Dolbeault cohomology of complex manifolds with torus action

Taras Panov

Steklov Mathematical Institute
Moscow State University
Higher School of Economics, Moscow

Algebraic Topology, Group Actions and Combinatorics

Sirius, Sochi, 12–16 May 2026

The moment-angle complex

$\mathcal{K}$ an abstract simplicial complex on the set $[m] = \{1, 2, \ldots, m\}$
$I = \{i_1, \ldots, i_k\} \in \mathcal{K}$ a simplex; always assume $\varnothing \in \mathcal{K}$.

Consider the m-dimensional unit polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

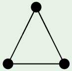
The moment-angle complex is

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{D} \times \prod_{i \notin I} \mathbb{S} \Big) \subset \mathbb{D}^m,$$

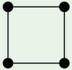
where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

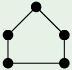
$\mathcal{Z}_{\mathcal{K}}$ has a natural action of the torus T^m.
When $\mathcal{K}$ is a simplicial subdivision of sphere (e.g., the boundary of a simplicial polytope), $\mathcal{Z}_{\mathcal{K}}$ is a topological manifold, called the moment-angle manifold.

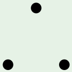
Example

1. Let $\mathcal{K} =$  (the boundary of a triangle). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{D} \times \mathbb{S} \times \mathbb{D}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{D}) = \partial(\mathbb{D}^3) \cong S^5.$$

2. Let $\mathcal{K} =$  (the boundary of a square). Then $\mathcal{Z}_{\mathcal{K}} \cong S^3 \times S^3$.

3. Let $\mathcal{K} =$  Then $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^4) \# \cdots \# (S^3 \times S^4)$ (5 times).

4. Let $\mathcal{K} =$  (three disjoint points). Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{S} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{D} \times \mathbb{S}) \cup (\mathbb{S} \times \mathbb{S} \times \mathbb{D}) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$$

(not a manifold).

We define an open subset $U(\mathcal{K}) \subset \mathbb{C}^m$ in a similar way:

$$U(\mathcal{K}) := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathbb{C}^\times \Big), \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

$U(\mathcal{K})$ is a toric variety with the corresponding fan given by

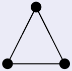
$$\Sigma_{\mathcal{K}} = \{ \mathbb{R}_\geqslant \langle e_i : i \in I \rangle : I \in \mathcal{K} \},$$

where e_i denotes the i-th standard basis vector of $\mathbb{R}^m$.

Theorem

(a) $U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1,\ldots,i_k\} \notin \mathcal{K}} \{ z_{i_1} = \cdots = z_{i_k} = 0 \}$
(the complement of an arrangement of coordinate subspaces);

(b) *There is a T^m-equivariant deformation retraction $U(\mathcal{K}) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}}$.*

E.g., $\mathcal{K} =$  Then $U(\mathcal{K}) = \mathbb{C}^3 \setminus \{ z_1 = z_2 = z_3 = 0 \} \xrightarrow{\simeq} S^5 = \mathcal{Z}_{\mathcal{K}}$.

Simplicial fans and exponential action

Suppose $\mathcal{K}$ is the underlying complex of a complete simplicial (not necessarily rational) fan Σ in $W^* \cong \mathbb{R}^n$.

Then the deformation retraction $U(\mathcal{K}) \to \mathcal{Z}_\mathcal{K}$ can be realised as the projection onto the orbit space of a smooth free and proper action of a non-compact subgroup $R \subset (\mathbb{C}^\times)^m$ isomorphic to $\mathbb{R}^{m-n}$, as described next.

A simplicial Σ fan is defined by generators $a_1, \ldots, a_m \in W^*$ of its one-dimensional cones and a simplicial complex $\mathcal{K}$ on $[m] = \{1, \ldots, m\}$.

For $I = \{i_1, \ldots, i_k\} \in [m]$ we let $\mathrm{A}_I = \{a_i \colon i \in I\}$. Then $\Sigma = \{\mathrm{cone}\, \mathrm{A}_I \colon I \in \mathcal{K}\}$ provided that the fan condition is satisfied: $\mathrm{relint}\, \mathrm{cone}\, \mathrm{A}_I \cap \mathrm{relint}\, \mathrm{cone}\, \mathrm{A}_J = \varnothing$ for $I, J \in \mathcal{K}$, $I \neq J$.

$$0 \longrightarrow V \xrightarrow{\Gamma^*} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0, \qquad A(e_i) = a_i, \quad V \cong \operatorname{Ker} A$$

Gale duality $\quad \updownarrow$

$$0 \longrightarrow W \longrightarrow \mathbb{R}^m \xrightarrow{\Gamma} V^* \longrightarrow 0, \qquad \Gamma(e_i) = \gamma_i$$

Gale dual vector configurations
$\mathrm{A} = \{a_1, \ldots, a_m\}$ in $W^* \cong \mathbb{R}^n$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ in $V^* \cong \mathbb{R}^{m-n}$.

Set

$$R = \exp(\operatorname{Ker} A) = \bigl\{(e^{\langle\gamma_1, v\rangle}, \ldots, e^{\langle\gamma_m, v\rangle})\colon v \in V\bigr\} \subset (\mathbb{R}^\times)^m \subset (\mathbb{C}^\times)^m$$

Theorem

(1) *The exponential action of R on $U(\mathcal{K}) \subset \mathbb{C}^m$ is free and proper iff $\{\mathcal{K}; a_1, \ldots, a_m\}$ defines a simplicial fan $\Sigma = A(\Sigma_{\mathcal{K}})$ in W^*.*

(2) *If Σ is a complete fan ($|\Sigma| = W^*$), then the quotient $U(\mathcal{K})/R$ is T^m-equivariantly homeomorphic to the moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$.*

Holomorphic exponential actions

$\widetilde{V} = (V, \mathcal{J}) \cong \mathbb{C}^\ell$, where $\mathcal{J}\colon V \to V$ is a complex structure on $V \cong \mathbb{R}^{m-n}$ (assuming $m - n = 2\ell$ is even).

$\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a vector configuration in $\widetilde{V}^* = \mathrm{Hom}_{\mathbb{C}}(\widetilde{V}, \mathbb{C})$.
Complex bilinear pairing $\widetilde{V}^* \times \widetilde{V} \to \mathbb{C}$

$$\langle \gamma, v \rangle_{\mathbb{C}} = \langle \gamma, v \rangle_{\mathbb{R}} - i \langle \gamma, \mathcal{J} v \rangle_{\mathbb{R}}.$$

The complex-analytic subgroup

$$H = \bigl\{ \bigl(e^{\langle \gamma_1, v \rangle_{\mathbb{C}}}, \ldots, e^{\langle \gamma_m, v \rangle_{\mathbb{C}}}\bigr) \colon v \in \widetilde{V} \bigr\} \subset (\mathbb{C}^\times)^m$$

acts holomorphically on

$$U(\mathcal{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathcal{K}} = \mathbb{C}^m \setminus \bigcup_{\{i_1, \ldots, i_p\} \notin \mathcal{K}} \{\mathbf{z} \colon z_{i_1} = \cdots = z_{i_p} = 0\}$$

Theorem

(1) *The holomorphic exponential action of $H \cong \mathbb{C}^\ell$ on $U(\mathcal{K}) \subset \mathbb{C}^m$ is free and proper iff $\{\mathcal{K}; a_1, \ldots, a_m\}$ defines a simplicial fan $\Sigma = A(\Sigma_\mathcal{K})$ in W^*.*

(2) *If Σ is a complete fan ($|\Sigma| = W^*$), then the quotient $U(\mathcal{K})/H$ is T^m-equivariantly homeomorphic to the moment-angle manifold $\mathcal{Z}_\mathcal{K}$.*

Corollary

Let $\mathcal{K}$ be the underlying complex of a complete simplicial fan (a star-shaped sphere triangulation) and $m - n = 2\ell$ is even.
Then the moment-angle complex $\mathcal{Z}_\mathcal{K} = (D^2, S^1)^\mathcal{K}$ has a structure of a compact complex manifold.

Example (holomorphic torus)

Exponential action of $H \cong \mathbb{C}$ on $\mathbb{C}^2$ defined by a configuration $\{\gamma_1, \gamma_2\}$:

$$H = \{(e^{\gamma_1 v} z_1, e^{\gamma_2 v} z_2)\colon v \in \mathbb{C}\}, \qquad (z_1, z_2) \mapsto (e^{\gamma_1 v} z_1, e^{\gamma_2 v} z_2).$$

Let $\mathcal{K} = \{\varnothing\}$ on $[2] = \{1, 2\}$, so that $U_{\mathbb{C}}(\mathcal{K}) = (\mathbb{C}^\times)^2$. If $\mathbb{R}\langle\gamma_1, \gamma_2\rangle = \mathbb{R}^2$, then the restriction of the action to $(\mathbb{C}^\times)^2$ is free and proper. The quotient

$$(\mathbb{C}^\times)^2/\widetilde{V} \cong \mathbb{C}/(\mathbb{Z}\langle\gamma_1, \gamma_2\rangle) \cong T^2$$

is a one-dimensional complex torus.

More generally, let $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ be a real basis in $\widetilde{V} \cong \mathbb{C}^\ell$. Then A is a configuration of m zero vectors in $W^* = \{0\}$, $\mathcal{K} = \{\varnothing\}$ on $[m] = [2\ell]$ and $U_{\mathbb{C}}(\mathcal{K}) = (\mathbb{C}^\times)^m$. The quotient $(\mathbb{C}^\times)^m/\widetilde{V} \cong \overline{\widetilde{V}}/\Lambda \cong T^{2\ell}$ is a holomorphic torus. Any holomorphic torus can be obtained in this way.

Proposition

A complex moment-angle manifold $\mathcal{Z}_K = U_{\mathbb{C}}(\mathcal{K})/\widetilde{V}$ is non-Kähler, unless it is a holomorphic torus.

Irrational toric varieties

If the space V is rational with respect to $\Gamma = \{\gamma_1, \dots, \gamma_m\}$, then

$$L = \{v \in V \colon \langle \gamma_k, v\rangle \in \mathbb{Z} \text{ for } k = 1, \dots, m\}.$$

is a full rank lattice in V, and $L^* = \mathbb{Z}\langle \gamma_1, \dots, \gamma_m\rangle \subset V^*$.

The Gale dual rational vector configuration $\mathrm{A} = \{a_1, \dots, a_m\}$ spans a full rank lattice $N = M^*$ in W^*.

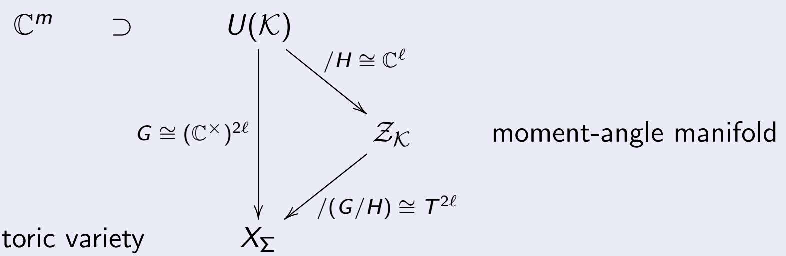
The algebraic torus $\mathbb{C}^\times_L = L \otimes_{\mathbb{Z}} \mathbb{C}^\times = V_{\mathbb{C}}/(2\pi i L)$ embeds in $(\mathbb{C}^\times)^m$ as the closed algebraic subgroup

$$\begin{aligned} G &= \exp \mathit{\Gamma}^*_{\mathbb{C}}(V_{\mathbb{C}}) = \exp(\mathrm{Ker}\, A_{\mathbb{C}} \colon \mathbb{C}^m \to W^*_{\mathbb{C}}) \\ &= \Bigl\{(t_1, \dots, t_m) \in (\mathbb{C}^\times)^m \colon \prod_{i=1}^m t_i^{\langle a_i, w\rangle} = 1, \quad w \in M\Bigr\}. \end{aligned}$$

The fan $\Sigma = A(\Sigma_{\mathcal{K}})$ is rational and defines a toric variety

$$X_\Sigma = U(\mathcal{K})/G = \mathcal{Z}_{\mathcal{K}}/H.$$

Rational case:



A perturbation of the vector configuration A destroys the rationality of the fan $\Sigma = \{\mathcal{K}, \mathrm{A}\}$, the subgroup $G \subset (\mathbb{C}^\times)^m$ ceases to be closed, the closed holomorphic tori in the fibres of the bundle $\mathcal{Z}_\mathcal{K} \to X_\Sigma$ "open up", and the fibre bundle turns into a holomorphic foliation $\mathcal{F}$ of $\mathcal{Z}_\mathcal{K}$ with noncompact leaves G/H.

The holomorphic foliated manifolds $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ model irrational deformations of toric varieties.

De Rham and Dolbeault cohomology (rational case)

The face ring (the Stanley–Reisner ring) of $\mathcal{K}$ is

$$\mathbb{C}[\mathcal{K}] := \mathbb{C}[v_1, \ldots, v_m]/I_{\mathcal{K}} = \mathbb{C}[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \colon \{i_1, \ldots, i_k\} \notin \mathcal{K}),$$

where $\mathbb{C}[v_1, \ldots, v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_{\mathcal{K}}$ is the Stanley–Reisner ideal.

Proposition

The T^m-equivariant cohomology is given by

$$H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) = H^*_{T^m}(U(\mathcal{K})) \cong \mathbb{C}[\mathcal{K}].$$

The toric variety X_Σ is Kähler (equivalently, projective) if and only if Σ is the normal fan of a nonsingular (Delzant) polytope P.

Theorem (Danilov)

The Dolbeault cohomology of complete nonsingular X_Σ is given by

$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\bar\partial}(X_\Sigma)$, $I_\mathcal{K}$ is the Stanley–Reisner ideal,
J_Σ is the ideal generated by the linear forms $\sum_{k=1}^m \langle a_k, w\rangle v_k$,
$a_k = A(e_k)$ are the generators of 1-dim cones of Σ, $w \in W$.

The nonzero Hodge numbers are given by $h^{p,p}(X_\Sigma) = h_p$,
where $h(\Sigma) = (h_0, h_1, \ldots, h_n)$ is the h-vector of Σ.

Theorem (Buchstaber-P.)

The de Rham cohomology ring of $\mathcal{Z}_\mathcal{K}$ is given by

$$\begin{aligned}
H^*(\mathcal{Z}_\mathcal{K}) &\cong \operatorname{Tor}_{\mathbb{C}[v_1,\ldots,v_m]}(\mathbb{C}[\mathcal{K}],\mathbb{C})\\
&\cong H\big(\Lambda[u_1,\ldots,u_m]\otimes\mathbb{C}[\mathcal{K}],d\big) \qquad du_i=v_i,\ dv_i=0\\
&\cong H\big(\Lambda[t_1,\ldots,t_{m-n}]\otimes H^*(X_\Sigma),d\big) \qquad \Lambda[t_1,\ldots,t_{m-n}]=H^*(T_L)\\
&\cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-|I|-1}(\mathcal{K}_I).
\end{aligned}$$

Here $H\big(\Lambda[t_1,\ldots,t_{m-n}]\otimes H^*(X_\Sigma),d\big)$ is the E_3 term of the Serre spectral sequence of the principal bundle $\mathcal{Z}_\mathcal{K}\to X_\Sigma$ with fibre $T_L\cong T^{m-n}$.

Theorem (P.-Ustinovsky)

Let Σ *be a rational fan,* $\mathcal{Z}_\mathcal{K} \longrightarrow X_\Sigma$ *a holomorphic principal bundle with fibre* $\widetilde{T}_L = \widetilde{V}^*/L^* \cong T^{2\ell}$.

Then the Dolbeault cohomology ring of $\mathcal{Z}_\mathcal{K}$ *is given by*

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K}) \cong H\big(H^{*,*}_{\bar\partial}(\widetilde{T}_L) \otimes H^{*,*}_{\bar\partial}(X_\Sigma), d\big),$$

where

$$H^{*,*}_{\bar\partial}(\widetilde{T}_L) = \Lambda\big(\widetilde{V} \oplus \overline{\widetilde{V}}\big) = \Lambda[\xi_1, \ldots, \xi_\ell, \eta_1, \ldots, \eta_\ell]$$
$$H^{*,*}_{\bar\partial}(X_\Sigma) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

$\xi_j \in H^{1,0}_{\bar\partial}(\widetilde{T}_L) = \widetilde{V}$, $\eta_j \in H^{0,1}_{\bar\partial}(\widetilde{T}_L) = \overline{\widetilde{V}}$, $dv_j = d\eta_j = 0$, $d\xi_j = c(\xi_j)$,
$c\colon H^{1,0}_{\bar\partial}(\widetilde{T}_L) \to H^{1,1}_{\bar\partial}(X_\Sigma)$ *is the holomorphic first Chern class map.*

Corollary

(a) *The Borel spectral sequence of the holomorphic fibration* $\mathcal{Z}_{\mathcal{K}} \xrightarrow{\widetilde{T}_L} X_\Sigma$ *(converging to Dolbeault cohomology of* $\mathcal{Z}_{\mathcal{K}}$*) collapses at the* E_3 *page;*

(b) *The Frölicher spectral sequence (with* $E_1 = H_{\bar\partial}^{*,*}(\mathcal{Z}_{\mathcal{K}})$*, converging to* $H^*(\mathcal{Z}_{\mathcal{K}})$*) collapses at* E_2*.*

Basic cohomology

M a manifold with an action of a connected Lie group G, $\mathfrak{g} = \mathrm{Lie}\, G$.

$$\Omega(M)_{\mathrm{bas},\,G} = \{\omega \in \Omega(M)\colon \iota_\xi\omega = L_\xi\omega = 0 \text{ for any } \xi \in \mathfrak{g}\},$$

$H^*_{\mathrm{bas},\,G}(M) = H(\Omega(M)_{\mathrm{bas},\,G}, d)$ the basic cohomology of M.

$S(\mathfrak{g}^*)$ the symmetric algebra on $\mathfrak{g}^*$ with generators of degree 2.
The Cartan model is

$$\mathcal{C}_\mathfrak{g}(\Omega(M)) = ((S(\mathfrak{g}^*) \otimes \Omega(M))^\mathfrak{g}, d_\mathfrak{g}),$$

where $(S(\mathfrak{g}^*) \otimes \Omega(M))^\mathfrak{g}$ denotes the $\mathfrak{g}$-invariant subalgebra.

An element $\omega \in \mathcal{C}_\mathfrak{g}(\Omega(M))$ is a "$\mathfrak{g}$-equivariant polynomial map from $\mathfrak{g}$ to $\Omega(M)$". The differential $d_\mathfrak{g}$ is given by

$$d_\mathfrak{g}(\omega)(\xi) = d(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

Theorem

$$H^*_{\mathrm{bas},\,G}(M) \cong H\bigl(\mathcal{C}_{\mathfrak{g}}(\Omega(M)), d_{\mathfrak{g}}\bigr).$$

If in addition G is a compact, then

$$H^*_{\mathrm{bas},\,G}(M) \cong H^*_G(M) = H^*(EG \times_G M) \quad \text{the equivariant cohomology.}$$

Basic de Rham and Dolbeault cohomology of the foliated manifold $(\mathcal{Z}_\mathcal{K}, \mathcal{F})$ (irrational toric variety) is like the cohomology of the toric quotient $X_\Sigma = \mathcal{Z}_\mathcal{K}/\widetilde{T}_L$ in the rational case:

Theorem (Ishida–Krutowski–P)

There is an isomorphism of algebras:

$$H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F}) \cong \mathbb{C}[v_1, \ldots, v_m]/(I_\mathcal{K} + J_\Sigma),$$

where $v_i \in H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K}, \mathcal{F})$, $i = 1, \ldots, m$,
$I_\mathcal{K}$ is the Stanley–Reisner ideal, generated by

$$v_{i_1} \cdots v_{i_k} \quad \textit{with } \{i_1, \ldots, i_k\} \notin \mathcal{K},$$

and J_Σ is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle a_i, w \rangle v_i, \quad w \in W.$$

The proof of the theorem is based on the following formality result. Let $\mathfrak{t} = \mathrm{Lie}(T^m) \cong \mathbb{R}^m$ and consider the Cartan model

$$\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K})) = \big((S(\mathfrak{t}^*) \otimes \Omega(\mathcal{Z}_\mathcal{K}))^{T^m}, d_\mathfrak{t}\big).$$

Since T^m is compact, we get

$$H(\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))) = H^*_{T^m}(\mathcal{Z}_\mathcal{K}) = \mathbb{C}[v_1, ..., v_m]/I_\mathcal{K}.$$

Lemma

The DGA $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ is formal. Furthermore, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_\mathfrak{t}(\Omega(\mathcal{Z}_\mathcal{K}))$ and $H_{T^m}(\mathcal{Z}_\mathcal{K})$ which respect the $S(\mathfrak{t}^*)$-module structure.

Dolbeault cohomology in the irrational case

Theorem (P–Ustinovsky, Krutowski–P)

The Dolbeault cohomology algebra of $\mathcal{Z}_\mathcal{K}$ is given by

$$H^{*,*}_{\bar\partial}(\mathcal{Z}_\mathcal{K}) \cong H\bigl(\Lambda(\widetilde{V} \oplus \overline{\widetilde{V}}) \otimes H^{*,*}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K},\mathcal{F}), d\bigr),$$

where $d(\overline{\widetilde{V}}) = 0$ and $d\colon \widetilde{V} \to H^{1,1}_{\mathrm{bas},\bar\partial}(\mathcal{Z}_\mathcal{K},\mathcal{F})$ is the "foliated Chern class".

References

[1] Taras Panov, Yuri Ustinovsky and Misha Verbitsky. *Complex geometry of moment-angle manifolds*. Math. Zeitschrift 284 (2016), no. 1, 309–333.

[2] Hiroaki Ishida, Roman Krutowski and Taras Panov. *Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds*. Internat. Math. Research Notices 2022 (2022), no. 7, 5541–5563.

[3] Taras Panov. *Exponential actions defined by vector configurations, Gale duality, and moment-angle manifolds*. Bulletin of the London Mathematical Society 57 (2025), no. 9, 2571–2629.