Automorphism groups of moment-angle manifolds

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Algebraic Topology, Group Actions, and Combinatorics

Sochi

$\mathcal{Z}_{\mathcal{K}}$ and $U_{\mathbb{C}}(\mathcal{K})$

Let $\mathcal{K}$ be a simplicial complex on $[m] = \{1, 2, \ldots, m\}$.

Moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} = \{(z_1, \ldots, z_m) \in \mathbb{D}^m \colon \ \{i \colon \ |z_i| < 1\} \in \mathcal{K}\},$$

where $\mathbb{D} = \{z \in \mathbb{C} \colon \ |z| \leq 1\}$.

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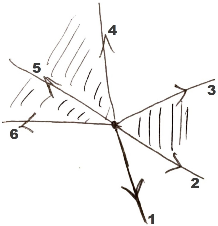
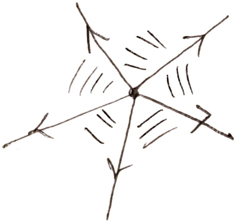
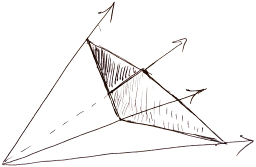
$$U_\mathbb{C}(\mathcal{K}) = \{(z_1, \dots, z_m) \in \mathbb{C}^m \colon \ \{i \colon \ z_i = 0\} \in \mathcal{K}\} =$$
$$= \mathbb{C}^m \setminus \bigcup_{\{i_1,\dots,i_k\} \notin \mathcal{K}} \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\}.$$

Fans







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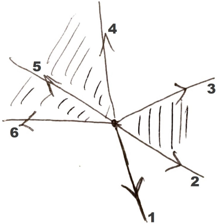

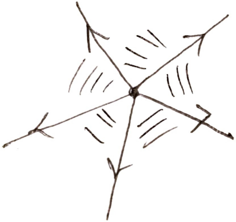

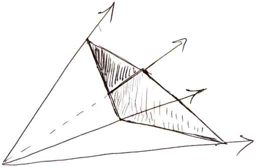

Σ is simplicial $\rightsquigarrow \mathcal{K} = \mathcal{K}_\Sigma = \{I \subset [m] : \mathrm{Cone}(a_i : \ i \in I) \in \Sigma\}$

Linear Gale duality

Configuration A of vectors $a_1, \ldots, a_m$ in $W^* \cong \mathbb{R}^n$

$$0 \longrightarrow V \longrightarrow \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0$$

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$$\operatorname{Ker} A = \Gamma^*(V),$$
$$\Gamma^*(v) = (\langle \gamma_1, v \rangle, \dots, \langle \gamma_m, v \rangle) \in \mathbb{R}^m$$

Real moment-angle manifolds

$$V \times U_{\mathbb{C}}(\mathcal{K}) \to U_{\mathbb{C}}(\mathcal{K}),$$
$$v \cdot z = (e^{\langle \gamma_1, v \rangle} z_1, \ldots, e^{\langle \gamma_m, v \rangle} z_m) = \exp \Gamma^*(v) \cdot z$$

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$U_{\mathbb{C}}(\mathcal{K})/R_\Sigma$ is a smooth manifold of dim $= m + n$, homeomorphic to $\mathcal{Z}_{\mathcal{K}}$.

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For a normal fan Σ of a polytope $P = \{w \in W\colon\ \langle a_i, w\rangle + b_i \geq 0, \quad i = 1, \ldots, m\}$, this is the same smooth structure as the one defined by

$$\mathcal{Z}_P = \{(z_1, \ldots, z_m) \in \mathbb{C}^m\colon \quad \gamma_1 |z_1|^2 + \ldots + \gamma_m |z_m|^2 = \delta\}$$

for $\delta = \sum_{i=1}^m b_i \gamma_i$.

Complex moment-angle manifolds

Assume $m+n$ is **even**, $l=\frac{m-n}{2}$, $J\colon\ V\to V$ is an operator of complex structure.

$$V_{\mathbb{C}} = V^{1,0}\oplus V^{0,1},\quad V\cong V^{1,0},\ v\mapsto v-iJv$$

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The complexified action:

$$V_{\mathbb{C}} \times U_{\mathbb{C}}(\mathcal{K}) \to U_{\mathbb{C}}(\mathcal{K}),$$

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$$H = \Gamma^*_{\mathbb{C}}(V^{1,0}) \subset (\mathbb{C}^{\times})^m \subset \mathrm{Aut}(U_{\mathbb{C}}(\mathcal{K})), \quad H \cong \mathbb{C}^l$$

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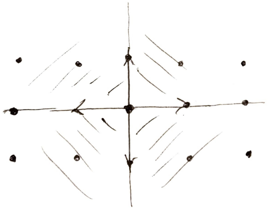
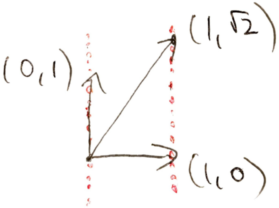
$U_{\mathbb{C}}(\mathcal{K})/H$ is a complex manifold diffeomorphic to $\mathcal{Z}_{\mathcal{K}}$.

Rational and regular fans

$N = \mathbb{Z}\langle a_1, \dots, a_m \rangle \subset W^*$.





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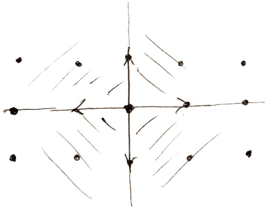

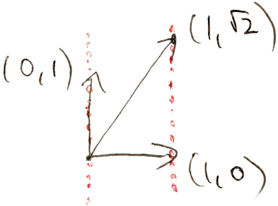

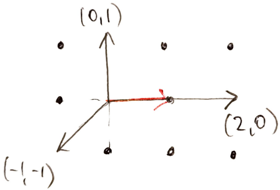

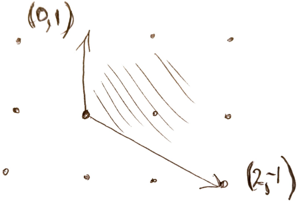

Rational case

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We have a principal F-bundle $\mathcal{Z}_{\mathcal{K}} \xrightarrow{/F} X_\Sigma$, where $F = G_\Sigma/H \cong T^l_{\mathbb{C}}$ is a complex compact torus.

The main statement

Recall that for complete, rational, simplicial Σ

$$1 \to G_\Sigma \to \mathfrak{N}(\mathrm{Aut}(U_{\mathbb{C}}(\mathcal{K}), G_\Sigma) \to \mathrm{Aut}(X_\Sigma) \to 1.$$

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$$1 \to H \to \mathfrak{N}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K})), H) \to \mathrm{Aut}(\mathcal{Z}_\mathcal{K}) \to 1,$$
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Moreover, $\mathfrak{C}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K})), H) = \mathfrak{C}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K}))G_\Sigma)$ *and*

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$\mathfrak{N}(\mathrm{Aut}(U_{\mathbb{C}}(\mathcal{K})), G_\Sigma)$ is generated by $\mathfrak{C}(\mathrm{Aut}(U_{\mathbb{C}}(\mathcal{K})), G_\Sigma)$ (that is a connected affine algebraic group) and $\mathrm{Aut}(N, \Sigma)$ (fan automorphisms).

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$$\mathfrak{f} \cong V_{\mathbb{C}}/V^{1,0} \cong V^{0,1}, \quad H^{1,0}(F) \cong \mathfrak{f}^* \cong \overline{V^{0,1}} \cong V^{1,0}$$
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$$H^{*,*}(\mathcal{Z}_\mathcal{K}) \cong [\Lambda(V^{1,0} \oplus V^{0,1}) \otimes H^{*,*}(X_\Sigma), d],$$

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$$c\colon V^{1,0} \cong H^{1,0}(F) \to H^{1,1}(X_\Sigma)$$

of $\mathcal{Z}_{\mathcal{K}} \to X_\Sigma$ and

$$c\colon \; V^{1,0} \hookrightarrow V^{0,1} \oplus V^{0,1}.$$

Example 1

$\mathrm{Aut}(X_\Sigma)/\mathrm{Aut}^0(X_\Sigma) \cong \frac{\mathrm{Aut}(N,\Sigma)}{\mathrm{Ker}\,\alpha}$, where $\alpha\colon\ Aut(N,\Sigma) \to \mathrm{Aut}(H^2(X_\Sigma;\mathbb{Z}))$.

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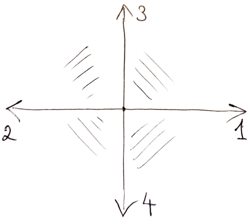

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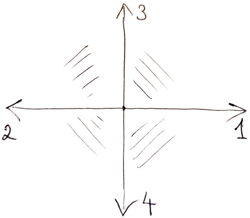


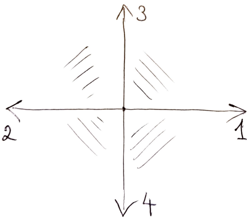
We have $\mathrm{Aut}(N,\Sigma) = D_4,\ \mathrm{Aut}(X_\Sigma)/\mathrm{Aut}^0(X_\Sigma) \cong \mathbb{Z}/2$.

$$\mathfrak{N}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K})), G_\Sigma) \cong GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \rtimes \mathbb{Z}/2$$

Example 1

$\mathrm{Aut}(X_\Sigma)/\mathrm{Aut}^0(X_\Sigma) \cong \frac{\mathrm{Aut}(N,\Sigma)}{\mathrm{Ker}\,\alpha}$, where $\alpha\colon\ Aut(N,\Sigma) \to \mathrm{Aut}(H^2(X_\Sigma;\mathbb{Z}))$.

$\mathcal{Z}_\mathcal{K} \cong S^3 \times S^3,\ X_\Sigma \cong \mathbb{C}P^1 \times \mathbb{C}P^1,\ F = T^1_\mathbb{C}$.



We have $\mathrm{Aut}(N,\Sigma) = D_4,\ \mathrm{Aut}(X_\Sigma)/\mathrm{Aut}^0(X_\Sigma) \cong \mathbb{Z}/2$.

$$\mathfrak{N}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K})), G_\Sigma) \cong GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \rtimes \mathbb{Z}/2$$

$$\mathrm{Aut}(\mathcal{Z}_\mathcal{K}) \cong \mathfrak{N}(\mathrm{Aut}(U_\mathbb{C}(\mathcal{K})), H)/H \cong \frac{GL(2,\mathbb{C}) \times GL(2,\mathbb{C})}{\mathbb{C}}.$$

Example 2

We may obtain different complex structures on $U_{\mathbb{C}}(\mathcal{K})/R_{\Sigma} \cong \mathcal{Z}_{\mathcal{K}}$ (which are not biholomorphic to each other).

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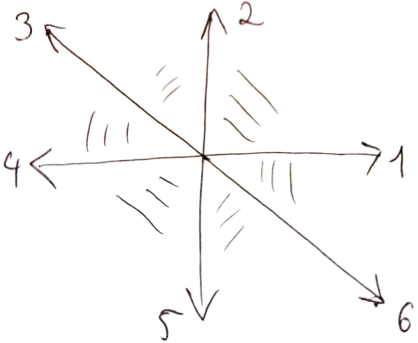
We may obtain different complex structures on $U_{\mathbb{C}}(\mathcal{K})/R_\Sigma \cong \mathcal{Z}_\mathcal{K}$ (which are not biholomorphic to each other).

$\mathcal{Z}_\mathcal{K} \cong (S^3 \times S^5)^{\#9} \# (S^4 \times S^4)^{\#8}$, $X_\Sigma \cong Bl_{p_{24},\ p_{15}}(\mathbb{C}P^1 \times \mathbb{C}P^1)$, $F \cong T^2_{\mathbb{C}}$.



Thank you for your attention!

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