial monoids.

Shufflings

For two maps in Δ $\tau : [p] \to [m],\ \pi : [q] \to [m]$ their shuffling of length n is a decomposition of their images into disjoint unions

$$Im(\tau)=A_1 \cup \dots \cup A_s,\ \ A_1 < \dots < A_s$$

$$Im(\pi)=B_1 \cup \dots \cup B_t,\ \ B_1 < \dots < B_t\ ,\ \ n = s + t - 1$$

such that

$$A_1 \leq B_1 \leq A_2 \leq B_2 \leq \dots$$

or vice versa

$$B_1 \leq A_1 \leq B_2 \leq A_2 \leq \dots$$

Here $A \leq B$ means that each element of a is less or equal to every element of B. Similarly for $<$.

All shufflings for $\tau{=}\pi{=}id : [2] \to [2]$.

1. $A_1{=}\{0\}, A_2{=}\{1\}, A_3{=}\{2\}, B_1{=}\{0\}, B_2{=}\{1\}, B_3{=}\{2\}.$
$$A_1 \ \leq \ B_1 \ \leq \ A_2 \ \leq \ B_2 \ \leq \ A_3 \ \leq \ B_3 \ .$$
2. $A_1{=}\{0\}, A_2{=}\{1\}, A_3{=}\{2\}, B_1{=}\{0\}, B_2{=}\{1\}, B_3{=}\{2\},$
$$B_1 \ \leq \ A_1 \ \leq \ B_2 \ \leq \ A_2 \ \leq \ B_3 \ \leq \ A_3 \ .$$
3. $A_1{=}\{0\},\ A_2{=}\{1\},\ A_3{=}\{2\}, B_1{=}\{0,1\},\ B_2{=}\{2\},$
$$A_1 \ \leq \ B_1 \ \leq \ A_2 \ \leq \ B_2 \ \leq \ A_3 \ .$$
4. $A_1{=}\{0,1\},\ A_3{=}\{2\}, B_1{=}\{0\},\ B_2{=}\{1\},\ B_3{=}\{2\},$
$$B_1 \ \leq \ A_1 \ \leq \ B_2 \ \leq \ A_2 \ \leq \ B_3 \ \ .$$

All shufflings for $\tau = \pi = id : [2] \to [2]$.

5 $A_1 = \{0\},\ A_2 = \{1\},\ A_3 = \{2\}, B_1 = \{0\},\ B_2 = \{1, 2\},$

$$A_1 \le B_1 \le A_2 \le B_2 \le A_3 .$$

6 $A_1 = \{0\},\ A_2 = \{1, 2\}, B_1 = \{0\},\ B_2 = \{1\},\ B_3 = \{2\},$

$$B_1 \le A_1 \le B_2 \le A_2 \le B_3 .$$

7 $A_1 = \{0\},\ A_2 = \{1, 2\}, B_1 = \{0, 1\},\ B_2 = \{2\} ,$

$$A_1 \le B_1 \le A_2 \le B_2 .$$

8 $A_1 = \{0, 1\},\ A_2 = \{2\}, B_1 = \{0\},\ B_2 = \{1, 2\} ,$

$$B_1 \le A_1 \le B_2 \le A_2 .$$

Linking number

Definition

The *linking number* $\mathbf{lk}(\tau,\pi)$ of two maps of ordinals $\tau : [p] \to [m], \ \pi : [q] \to [m]$ is the smallest natural number n for which there exists a shuffling of τ and π of length $n+1$.

Example

In the example above $\mathbf{lk}(id_{[2]}, id_{[2]}) = 3$.

Example

In general, $\mathbf{lk}(id_{[n]}, id_{[n]}) = n+1$.

Linking number

Example (Generic pair with linking number 1)

The monomorphism $\tau_{p,q} : [p] \to [p+q]$, which does not take the values $p+1, \ldots, p+q$ and the monomorphism $\pi_{p,q} : [q] \to [p+q]$, which does not take the values $0, \ldots, q-1$.

Example (Generic pair with linking number 2)

The monomorphism $\tau^i_{p,q} : [p] \to [p+q-1]$, which does not take the values $i+1, \ldots, i+q-1$ and the monomorphism $\pi^i_{p,q} : [q] \to [p+q-1]$, which does not take the values $0, \ldots, i-1$ and $i+q+1, \ldots, p+q-1$.

n-commutative cosimplicial monoids

Definition

A cosimplicial monoid M^* in a symmetric monoidal category V is called n-commutative if for any $\phi : [p] \to [m],\ \psi : [q] \to [m]$ with $\mathbf{lk}(\phi, \psi) \leq n$ the following diagram commutes:

$$\begin{tikzcd}
M^p \otimes M^q \arrow[rr, "\tau"] \arrow[d, "M(\phi)\otimes M(\psi)"'] & & M^q \otimes M^p \arrow[d, "M(\psi)\otimes M(\phi)"] \\
M^m \otimes M^m \arrow[dr, "\mu"] & & M^m \otimes M^m \arrow[dl, "\mu"'] \\
& M^m &
\end{tikzcd}$$

Herre, τ is the symmetry in V.

n-commutative cosimplicial monoids

Example

- 0-commutative cosimplicial monoids are just cosimplicial monoids.
- ∞-commutative cosimplicial monoids are cosimplicial commutative monoids.

Theorem (Batanin-Davydov)

1. *For a tensor functor $F : C \to D$ the cosimplicial monoid $E^*(F)$ is 1-commutative.*
2. *For a tensor category C the cosimplicial monoid $E^*(C)$ is 2-commutative.*

n-commutative cosimplicial monoids

Example

Let $\pi : [1] \to [2]$ and $\tau : [2] \to [2]$ be the following maps in Δ :

$$\pi(0) = 0, \pi(1) = 2 \ , \ \ \tau = id_{[2]}.$$

Then $\mathbf{lk}(\pi, \tau) = 2$. Let us show that the cosimplicial complex of an arbitrary tensor category C satisfies commutativity condition on these two maps.

n-commutative cosimplicial monoids

Example

$E^1(C) = End(Id_C)$ and $E^2(C) = End(Id_C^2)$. By definition, $E(\pi)$ on a natural transformation $\psi : Id_C \to Id_C$ is the natural transformation from $E^2(C)$ whose components are

$$\phi_{X,Y} = \psi_{X\otimes Y} : X \otimes Y \to X \otimes Y.$$

Then the commutativity condition amounts to the commutativity of the diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\eta_{X,Y}"] \arrow[d, "\psi_{X\otimes Y}"'] & X \otimes Y \arrow[d, "\psi_{X\otimes Y}"] \\
X \otimes Y \arrow[r, "\eta_{X,Y}"] & X \otimes Y
\end{tikzcd}$$

for any natural transformation $\eta_{X,Y}$. This follows from naturality of ψ.

E_{n+1}-algebras and n-commutative cosimplicial monoids

Theorem (Batanin-Davydov)

1. *Let* $(V, \otimes, I)$ *be a simplicial monoidal model category satisfying some technical assumptions. The object* $Tot(E^*)$ *of an n-commutative monoid in* V *has a natural* E_{n+1}*-algebra structure.*
2. *For* $V = Ch(Mod_R)$ *this structure induces Steenrod* $\cup_i$*-products for each* $0 \leq i \leq n$ *on* $Tot(E^*)$*. Their commutators induce the Poisson brackets in cohomology.*

Proof of Davydov's conjecture

Theorem (Batanin-Davydov)

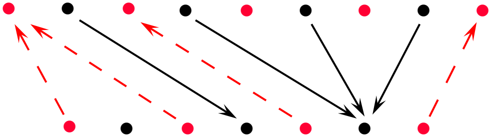
1. *For a tensor functor F the cup product and bracket operations on $\mathcal{D}(F)$ induced by E_2-action above coincide with the Davydov-Yetter operations.*
2. *For a tensor category C the primary bracket on the Davydov-Yetter cohomology of C is homotopically trivial but the secondary bracket can be nontrivial.*

Proof

Joyal's duality is an isomorphism of categories

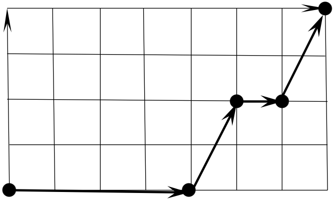
$$\widetilde{(-)} : \Delta \to \mathrm{Int}^{op},\ \widetilde{[n]} = \langle n+1 \rangle.$$

Where Int is the category of nondegenerate finite intervals $\langle n \rangle = (0 < 1 < \ldots < n)$ and morphisms preserving minimum and maximum elements.



Proof

Let $\tau : [p] \to [m],\ \pi : [q] \to [m]$ be two maps in Δ. After Joyal duality we have $\widetilde{\tau} : \langle m+1\rangle \to \langle p+1\rangle,\ \widetilde{\pi} : \langle m+1\rangle \to \langle q+1\rangle$ or the same as a single map $\widetilde{(\tau,\pi)} : \langle m+1\rangle \to \langle p+1\rangle \times \langle q+1\rangle$ which preserves minimum and maximum elements. We interpret it as a path on a commutative lattice.

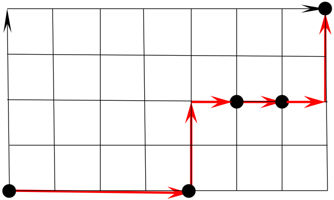


Proof

Let $\langle p+1\rangle \Box \langle q+1\rangle$ be a noncommutative lattice. Here $\Box$ is the funny tensor product of categories. There is a canonical quotient map $\langle p+1\rangle \Box \langle q+1\rangle \to \langle p+1\rangle \times \langle q+1\rangle$.

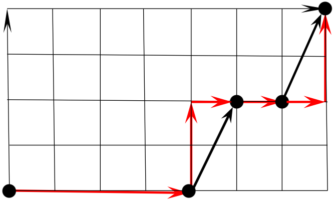
Definition

An end points preserving path $\lambda : \langle m+1\rangle \to \langle p+1\rangle \Box \langle q+1\rangle$ on a noncommutative lattice has complexity n if it has exactly n corners.



Shufflings and paths liftings

A lattice path $\psi : \langle m+1\rangle \to \langle p+1\rangle \Box \langle q+1\rangle$ is a lifting of path $\phi : \langle m+1\rangle \to \langle p+1\rangle \times \langle q+1\rangle$ provided ϕ admits a factorisation $\langle m+1\rangle \xrightarrow{\psi} \langle p+1\rangle \Box \langle q+1\rangle \to \langle p+1\rangle \times \langle q+1\rangle$.

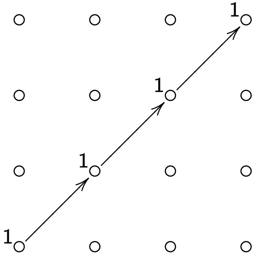


Theorem

For two maps $\tau : [p] \to [m]$, $\pi : [q] \to [m]$ in Δ there is a one to one correspondence between shufflings of τ and ϕ and liftings of $\widetilde{(\tau,\pi)}$. Under this correspondence the shufflings of length $n+1$ correspond to liftings of complexity n and vice versa.

Example

Let $\pi = \tau = id : [2] \to [2]$. The corresponding path ϕ is the diagonal path $\delta : \langle 3 \rangle \to \langle 3 \rangle \times \langle 3 \rangle$ as in the following picture:

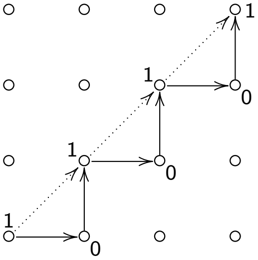


Example

A shuffling of length 6:

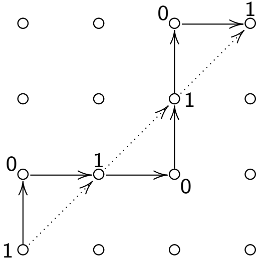
$A_1 = \{0\}, A_2 = \{1\}, A_3 = \{2\},\ B_1 = \{0\}, B_2 = \{1\}, B_3 = \{2\}.$

Corresponding lifting has complexity 5 :



Example

A lifting of complexity 3:



Corresponding shuffling of length 4 :

$$A_1 = \{0,1\},\ A_2 = \{2\}, B_1 = \{0\},\ B_2 = \{1,2\}\ .$$

Proof continues

Lattice paths on all noncommutative hypercubes can be assembled to a $\mathbb{N}$-colored *Set*-operad called lattice path operad $\mathcal{L}$ (Batanin-Berger 2009). The set of operations of $\mathcal{L}$:

$$\mathcal{L}(n_1, \dots, n_k; n) = \mathbf{Cat}_{*,*}(\langle n+1\rangle, \langle n_1+1\rangle \square \cdots \square \langle n_k+1\rangle).$$

This set is a functor on Δ with respect to the last variables and on $(\Delta^{op})^k$ with respect to other variables.
We can form a 'geometric realisation" of $\mathcal{L}$ with respect to a fixed standard system of simplices :

$$\delta : \Delta \to V.$$

The result is a one coloured symmetric operad $\mathcal{L}_\delta$.

End of Proof.

Paths of complexity less or equal to n form a suboperad $\mathcal{L}^{(n)} \subset \mathcal{L}$ so we have the geometric realisation $\mathcal{L}^{(n)}_{\delta}$.

Theorem (Batanin-Berger 2009)

- *For any $1 \leq n \leq \infty$ and an $\mathcal{L}^{(n)}$-algebra X the cosimplicial totalisation $Tot_{\delta}(X)$ is naturally a $\mathcal{L}^{(n)}_{\delta}$-algebra.*
- *The operad $\mathcal{L}^{(n)}_{\delta}$ has the homotopy type of the little n-cube operad.*

End of proof of Batanin-Davydov Theorem. Paths lifting condition for n-commutative cosimplicial monoid implies that it has a natural $\mathcal{L}^{(n+1)}$-algebra structure. Hence, its cosimplicial totalisation has a natural structure of an E_{n+1}-algebra.

Explicit formulas for Steenrod $\cup_i$-product

Let E^* be an n-commutative cosimplicial monoid in Mod_R. For each $0 \leq i \leq n$ the Steenrod product $\cup_i$ on the normalised totalisation of E^* is given by the formula:

$$a \cup_i b =$$

$$= \sum_{\psi \in \mathbf{slp}_{+}^{(i+1)}(p,q)} (-1)^{(p-1)(q-1)} \mathrm{sgn}(\psi)\psi(a \otimes b), a \in E^p, b \in E^q,$$

where $\psi(a \otimes b)$ is the action of the operation ψ on $a \otimes b$ and $\mathbf{slp}_{+}^{(i+1)}(p,q)$ is the set of even noncommutative liftings of complexity $i+1$ of smooth Delannoy paths. We call such liftings smooth lattice paths.

Explicit formulas for Poisson bracket

For $n \geq 1$ the degree $-n$ bracket:

$$\beta^{(n)} : E(p) \otimes E(q) \to E(p+q-n)$$

is given by

$$\beta^{(n)}(a \otimes b) = (-1)^{n+1} a \cup_n b - (-1)^{(p-1)(q-1)} b \cup_n a =$$

$$= \sum_{\psi \in \mathbf{slp}_{+}^{(n+1)}(p,q)} (-1)^{(p-1)(q-1)} \mathrm{sgn}(\psi)\psi(\mathrm{a} \otimes \mathrm{b}) +$$

$$+(-1)^{(n)} \sum_{\psi \in \mathbf{slp}_{-}^{(n+1)}(p,q)} \mathrm{sgn}(\psi)\psi(\mathrm{a} \otimes \mathrm{b}).$$

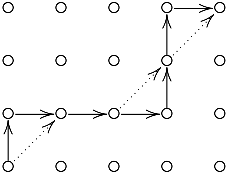
Action of a smooth lattice path

An odd smooth lattice path

$$\psi : \langle 4 \rangle \to \langle 4 \rangle \Box \langle 3 \rangle$$

of complexity 3:

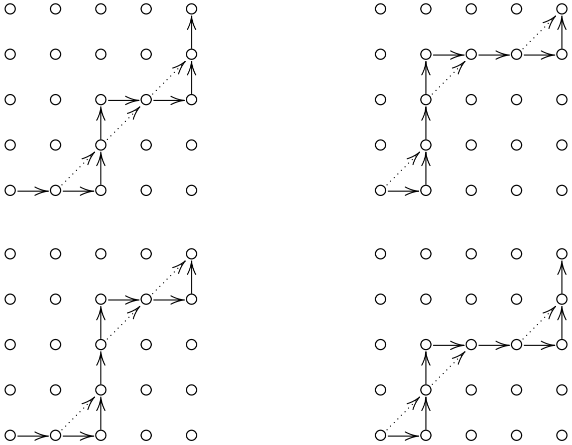
$\psi \ =$



acts as $\psi(a \otimes b) = E(\pi(b)) \cdot E(\tau(a))$, for $a \in E_3, b \in E_2$, $\tau(a) = \partial_1(a), \pi(b) = b$, $\mathrm{sgn}(\psi) = -1$. That is this operation provides an entry $(-1)b \cdot \partial_1(a)$ in the Poisson bracket $\beta^{(2)}$.

Example

There are eight smooth lattice paths $\langle 5\rangle \to \langle 4\rangle \square \langle 4\rangle$ of complexity 3, four of each are even:



Example

The degree 2 bracket $\beta^{(2)} : E^3 \otimes E^3 \to E^4$ is given by

$$\{\{a, b\}\} = \partial_2(a)\partial_0(b) + \partial_4(a)\partial_0(b) + \partial_4(a)\partial_2(b) - \partial_1(a)\partial_3(b) +$$

$$+\partial_2(b)\partial_0(a) + \partial_4(b)\partial_0(a) + \partial_4(b)\partial_2(a) - \partial_1(b)\partial_3(a)\ .$$

Symmetric cosimplicial monoids

Let Fin_* be the skeletal category of pointed finite sets. The objects of it are finite ordinals $[n] = \{0, \dots, n\}$, where we consider 0 as a based point, and morphisms are any maps which preserve base points. There is a functor $D : Int \to Fin_*$ defined by:

$$D(\langle n \rangle) = [n-1]$$

and for an interval map $\phi : \langle n \rangle \to \langle m \rangle$:

$$D(\phi)(i) = \phi(i) \text{ if } \phi(i) \neq m \text{ , and } D(\phi)(i) = 0 \text{ otherwise.}$$

Let $\Gamma = Fin_*^{op}$ be the Segal category. We have then the following functor

$$C^{op} : \Delta \overset{\widetilde{(-)}}{\to} Int^{op} \overset{D^{op}}{\to} \Gamma.$$

Definition

A *symmetric cosimplicial monoid* is a functor $E : \Gamma \to Mon(V)$.

Symmetric cosimplicial monoids

Definition

A symmetric cosimplicial monoid is *n-commutative* if its underlying cosimplicial monoid is n-commutative.

Theorem

- *The cosimplicial monoid $E(F)$ of a symmetric tensor functor $F : \mathbb{C} \to \mathbb{D}$ is a 1-commutative symmetric cosimplicial monoid.*
- *The cosimplicial monoid $E(C)$ of a symmetric tensor category C is a 2-commutative symmetric cosimplicial monoid.*

Symmetric cosimplicial monoids

The symmetric group actions on the components E^n of a symmetric cosimplicial monoid in characteristic 0 give rise to the so-called *Hodge decomposition* of the cohomology

$$H^n(E) \ = \ H^{n,0}(E) \ \oplus \ \dots \ \oplus \ H^{n,n}(E) \ .$$

The decomposition is compatible with the cup-product

$$\cup : H^{m,s}(E) \otimes H^{n,t}(E) \to H^{m+n,s+t}(E) \ .$$

In particular, the top components $H^{n,n}(E)$ is the cohomology of the sub-complex of $C^*(E)$ of its anti-symmetric elements.
If E is also n-commutative this decomposition interacts with the higher order operations coming from E_{n+1}-algebra structure. And this is a powerful tool for calculations.
We don't understand, yet, on categorical level how to describe a full ambient operadic (or other higher categorical) structure on an n-commutative symmetric cosimplicial monoid.

THANK YOU