Universal Spaces of Parameters for $G_{n,2}$ and Stable Rational Curves $\overline{\mathcal{M}}_{0,n}$

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Plan

1. Toric Topology of $G_{n,2}$
   - Strata and Moment Map
   - Spaces of Parameters
   - Universal Space of Parameters
2. Moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves
   - Gel'fand-MacPhearson correspondence
   - Virtual Spaces of Parameters and Stable Rational Curves
   - Constructing the projection

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Complex Grassmann manifolds $G_{n,2}$

- Points in $G_{n,2} = G_2(\mathbb{C}^n)$ are represented by $n \times 2$ complex matrices

$$X = \begin{pmatrix} z_1 & w_1 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix}, \quad \operatorname{rk} X = 2,$$

  up to the right $GL(2, \mathbb{C})$-action. The Plücker coordinates

$$P^{ij} = z_i w_j - z_j w_i, \quad 1 \leq i < j \leq n$$

  are defined up to $\mathbb{C}^*$-action.
- There are canonical left actions of $\mathbb{T}^n = (S^1)^n$ and $(\mathbb{C}^*)^n$ on $G_{n,2}$.
- The effective actions of $T^{n-1} = \mathbb{T}^n / \operatorname{diag}(\mathbb{T}^n)$ on $G_{n,2}$ have complexity $n - 3$.

The $\mathbb{T}^n$-actions on $G_{n,2}$ for $n \geq 4$ served as model examples for the new theory of torus actions of positive complexity developed by V. M. Buchstaber and S. Terzić.

For each pair (i, j) with $1 \leq i < j \leq n$, define the chart

$$M_{ij} = \{L \in G_{n,2} \mid P^{ij}(L) \neq 0\}$$

with

$$\varphi_{ij} \colon M_{ij} \to \mathbb{C}^{2(n-2)}, \quad \varphi_{ij}(L) = (z_k, w_k)_{k \neq i,j} \in \mathbb{C}^{2(n-2)},$$

where we set $(z_i, w_i) = (1, 0)$ and $(z_j, w_j) = (0, 1)$ via the $GL(2, \mathbb{C})$.

- Each M_{ij} is $\mathbb{T}^n$-invariant.
- Each M_{ij} contains exactly one $\mathbb{T}^n$-fixed point $x_{ij} = \mathrm{span}_{\mathbb{C}}\{e_i, e_j\}$.
- The charts $\{M_{ij}\}$ form a $\mathbb{T}^n$-invariant atlas on $G_{n,2}$.

Stratification of $G_{n,2}$

Let σ be a set of 2-indices (i, j) with $1 \leq i < j \leq n$. Then define

$$W_\sigma = \bigcap_{(i,j)\in\sigma} M_{ij} \cap \bigcap_{(i,j)\notin\sigma} (G_{n,2} \setminus M_{ij}).$$

Equivalently,

$$W_\sigma = \{L \in G_{n,2} \mid P^{ij}(L) \neq 0 \Leftrightarrow (i, j) \in \sigma\}.$$

If $W_\sigma \neq \varnothing$, then σ is admissible and W_σ is a stratum.

Example

- $\sigma = \binom{[n]}{2}$ is admissible and $W = W_{\binom{[n]}{2}}$ is an open dense stratum,
- $\sigma = \varnothing$ is not admissible.

- Strata W_σ are invariant under the actions of the tori $\mathbb{T}^n$ and $(\mathbb{C}^*)^n$.

- $G_{n,2} = \bigsqcup_{\sigma \text{ admissible}} W_\sigma, \quad W_\sigma \cap W_{\sigma'} = \varnothing$ for $\sigma \neq \sigma'$.

Moment Map and Admissible Polytopes

Moment map for the $\mathbb{T}^n$-action on $G_{n,2}$ is defined as

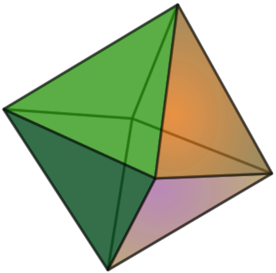
$$\mu\colon G_{n,2} \to \mathbb{R}^n, \quad L \mapsto \frac{1}{\sum_{i<j} |P^{ij}(L)|^2} \sum_{i<j} |P^{ij}(L)|^2 \Lambda_{ij}, \quad \Lambda_{ij} = e_i + e_j.$$

- The hypersimplex $\Delta_{n,2}$ is the image $\mu(G_{n,2})$ and

$$\Delta_{n,2} = \mathrm{conv}\{\Lambda_{ij} \mid 1 \le i < j \le n\}.$$

- For an admissible set σ, define the admissible polytope

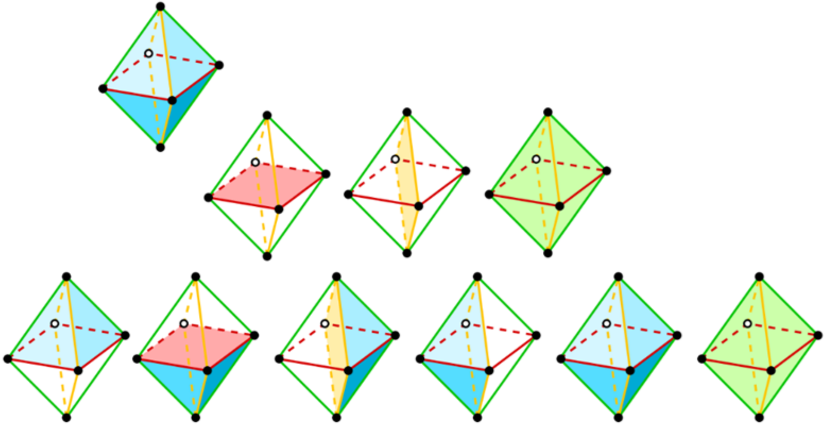
$$P_\sigma = \mathrm{conv}\{\Lambda_{ij} \mid (i,j) \in \sigma\}.$$



Proposition

$$\mu(W_\sigma) = \mathring{P}_\sigma.$$

Admissible Polytopes in $\Delta_{4,2}$



Spaces of Parameters

For a stratum W_σ we have:

- The orbit space $W_\sigma/\mathbb{T}^n$ together with the moment map projection

$$\bar{\mu}\colon W_\sigma/\mathbb{T}^n \longrightarrow \mathring{P}_\sigma.$$

- Spaces of parameters $F_\sigma = W_\sigma/(\mathbb{C}^*)^n$ together with the projection

$$q_\sigma\colon W_\sigma/\mathbb{T}^n \longrightarrow F_\sigma.$$

Theorem (Buchstaber–Terzić)

The map

$$h_\sigma\colon W_\sigma/\mathbb{T}^n \longrightarrow \mathring{P}_\sigma \times F_\sigma, \quad h_\sigma = (\bar{\mu}, q_\sigma)$$

is a homeomorphism.

On the topological model for the orbit space $G_{n,2}/\mathbb{T}^n$

For $\sigma = \binom{[n]}{2}$ we have an open dense subset

$$W/\mathbb{T}^n \cong \mathring{\Delta}_{n,2} \times F \quad \text{in} \quad G_{n,2}/\mathbb{T}^n.$$

The idea of model spaces goes back to E. B. Vinberg.

Problem (Buchstaber-Terzić)

For $n \in \mathbb{N}$, construct a topological model for $G_{n,2}/\mathbb{T}^n$: find a smooth compact manifold $\mathcal{F}_n$ and a continuous projection

$$\pi\colon \Delta_{n,2} \times \mathcal{F}_n \longrightarrow G_{n,2}/\mathbb{T}^n$$

such that π is compatible with the homeomorphisms h_σ for all admissible σ.

If such projection exists, then

$$G_{n,2}/\mathbb{T}^n \cong \Delta_{n,2} \times \mathcal{F}_n/\sim .$$

Smooth manifold $\mathcal{F}_n$ is a universal space of parameters iff:

- $\mathcal{F}_n$ is a compactification of F.
- There exists a projection

$$\pi\colon \Delta_{n,2} \times \mathcal{F}_n \longrightarrow G_{n,2}/\mathbb{T}^n$$

  extending the homeomorphism $h^{-1}\colon \mathring{\Delta}_{n,2} \times F \to W/\mathbb{T}^n$.

- $\bar{\mu} \circ \pi = \mathrm{pr}_1$ for $\mathrm{pr}_1\colon \Delta_{n,2} \times \mathcal{F}_n \to \Delta_{n,2}$.
- For each $\mathrm{pt}_\sigma \in \mathring{P}_\sigma$, the space $\widetilde{F}_\sigma = \pi^{-1}(\mathrm{pt}_\sigma \times F_\sigma)$ does not depend on the choice of pt_σ, and neither does the projection $p_\sigma = q_\sigma \circ \pi$.

Virtual Spaces of Parameters

The spaces $\widetilde{F}_\sigma$ are the virtual spaces of parameters equipped with the projections

$$p_\sigma : \widetilde{F}_\sigma \longrightarrow F_\sigma .$$

Observe that

$$\mathcal{F}_n = \bigcup_\sigma \widetilde{F}_\sigma$$

and for $x \in \mathring{P}_\sigma$ and $c \in \widetilde{F}_\sigma$

$$\pi(x, c) = h_\sigma^{-1}\big(x, p_\sigma(c)\big).$$

Question

How can such a projection be constructed?

Buchstaber–Terzić Approach

- Fix a chart M_{ij} and take coordinates $(z_k, w_k)_{k\neq i,j} \in \mathbb{C}^{2(n-2)}$.
- For an index pair (p,q) with $p,q \neq i,j$, define $[c_{pq} : c'_{pq}] \in \mathbb{CP}^1$ by

$$c'_{pq} z_p w_q = c_{pq} z_q w_p.$$

- Take an embedding $f_{ij} \colon F \hookrightarrow (\mathbb{CP}^1)^{\binom{n-2}{2}}$ given by

$$[L] \longmapsto \left([c_{pq} : c'_{pq}]\right)_{1\leq p<q\leq n,\ p,q\neq i,j} \in (\mathbb{CP}^1)^{\binom{n-2}{2}}.$$

Proposition

The closure $\bar{F}$ of the image $f_{ij}(F)$ in $(\mathbb{CP}^1)^{\binom{n-2}{2}}$ is given by $\binom{n-2}{3}$ cubic equations

$$c'_{pq} c_{pk} c'_{qk} = c_{pq} c'_{pk} c_{qk}, \quad 1 \leq p < q < k \leq n, \quad p,q,k \neq i,j.$$

Buchstaber–Terzić Approach

- For charts M_{ij} and M_{pq}, define the coordinate automorphism

$$f_{ij,pq} = f_{pq} \circ f_{ij}^{-1}.$$

- Find a compactification $\widehat{\mathcal{F}}_n$ of $F \subseteq (\mathbb{CP}^1)^{\binom{n-2}{2}}$ such that the maps $f_{ij,pq}$ extend to homeomorphisms on $\widehat{\mathcal{F}}_n$.

- Apply the wonderful compactification to $F \subseteq (\mathbb{CP}^1)^{\binom{n-2}{2}}$ along the submanifolds where $f_{ij,pq}$ does not extend.

Theorem (Buchstaber–Terzić, 2023)

The space $\mathcal{F}_n = \widehat{\mathcal{F}}_n$ is a universal space of parameters for the $\mathbb{T}^n$-action on $G_{n,2}$. Moreover, $\mathcal{F}_n$ is diffeomorphic to the Chow quotient $G_{n,2}//(\mathbb{C}^*)^n$ and hence to the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves.

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Moduli Spaces $\mathcal{M}_{0,n}$ of Rational Curves with n points

- A rational curve is a complex algebraic curve biholomorphic to $\mathbb{CP}^1$.
- Moduli spaces $\mathcal{M}_{0,n}$ are smooth non-compact manifolds defined as

$$\mathcal{M}_{0,n} = \frac{(\mathbb{CP}^1)^n \setminus \Delta}{PGL(2,\mathbb{C})}, \quad \Delta = \{c_i = c_j \mid 1 \leq i < j \leq n\},$$

with the pointwise action of $PGL(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$.

Gel'fand-MacPhearson correspondence

- For a fixed σ, define $\mathrm{Mat}^{\sigma}_{n,2}(\mathbb{C})$ as the set of complex $n \times 2$ matrices X with rows (z_k, w_k) for $1 \leq k \leq n$ such that

$$z_i w_j - z_j w_i \neq 0 \Leftrightarrow (i,j) \in \sigma.$$

- Two commuting actions: $(\mathbb{C}^*)^n \curvearrowright \mathrm{Mat}^{\sigma}_{n,2}(\mathbb{C}) \curvearrowleft GL(2,\mathbb{C})$.

- By definitions, $W_\sigma = \mathrm{Mat}^{\sigma}_{n,2}(\mathbb{C})/GL(2,\mathbb{C})$ and $F_\sigma = W_\sigma/(\mathbb{C}^*)^n$.

- By changing the order of the factorizations, we obtain

$$F_\sigma \cong \frac{\mathrm{Mat}^{\sigma}_{n,2}(\mathbb{C})/(\mathbb{C}^*)^n}{PGL(2,\mathbb{C})}.$$

  The nonzero rows (z_k, w_k) are now viewed as points $[z_k : w_k] \in \mathbb{CP}^1$.

Spaces of Parameters as Moduli Spaces of Rational Curves

For an admissible σ, the nonzero rows (z_k, w_k) split into $N = N(\sigma)$ groups of pairwise proportional vectors in $\mathbb{C}^2$. Thus, we have proved

Theorem

F_σ is homeomorphic to $\mathcal{M}_{0,N(\sigma)}$. For the main stratum W, the parameter space F is homeomorphic to $\mathcal{M}_{0,n}$.

Rational Normal Curves

- Recall a well-known Veroneze map:

$$v_n \colon \mathbb{CP}^1 \longrightarrow \mathbb{CP}^{n-2}, \quad [t_0 : t_1] \mapsto [t_0^{n-2} : t_0^{n-1} t_1 : t_0^{n-3} t_1^2 : \ldots : t_1^{n-2}].$$

- $C = v_n(\mathbb{CP}^1)$ is isomorphic to $\mathbb{CP}^1$.
- $PGL(n-1, \mathbb{C})$-action on $\mathbb{CP}^{n-2}$ defines rational (normal) curves.

Theorem

The family of rational normal curves passing through n points $x_1, \ldots, x_n$ in general position in $\mathbb{CP}^{n-2}$ is isomorphic to the moduli space $\mathcal{M}_{0,n}$.

Stable Rational Curves

- If we allow rational curves to have nodal singularities ($xy = 0$), we obtain a nodal rational curves.
- A nodal rational curve C with marked points $x_1, \dots, x_n \in C$ is stable if the points x_i do not lie at the nodes and $\mathrm{Aut}(C)$ is finite.

Theorem

Every stable rational curve with n marked points is uniquely realized as a stable curve passing through a given set of points $x_1, \dots, x_n$ in general position in $\mathbb{CP}^{n-2}$.

- The Deligne–Mumford compactification $\overline{\mathcal{M}}_{0,n}$ is realized as the moduli space of stable rational curves.

Virtual Spaces of Parameters

- For admissible σ we have $F_\sigma \cong \mathcal{M}_{0,N(\sigma)}$.
- Define $\widetilde{F}_\sigma = \overline{\mathcal{M}}_{0,N(\sigma)}$.
- There is an embedding

$$\iota_\sigma : \overline{\mathcal{M}}_{0,N(\sigma)} \hookrightarrow \overline{\mathcal{M}}_{0,n}.$$

- There is a projection

$$p_\sigma : \overline{\mathcal{M}}_{0,N(\sigma)} \longrightarrow \mathcal{M}_{0,N(\sigma)}.$$

Embedding $\iota_\sigma\colon \overline{\mathcal{M}}_{0,N(\sigma)} \hookrightarrow \overline{\mathcal{M}}_{0,n}$

An admissible set σ defines a disjoint partition

$$[n] = A_1 \sqcup \cdots \sqcup A_{N(\sigma)} \sqcup B, \quad A_i \neq \varnothing,\ N(\sigma) \geq 2.$$

Step 1. Take marked points $y_1, \ldots, y_{N(\sigma)}$ in general position in $\mathbb{CP}^{N(\sigma)-2}$.

Step 2. Take rational stable curve C with marked points $y_1, \ldots, y_{N(\sigma)}$.

Step 3. Blow up C at y_i and replace y_i with $|A_i|$ marked points x_j for $j \in A_i$.

Proposition

$\bigcup_\sigma \iota_\sigma(\overline{\mathcal{M}}_{0,N(\sigma)}) = \overline{\mathcal{M}}_{0,n}$.

Projection $p_\sigma : \overline{\mathcal{M}}_{0,N(\sigma)} \longrightarrow \mathcal{M}_{0,N(\sigma)}$

Step 1. Take marked points $y_1, \dots, y_{N(\sigma)}$ in general position in $\mathbb{CP}^{N(\sigma)-2}$.

Step 2. Take a stable rational curve C' with marked points $x_1, \dots, x_n$ obtained by blowing up C according to σ.

Step 3. Contract the components containing the marked points corresponding to $A_1, \dots, A_N, B$ separately.

Projection $\pi\colon \Delta_{n,2} \times \mathcal{F}_n \to G_{n,2}/\mathbb{T}^n$

Conjecture (work in progress)

The projection π defined via $\widetilde{F}_\sigma$ and p_σ is well defined.

Список литературы I

- M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.
- V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513.
- I. M. Gelfand and V. V. Serganova, *Combinatorial geometries and torus strata on homogeneous compact manifolds*, Russian Math. Surveys **42** (1987), no. 2, 133–168.
- M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 14, Springer-Verlag, Berlin, 1988.
- W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 1, 183–225.

Список литературы II

- C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494.
- M. M. Kapranov, *Chow quotients of Grassmannians. I*, in: I. M. Gelfand Seminar, Adv. Soviet Math., vol. 16, Part 2, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.
- S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574.
- B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.
- A. Losev and Yu. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. **48** (2000), 443–472.
- Li Li, *Wonderful compactification of an arrangement of subvarieties*, Michigan Math. J. **58** (2009), no. 2, 535–563.

Список литературы III

V. M. Buchstaber and S. Terzić, *$(2n, k)$-manifolds and applications*, in: *Okounkov bodies and applications*, Oberwolfach Rep. **11** (2014), no. 2, 1469–1472.

V. M. Buchstaber and S. Terzić, *Toric topology of the complex Grassmann manifolds*, Mosc. Math. J. **19** (2019), no. 3, 397–463.

V. M. Buchstaber and S. Terzić, *Resolution of singularities of the orbit spaces $G_{n,2}/T^n$*, Proc. Steklov Inst. Math. **317** (2022), 21–54.

V. M. Buchstaber and S. Terzić, *The orbit spaces $G_{n,2}/T^n$ and the Chow quotients $G_{n,2}//(\mathbb{C}^*)^n$ of the Grassmann manifolds $G_{n,2}$*, Sb. Math. **214** (2023), no. 12, 1694–1720.

V. M. Buchstaber and S. Terzić, *Moduli spaces of weighted pointed stable curves and toric topology of Grassmann manifolds*, J. Geom. Phys. **215** (2025), 105533.

Список литературы IV

V. M. Buchstaber and S. Terzić, *Cohomology of symplectic T^n-reductions and compactifications of $\mathcal{M}_{0,n}$*, arXiv:2602.21751 (2026).