Cohomology of symplectic T^n-reductions on $G_{n,2}$ and problem of compactification of the moduli spaces $\mathcal{M}_{0,n}$

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Complex Grassmann manifolds $G_{n,2} = G_{n,2}(\mathbb{C})$

- $G_{n,2}$ consists of complex 2-dimensional subspaces in $\mathbb{C}^n$
- The effective action of the torus $T^{n-1} = T^n/\mathrm{diag}(T^n)$ on $G_{n,2}$ induced by coordinate wise action of $T^n \subset U(n)$ on $\mathbb{C}^n$.

The Plücker embedding

$$p : G_{n,2} \to \mathbb{C}P^N, \;\; N = \binom{n}{2} - 1,$$

is defined by $p(L) = (P^{ij}(L))_{1\leq i<j\leq n}$

- Let $\rho^2 : T^n \to T^{N+1}$ is the second exterior representation, $N = \binom{n}{2} - 1$.
- The weight vectors for ρ^2 are $\Lambda_{ij} \in \mathbb{Z}^n$:

$$\Lambda_{ij}(i) = \Lambda_{ij}(j) = 1, \; \Lambda_{ij}(k) = 0, \; k \neq i, j, \;\; 1 \leq i < j \leq n$$

- T^n-action on $\mathbb{C}P^N$ given as the composition of ρ^2 and the canonical T^{N+1}-action on $\mathbb{C}P^N$
- $p(G_{n,2}) \subset \mathbb{C}P^N$ is invariant for this T^n-action.

The moment map $\mu : G_{n,2} \to \mathbb{R}^n$ for $T^n \curvearrowright G_{n,2}$ is defined by:

$$\mu(L) = \frac{1}{\sum_{\{i,j\}\in\binom{n}{2}} |P^{ij}(L))|^2} \sum_{\{i,j\}\in\binom{n}{2}} |P^{ij}(L)|^2 \Lambda_{ij}$$

- $\mathrm{Im}(\mu) = \mathrm{convhull}(\Lambda_I, I \in \binom{n}{2}) = \Delta_{n,2}$ – hypersimplex
- $\Delta_{n,2} = \{\mathbf{x} \in \mathbb{R}^n | 0 \leq x_i \leq 1,\ i = 1, \ldots, n,\ \ x_1 + \ldots + x_n = 2\}$

The moment map $\tilde{\mu} : \mathbb{C}P^N \to \Delta_{n,2}$ for $T^n \curvearrowright \mathbb{C}P^N$ is given by

$$\tilde{\mu}(\mathbf{z}) = \frac{1}{\sum\limits_{i=0}^{N} |z_i|^2} \sum_{i=0}^{N} |z_i|^2 |\Lambda_I,$$

where $\mathbf{z} = (z_0 : \ldots : z_N)$.

The sets $\{0, \ldots, N\}$ and $\{I, I \in \binom{n}{2}\}$ are related by: to the number i, it corresponds the pair kl at the i-th place in the lexicographical order, that is $(0, 1, \ldots, N) = (12, 13, \ldots, n-1n)$.

It holds

$$\mu = \tilde{\mu} \circ p.$$

Stratification for $\mathbb{C}P^N$ (B-T, 2019)

$(\mathbb{C}^*)^n$-action on $\mathbb{C}P^N$ defines its stratification $\mathbb{C}P^N = \cup W_\sigma$:

- for any $\sigma \subset \{0, \dots, N\}$,
  the set $W_\sigma = \{\mathbf{z} \in \mathbb{C}P^N | z_i \neq 0 \text{ iff } i \in \sigma\}$ is a stratum
- All points from W_σ have the same stabilizer T_σ
- The torus $T^\sigma = T^n / T_\sigma$ acts freely on W_σ
- $\tilde{\mu}(W_\sigma) = \overset{\circ}{P}_\sigma =$ convexhull(Λ_{kl}), where $kl \leftrightarrow i \in \sigma$
- The set $\{P_\sigma\}$ consists of all polytopes that are convex hulls over subsets of vertices for $\Delta_{n,2}$.

Theorem (B-T, 2019)

It holds:

- $dimP_\sigma = dimT^\sigma$.
- $rank\ d\tilde{\mu}_{\mathbf{z}} = dimP_\sigma$, *where* $\mathbf{z} \in W_\sigma$.

Corollary

$\mathbf{x} \in \Delta_{n,2}$ is a regular value of $\tilde{\mu} : \mathbb{C}P^N \to \Delta_{n,2}$ iff the stabilizer of a point from $\tilde{\mu}^{-1}(\mathbf{x}) \subset \mathbb{C}P^N$ for the given T^n-action is $S^1 = \mathrm{diag}(T^n)$.

$\{P_\sigma\}$ define the chamber decomposition $\{C_\omega\}$ of $\Delta_{n,2}$, by all possible non-empty intersections of P_σ.

The regular values of $\tilde{\mu}$ are the points $\mathbf{x} \in C_\omega$, $\dim C_\omega = (n-1)$ and

$$M_{\mathbf{x}}^{2N-n+1} = \tilde{\mu}^{-1}(\mathbf{x}) \subset \mathbb{C}P^N$$

is a smooth $(2N-n+1)$-dimensional manifold

Stratification for $G_{n,2}$

$(\mathbb{C}^*)^n$-action on $G_{n,2}$ defines its stratification $G_{n,2} = \cup W_\sigma$:

- a stratum is an non-empty set $W_\sigma = \{L \in G_{n,2} | P^{ij}(L) \neq 0 \text{ iff } \{i,j\} \in \sigma\}$, where $\sigma \subset \{\binom{n}{2}\}$
- All points from W_σ have the same stabilizer T_σ
- The torus $T^\sigma = T^n / T_\sigma$ acts freely on W_σ
- $\mu(W_\sigma) = \overset{\circ}{P}_\sigma =$ convexhull(Λ_{ij}), $\{i,j\} \in \sigma$

Theorem (B-T, 2019)

It holds:

- $dim P_\sigma = dim T^\sigma$.
- $rank\ d\mu_L = dim P_\sigma$, *where* $L \in W_\sigma$.

Corollary

$\mathbf{x} \in \Delta_{n,2}$ is a regular value of $\mu : G_{n,2} \to \Delta_{n,2}$ iff the stabilizer of a point from $\mu^{-1}(\mathbf{x}) \subset G_{n,2}$ for the canonical T^n-action is $S^1 = \text{diag}(T^n)$.

$\{P_\sigma\}$ define the chamber decomposition $\{C_\omega\}$ of $\Delta_{n,2}$, by all possible non-empty intersections of P_σ.

The regular values of μ are the points $\mathbf{x} \in C_\omega$, $\dim C_\omega = (n-1)$ and

$$M_{\mathbf{x}}^{3n-7} = \mu^{-1}(\mathbf{x}) \subset G_{n,2}, \quad M_{\mathbf{x}}^{3n-7} = M_{\mathbf{x}}^{2N-n-1} \cap p(G_{n,2})$$

for the Plücker embedding $p : G_{n,2} \to \mathbb{C}P^N$.

The chamber decomposition for $\Delta_{n,2}$ defined by T^n-action $G_{n,2}$ can be described by the hyperplane arrangement in $\mathbb{R}^n$:

- $\mathcal{A} = \{x_{i_1} + \ldots + x_{i_s} = 1, 1 \leq i_1 < \ldots < i_s \leq n,\ 2 \leq s \leq [\frac{n}{2}]\}$
- $\mathcal{L}(\mathcal{A})$ -lattice of $\mathcal{A}$
- $\hat{\mathcal{A}} = \mathcal{L}(\mathcal{A}) \cap \Delta_{n,2}$

B-T, 2021

The lattice $\hat{\mathcal{A}}$ coincides with the chamber decomposition of $\Delta_{n,2}$ defined by the admissible polytopes for $G_{n,2}$.

Remark

The admissible polytopes and the chamber decompositions of $\Delta_{n,2}$ for T^n-actions on $G_{n,2}$ and $\mathbb{C}P^N$ do not coincide for $n \geq 5$.

Symplectic reduction

- Let (M, ω) be a symplectic manifold
- T acts on M preserving the symplectic form ω.
- For $v \in \mathbf{t}$, by X_v is denoted the corresponding T-invariant vector field.
- T - action is said to be Hamiltonian if one form $\omega(X_v, \cdot)$ is exact.
- There exists a Hamiltonian H_v, such that

$$\omega(X_v, Y) = dH_v(Y)$$

for any vector field Y on M.

For a fixed basis $\{e_i\}$ for $\mathbf{t}$ and the corresponding Hamiltonians H_{e_i}, it is defined the moment map

$$\mu : M \to \mathbf{t}^*, \ \ \mu(x)(e_i) = H_{e_i}(x).$$

Theorem (Atiyah, Guillemin-Sternberg)

The image $\mu(M)$ is a convex polytope in $\mathbf{t}^*$ for a compact M.

Let μ is proper, let $\mathbf{x}$ be a regular value of μ and T acts freely on $\mu^{-1}(\mathbf{x})$.

Marsden-Weinstein

The quotient manifold $\mu^{-1}(\mathbf{x})/T$ is endowed with a unique symplectic form $\hat{\omega}$ such that

$$p^*\hat{\omega} = i^*\omega,$$

$i : \mu^{-1}(\mathbf{x}) \to M$ - inclusion and $p : \mu^{-1}(\mathbf{x}) \to \mu^{-1}(\mathbf{x})/T$ - projection.

- Symplectic manifold $(\mu^{-1}(\mathbf{x})/T, \hat{\omega})$ is referred to as the symplectic reduction or symplectic quotient of M by T-action.
- Symplectic quotient in general depends on a regular value $\mathbf{x}$.

Symplectic manifolds $G_{n,2}$ and $\mathbb{C}P^N$

- $G_{n,2} = U(n)/(U(2) \times U(n-2))$ admits Kähler metric invariant for the canonical $U(n)$-action, thus $U(n)$-invariant symplectic form.
- The moment map μ comes from this symplectic form.
- The Plücker embedding $p : G_{n,2} \to \mathbb{C}P^N$ is isometric for the Kähler metric on $G_{n,2}$ and Fubini-Study metric on $\mathbb{C}P^N$.

Related problems

For a regular value $\mathbf{x} \in \Delta_{n,2}$ describe:

- symplectic reductions $M_{\mathbf{x}}^{2N-n-1}/T^{n-1}$ and $M_{\mathbf{x}}^{3n-7}/T^{n-1}$ for $T^n \curvearrowright \mathbb{C}P^N$ and $T^n \curvearrowright G_{n,2}$.
- smooth manifolds

$$M_{\mathbf{x}}^{2N-n-1} \subset \mathbb{C}P^N \text{ and } M_{\mathbf{x}}^{3n-7} = M_{\mathbf{x}}^{2N-n-1} \cap p(G_{n,2}) \subset G_{n,2}$$

which produce symplectic reduction.

For $n = 4$ and $N = 5$ it holds:

B-T, 2016

A symplectic reduction on $\mathbb{C}P^5$ and $G_{4,2}$ does not depend on a regular value $\mathbf{x} \in \Delta_{n,2}$:

$$M^7_{\mathbf{x}}/T^3 \cong \mathbb{C}P^2, \quad M^5_{\mathbf{x}}/T^3 \cong \mathbb{C}P^1.$$

Theorem (B-T, 2025)

Smooth manifolds $M^7_{\mathbf{x}}$ and $M^5_{\mathbf{x}}$ do not depend on a regular value $\mathbf{x} \in \Delta_{n,2}$:

$$M^7_{\mathbf{x}} \cong S^5 \times T^2 \subset \mathbb{C}P^5$$

$$M^5_{\mathbf{x}} \cong S^3 \times T^2 \subset G_{4,2}$$

Cohomology of a symplectic reduction

The T - equivariant cohomology of M is defined by

$$H_T^*(M) = H^*(M_T), \;\; M_T = ET \times_T M = (ET \times M)/T,$$

- $ET \to BT$ is the universal bundle for T,
- T acts freely $ET \times M$ by the diagonal action.

- T acts freely on $\mu^{-1}(\xi)$, which implies:

$$ET \times_T \mu^{-1}(\xi) \cong ET \times (\mu^{-1}(\xi)/T) \approx \mu^{-1}(\xi)/T.$$

- The inclusion $\mu^{-1}(\xi) \to M$ induces inclusion $\mu^{-1}(\xi)/T \to M_T$, which defines homomorphism

$$\mathfrak{k}_\xi : H_T^*(M) \to H^*(\mu^{-1}(\xi)/T).$$

By generalizing the techniques from Morse-Bott theory F. Kirwan proved that the homomorphism $\mathfrak{k}_\xi$ is an epimorphism.

On R. Goldin's results

R. Goldins described the kernel of the Kirwan's map for $Fl_n(\mathbb{C})$ and $G_{n,k}(\mathbb{C})$, that is the cohomology ring with complex coefficients of their symplectic reductions by the canonical action of T^n.

- $\mathcal{H}$ - $(n \times n)$ complex Hermitian matrices, $U(n) \curvearrowright \mathcal{H}$ by

$$A \cdot H = AHA^{-1}$$

- Orbits $\mathcal{O}_\lambda$ - manifolds with given spectrum $\lambda = (\lambda_1, \dots, \lambda_n)$- real
-
  1. If $\lambda_i \neq \lambda_j$, $i \neq j$ (generic case) then

$$\mathcal{O}_\lambda \cong_{\text{diff}} Fl_n(\mathbb{C})$$

  2. Otherwise, the stabilizer for $\mathcal{O}_\lambda$ is $U(k_1) \times \cdots \times U(k_l)$, where $k_1, \dots, k_l$ are the number of λ_i's equal to each other and $k_1 + \ldots + k_l = n$

$$\mathcal{O}_\lambda \cong_{\text{diff}} Fl_{n,k_1,\dots,k_n} = U(n)/(U(k_1) \times \dots U(k_l))$$

- In particular, for $l = 2$, that is $\lambda = (\lambda_1^{\times k}, \lambda_2^{\times (n-k)})$

$$\mathcal{O}_\lambda \cong_{\text{diff}} G_{n,k}(\mathbb{C}) \ \ (\text{notation } G_{n,k}(\lambda))$$

- $\mathcal{O}_\lambda$ are symplectic manifolds with the standard Kirilov-Konstant-Souriau symplectic form ω_λ
- $\mu_\lambda : \mathcal{O}_\lambda \to \mathbb{R}^n$ - assigns to a Herimitian matrix its diagonal entries, is a moment map defined by T^n-action and ω_λ

In this way there are exhibited a lot moment maps on $G_{n,k}$, one for each pair of distinct real numbers.

Theorem

Let $\mathcal{O}_\lambda$ be a generic orbit of $U(n)$. The cohomology of $\mathcal{O}_\lambda // T(\xi)$ is isomorphic to the ring

$$\frac{\mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]}{\langle \sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n), \sum_{i=1}^{n} u_i, \partial_{\nu^{-1}} \Delta(x, u_\tau) \rangle}, \tag{1}$$

for all ν and τ from S_n such that $\sum_{i=k+1}^{n} \lambda_{\nu(i)} < \sum_{i=k+1}^{n} \xi_{\tau(i)}$ for some $k = 1, \dots, n-1$ and $\deg x_i = \deg u_i = 2$.

Theorem

The cohomology of $G_{n,k}(\lambda)//T(\xi)$ *is isomorphic to the ring*

$$\frac{\mathbb{C}[\sigma_i(x_1,\ldots,x_k),\sigma_i(x_{k+1},\ldots,x_n),u_1,\ldots,u_n]}{\langle\sigma_i(x_1,\ldots,x_n)-\sigma_i(u_1,\ldots,u_n),\sum_{i=1}^{n}u_i,\partial_{\nu^{-1}}\Delta(x,u_\tau)\rangle}, \qquad (2)$$

for all ν *and* τ *from* S_n *such that* $\sum_{i=k+1}^{n}\lambda_{\nu(i)}<\sum_{i=k+1}^{n}\xi_{\tau(i)}$ *for some* k*, and* $\partial_{\nu^{-1}}\Delta(x,u_\tau)$ *is symmetric in* $x_1,\ldots,x_k$ *and* $x_{k+1},\ldots,x_n$.

Here : ξ is a regular value for μ, $\Delta(x,u)=\prod_{i<j}(x_i-u_j)$ and ∂_ω is divided difference operator.

Theorem

The T-equivariant cohomology of $Fl_n(\mathbb{C})$ and $G_{n,k}(\mathbb{C})$ is

$$H_T^*(Fl_n) = \frac{\mathbb{C}[x_1,\ldots,x_n,u_1,\ldots u_n]}{\langle \sigma_i(x_1,\ldots,x_n)-\sigma_i(u_1,\ldots,u_n),\sum_{i=1}^n u_i\rangle},$$

$$H_T^*(G_{n,k}) = \frac{\mathbb{C}[\sigma_i(x_1,\ldots,x_k),\sigma_i(x_{k+1},\ldots,x_n),u_1,\ldots u_n]}{\langle \sigma_i(x_1,\ldots,x_n)-\sigma_i(u_1,\ldots,u_n),\sum_{i=1}^n u_i\rangle},$$

where $\deg x_i = \deg u_i = 2$ *and* σ_i *is the i-th symmetric function,* $1 \leq i \leq n$.

$T = T^{n-1}$ torus in $SU(n)$ that acts effectively on Fl_n and $G_{n,k}$

The kernel of the Kirwan map $\mathfrak{k}_\xi$ is given by $\partial_{\nu^{-1}}\Delta(x,u_\tau)$ for appropriate choices of $\nu,\tau \in S_n$.

The case of $G_{n,2}$

Find $\lambda \in \mathcal{H}$ such that $\mathcal{O}_\lambda \cong_{\text{diff}} G_{n,2}$ and $\mu_\lambda = \mu$.

- $G_{n,2}$ can be identified with orthogonal projection operators $P : \mathbb{C}^n \to \mathbb{C}^n$, whose images are 2-dimensional subspaces:

$$P = (P^*)^T, \;\; \text{rank}(P) = \text{trace}(P) = 2$$

- If represent $L \in G_{n,2}$ by $(2 \times n)$ -matrix A_L it holds

$$P_L = A_L^{\#}(A_L A_L^{\#})^{-1} A_L,$$

where $A^{\#} = (A^*)^T$.

Let $P_L = (p_{ij}(L))$ is a $(n \times n)$-orthogonal projection matrix for L and $P^{ij}(L)$, $i < j$, are the Plücker coordinates for L

Lemma

The entries $p_{ij}(L)$ and $P^{ij}(L)$ are related by

$$p_{ij}(L) = \frac{\sum_{k=1}^{n} P^{ik}(L)P^{jk}(L)}{\sum_{I \in \{\binom{n}{2}\}} |P^I(L)|^2}.$$

Proposition

The moment map μ_λ for $\lambda = (1, 1, 0, \ldots, 0)$ on $\mathcal{O}_\lambda \cong G_{n,2}$ coincides with the standard moment map on $G_{n,2}$ given by the Plücker embedding $G_{n,2} \to \mathbb{C}P^{N-1}$, $N = \binom{n}{2}$.

For an arbitrary $\lambda = (\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_2)$ this implies:

Corollary

1. The moment map μ_λ on $\mathcal{O}_\lambda \cong G_{n,2}$ is given by

$$\mu_\lambda(L) = (\lambda_1 - \lambda_2)\mu(L) + (\lambda_2, \ldots, \lambda_2).$$

2. The image of μ_λ is a polytope $\Delta^\lambda_{n,2}$ - the convex hull of the vertices $\Lambda^\lambda_{ij} \in \mathbb{R}^n$ such that $\Lambda^\lambda_{ij}(i) = \Lambda^\lambda_{ij}(j) = \lambda_1$, while $\Lambda^\lambda_{ij}(k) = \lambda_2$ for $k \neq i, j$.

Cohomology of symplectic reduction for $G_{n,2}$

Theorem

Let ξ be a regular value of the standard moment map $\mu : G_{n,2} \to \mathbb{R}^n$. The cohomology ring $H^*(\mu^{-1}(\xi)/T)$ is the quotient of the ring of $H^*_T(G_{n,2})$ by the relations: $\partial_{\nu^{-1}}\Delta(x, u_\tau)$ for all $\nu, \tau \in S_n$ such that:

1. $\nu(n) \in \{3, \dots, n\}$ *and* τ *is arbitrary,*
2. $\{\nu(1), \nu(n)\} = \{1, 2\}$ *and* τ *is arbitrary,*
3. $\nu(n) \in \{1, 2\}$, $\nu(j) \in \{1, 2\}$ *for some* $2 \leq j \leq n-2$ *and* τ *is such that* $\xi_{\tau(j+1)} + \dots + \xi_{\tau(n)} > 1$.

Recall:

- $\xi \in \Delta_{n,2}$ is a regular value of μ iff ξ belongs to a chamber of a maximal dimension $(n-1)$ defined by the hyperplane $\hat{\mathcal{A}}$.
- All $\mu^{-1}(\xi)$ for ξ being from the same chamber C_ω are diffeomorphic.

Corollary

$H^*(\mu^{-1}(\xi_1)/T) = H^*(\mu^{-1}(\xi_2)/T)$ for any $\xi_1, \xi_2 \in C_\omega$.

S_n-action on $\mathbb{R}^n$ induces S_n-action on the set of all chambers in $\Delta_{n,2}$:

Corollary

If C_{ω_1} and C_{ω_2} of maximal dimension belong to the same S_n-orbit, then

$$H^*(\mu^{-1}(\xi_1)/T) \cong H^*(\mu^{-1}(\xi_2)/T),$$

for $\xi_1 \in C_{\omega_1}$ and $\xi_2 \in C_{\omega_2}$.

We prove: $H^*(\mu^{-1}(\xi)/T)$ is completely determined by the supporting hyperplanes of the walls of C_ω such that $\xi \in C_\omega$. These hyperplanes are from the arrangement $\mathcal{A}$.

Theorem

Let ξ be a regular value of μ and $\xi \in C_\omega$. Let $v_s = (i_{s_1}, \ldots, i_{s_{j_s}})$, $2 \leq j_2 \leq n-2$ are the normal vectors of the supporting hyperplanes of the walls of C_ω such that $C_\omega : \cap\{x_{i_{s_1}} + \ldots + x_{i_{s_{j_s}}} > 1\}$. The cohomology ring $H^*(\mu^{-1}(\xi)/T)$ is determined by $\nu, \tau \in S_n$ such that:

- *$\nu(n) \in \{3, \ldots, n\}$ and τ is arbitrary,*
- *$\{\nu(1), \nu(n)\} = \{1, 2\}$ and τ is arbitrary,*
- *$\nu(n) \in \{1, 2\}$, $\nu(j) \in \{1, 2\}$ for some $2 \leq j \leq n-2$ and τ is such that $\{i_{s_1}, \ldots, i_{s_{j_s}}\} \subset \{\tau(j+1), \ldots \tau(n)\}$ for some s.*

Compactifications of moduli space $\mathcal{M}_{0,n}$

- $\mathcal{M}_g$ - moduli space of smooth genus g curves, $\mathcal{M}_{g,n}$ - moduli space of smooth genus g curves with n marked distinct ordered points.
- Deligne and Mumford in 1969 obtained the compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$, $g \geq 2$. The main idea behind this compactification is to enlarge the notion of a smooth, proper curve, to allow nodal singularities. Such curves are called stable.
- Knudsen in 1983 worked out an extension $\overline{\mathcal{M}}_{g,n}$ which allows Deligne and Mumford construction to work for the spaces $\mathcal{M}_{g,n}$, $g \geq 0$, $2g - 2 + n > 0$, in particular for genus zero curves and low-genus cases.

- In 1992 Keel provided alternative construction of $\overline{\mathcal{M}}_{0,n}$ to that of Deligne-Mumford-Knudsen. The variety $\overline{\mathcal{M}}_{0,n}$ he described as iterated blow up of $(\mathbb{C}P^1)^{n-3}$.
- In 1993 Kapranov defined Chow quotient $G_{n,k}//(\mathbb{C}^*)^n$ as a compactification of $W_{n,k}/(\mathbb{C}^*)^n$ for the main stratum $W_{n,k}$ in $G_{n,k}$. He proved:
  1. An isomorphism between $G_{n,2}//(\mathbb{C}^*)^n$ and $\overline{\mathcal{M}}_{0,n}$ can be established.
  2. The variety $\overline{\mathcal{M}}_{0,n}$ can be obtained from $\mathbb{C}P^{n-3}$ as iterated blowups.

Many different problems led to many different compactification of $\mathcal{M}_{g,n}$.

- Losev and Manin in 2000 introduced the spaces $\overline{L}_{g,S}$ of genus g curves with marked points parametrized by S and painted by black and white, which satisfy certain stability conditions.
- Hasset in 2003 introduced a family of compactifications given by stable pointed weighted curves $\overline{\mathcal{M}}_{g,\mathcal{A}(S)}$.
- 1. Manin in 2004 showed that the spaces $\overline{L}_{g,S}$ are special case of Hassett's compactification.
  2. In particular, the compactifications $\overline{L}_{0,n,2}$ with exactly two white points can be exhibited as toric varieties over permutahedron.

On $(2n, k)$-manifolds

- In 2014 we introduced the class $(2n, k)$-manifold:
  1. given by smooth manifolds M^{2n} equipped with a smooth T^k-action, $k \leq n$,
  2. generalize properties of manifolds with torus action, both with zero and positive complexity.
- In 2019 we worked out in detail the concept and axioms of $(2n, k)$-manifolds:
  1. It was introduced the notion of universal space of parameters $\mathcal{F}_n$ as a compactification of the space of parameters F_n of the main stratum W_n for M^{2n}.
  2. This was done for purpose of describing the topology of the orbit space of a $(2n, k)$-manifold.
- It 2023 we presented a new explicit, constructive geometrical method for the construction of $\mathcal{F}_n$ for $G_{n,2}$.
  1. The method comes out our work on description of the orbit space $G_{n,2}/T^n$ nad it is essentially used.
  2. The space $\mathcal{F}_n$ is proved to coincide with $\overline{\mathcal{M}}_{0,n}$.

We recall:

- $F_\omega = \mu^{-1}(\xi)/T^n$ for $\xi \in C_\omega$, the spaces of parameters of the chambers, are core ingredients in construction of a model for $G_{n,2}/T^n$.
- For $\dim C_\omega = n-1$, the spaces F_ω are smooth manifolds with a symplectic structure obtained by symplectic reduction.

B-T, 2024

Such F_ω are isomorphic to Hassett spaces $\mathcal{M}_{0,\mathcal{A}}$ of weighted stable genus zero curves for an appropriate $\mathcal{A}$' s.

B-T introduced in 2024 the Hassett category $\overset{\circ}{\mathcal{H}}_{0,n}$:

1. objects are those Hassett spaces $\mathcal{M}_{0,\mathcal{A}}$, which are isomorphic to the spaces of parameters F_ω,
2. initial object $\mathcal{M}_{0,\mathcal{A}_0} \cong \overline{\mathcal{M}}_{0,n}$, where $\mathcal{A}_0 = (1, \dots, 1)$,
3. morphisms are given by the reduction morphisms $\rho_{\mathcal{A}_0,\mathcal{A}} : \mathcal{M}_{0,\mathcal{A}_0} \to \mathcal{M}_{0,\mathcal{A}}$.

Theorem (B-T, 2025)

This category is distinguished by the property that it can be topologically modeled on $G_{n,2}/T^n$.

The appropriate weights $\mathcal{A}$ are given as follows:

- $\mathcal{D}_{0,n} = \{(a_1, \ldots, a_n) | 0 < a_j \leq 1,\ a_1 + \ldots + a_n > 2\}$ - the domain of weights.
- The coarse chamber decomposition of $\mathcal{D}_{0,n}$ is defined by the hyperplane arrangement

$$\mathcal{W}_c = \{\sum_{j \in S} x_j = 1 | S \subset \{1, \ldots, n\}, 2 < |S| < n-2\},$$

  the corresponding chambers we denote by w_c.
- The boundary of $\mathcal{D}_{0,n}$ is defined by

$$\partial \mathcal{D}_{0,n} = \{(a_1, \ldots, a_n) | a_1 + \ldots + a_n = 2,\ 0 < a_i < 1,\ i = 1, \ldots, n\}.$$

A weight $\mathcal{A}$ is appropriate if :

$$\mathcal{A} \in w_c \subset \mathcal{D}_{0,n} \text{ such that } \dim(w_c \cap \partial \mathcal{D}_{0,n}) = n - 1.$$

Main effective theorem

Theorem

The cohomology ring of a space $\mathcal{M}_{0,\mathcal{A}}$, $\mathcal{A} \neq \mathcal{A}_0$ *from the Hassett category* $\overset{\circ}{\mathcal{H}}_0$ *is given by*

$$H^*(\mathcal{M}_{0,\mathcal{A}}) = \frac{\mathbb{C}[\sigma_i(x_1, x_2), \sigma_i(x_3, \dots, x_n), u_1, \dots, u_n]}{\langle \sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n), \sum_{i=1}^{n} u_i, \partial_{\nu^{-1}} \Delta(x, u_\tau) \rangle},$$

for all ν *and* τ *such that:*

- $\nu(n) \in \{3, \dots, n\}$ *and* τ *is arbitrary,*
- $\{\nu(1), \nu(n)\} = \{1, 2\}$ *and* τ *is arbitrary,*
- $\nu(n) \in \{1, 2\}$, $\nu(j) \in \{1, 2\}$ *for some* $2 \leq j \leq n-2$ *and* τ *is such that* $\{i_{s_1}, \dots, i_{s_{j_s}}\} \subset \{\tau(j+1), \dots \tau(n)\}$ *for some* s,

where $\{i_{s_1}, \dots, i_{s_{j_s}}\}$ *are the normal vectors of the supporting hyperplanes of the walls of the chamber* $C_\omega \subset \Delta_{n,2}$ *such that* $C_\omega \cap w_c \neq \emptyset$ *for the chamber* $w_c \subset \mathcal{D}_{0,n}$ *such that* $\mathcal{A} \in w_c$.

Computation for $n = 4$

Corollary

$$H^*(\mu^{-1}(\xi)/T^4) \cong \frac{\mathbb{C}[\sigma_i(x_1, x_2), \sigma_i(x_3, x_4), u_1, \dots, u_4]}{\langle \sigma_i(x_1, \dots, x_4) - \sigma_i(u_1, \dots, u_4), \sum_{i=1}^{4} u_i, \partial_{\nu^{-1}} \Delta(x, \tau) \rangle}$$

for $\nu, \tau \in S_4$ such that

- $\nu(4) \in \{3, 4\}$ and τ is arbitrary,
- $\{\nu(1), \nu(4)\} = \{1, 2\}$ and τ is arbitrary,
- $\{\nu(2), \nu(4)\} = \{1, 2\}$ and τ is such that $i \in \{\tau(3), \tau(4)\}$ for some fixed $1 \leq i \leq 4$.

Corollary

The cohomology ring of $\mu^{-1}(\xi)/T^4 \subset G_{4,2}/T^4$ is given by

$$H^*(\mu^{-1}(\xi)/T^4) \cong \mathbb{C}[x]/\langle x^2 \rangle.$$

Computation for $n = 5$

Note (B-T, 2025):

- $\xi = (\frac{2}{5}, \dots, \frac{2}{5}) \in \Delta_{5,2}$ is a regular value for μ,
- $\xi \in C_\omega = \cap_{1 \leq i < j \leq 5}\{x_i + x_j < 1\} = \cap_{1 \leq i < j < k \leq 5}\{x_i + x_j + x_k > 1\}$,
- $\mu^{-1}(\xi)/T^5$ is isomorphic to $\mathcal{F}_5 \cong \overline{\mathcal{M}}_{0,5}$.

Corollary

$$H^*(\overline{\mathcal{M}}_{0,5}) \cong \frac{\mathbb{C}[\sigma_i(x_1, x_2), \sigma_i(x_3, x_4, x_5), u_1, \ldots, u_5]}{\langle \sigma_i(x_1, \ldots, x_5) - \sigma_i(u_1, \ldots, u_5), \sum_{i-1}^{5} u_i, \partial_{\nu^{-1}}\Delta(x, u_\tau)\rangle},$$

for $\nu, \tau \in S_5$ such that

- $\nu(5) \in \{3, 4, 5\}$ and τ is arbitrary,
- $\{\nu(1), \nu(5)\} = \{1, 2\}$ and τ is arbitrary,
- $\{\nu(2), \nu(5)\} = \{1, 2\}$ and τ is arbitrary.

Proposition

$$H^*(\overline{\mathcal{M}}_{0,5}) \cong \frac{\mathbb{C}[x, u_1, \ldots, u_5]}{\langle \sum_{i=1}^{5} u_i, u_i u_j - x^2, u_l^2 + 4x^2, u_i x\rangle}.$$

Note (B-T, 2025):

- $\xi = (\frac{2}{3}, \frac{2}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}) \in \Delta_{5,2}$ is a regualr value of μ,
- $\xi \in C_\omega = \{x_1 + x_2 > 1\} \cap) \cap_{1 \leq i < j \leq 5, \{i,j\} \neq \{1,2\}} \{x_i + x_j < 1\}$,
- $\mu^{-1}(\xi)/T$ is isomorphic to the Losev-Manin space $\bar{L}_{0,5}$,
- $\bar{L}_{0,5}$ is isomorphic to the toric manifold over the hexagon.

Proposition

$$H^*(\bar{L}_{0,5}) \cong \frac{\mathbb{C}[u_2, u_3, u_4, u_5]}{\langle \text{relations} \rangle},$$

where the relations are:

$$u_2 u_3 = u_2 u_4 = u_2 u_5, \; u_3 u_4 = u_3 u_5 = u_4 u_5 = -9 u_2 u_3,$$

$$u_2^2 = 11 u_2 u_3, \; u_3^2 = u_4^2 = u_5^2 = 16 u_2 u_3.$$

Comparison with the Chow ring of $A^*(\overline{\mathcal{M}}_{0,5})$

- $A^*(\overline{\mathcal{M}}_{0,n})$ was computed by Keel (1992) and then by Tavakol (2017),
- The generators - the divisors in a compactification of $\mathcal{M}_{0,n}$ to $\overline{\mathcal{M}}_{0,n}$.
- It coincides with cohomology ring $H^*(\overline{\mathcal{M}}_{0,n})$.

Corollary

$H^*(\overline{\mathcal{M}}_{0,5})$ has five 2-degree generators $D^{12}, D^{13}, D^{14}, D^{15}, D^{23}$ and

$$D^{1i}D^{1j} = 0,\ i \neq j,\ D^{12}D^{23} = D^{13}D^{23} = 0,$$

$$(D^{ij})^2 = -D^{14}D^{23} = -D^{15}D^{23}.$$

Lemma

The relations between the two generators for $H^*(\overline{\mathcal{M}}_{0,5})$ are

$$D^{12} = u_2 - u_3,\ D^{13} = u_4 - u_5,\ D^{14} = \sqrt{5}(2x + u_2 + u_3),$$

$$D^{15} = \sqrt{5}(-2x + u_4 + u_5),\ D^{23} = \frac{1}{\sqrt{5}}(a(u_2 + u_3) + b(u_4 + u_5) + cx),$$

where $b = -a - 10,\ c = 5a + 25$ and $a^2 + 10a + 15 = 0$.

Comparison with the Chow ring $A^*(\mathcal{M}_{0,\mathcal{A}})$

- $\mathcal{M}_{0,\mathcal{A}}$ for $\mathcal{A} = (\underbrace{1,\dots,1}_{m},\underbrace{\varepsilon,\dots,\varepsilon}_{n-m})$, $m \geq 2$ and $\varepsilon < \frac{1}{n-m}$, - heavy/light spaces. $A^*(\mathcal{M}_{0,\mathcal{A}})$ is explicitly described (Kannan, Karp, Li, 2021)
- The Losev-Manin space $\bar{L}_{0,5} = \mathcal{M}_{0,\mathcal{A}}$ for $\mathcal{A} = (1, 1, a_3, a_4, a_5)$, $a_i < \frac{1}{3}$.

Corollary

$H^*(\bar{L}_{0,5})$ has four 2-degree generators $D^{23}, D^{24}, D^{25}, D^{234}$ and

$$D^{23}D^{24} = D^{23}D^{25} = D^{24}D^{25} = D^{25}D^{234} = 0,$$

$$(D^{2i})^2 = (D^{234})^2 = -D^{23}D^{234} = -D^{24}D^{234}.$$

Lemma

The relation between the two generators for $H^*(\bar{L}_{0,5})$ are

$$D^{23} = u_2 - u_3,\ D^{24} = u_2 - u_4,\ D^{25} = u_2 - u_5,\ D^{234} = -u_2 - u_3 - u_4 - 2u_5.$$

Comparison with a toric manifold X

- $$A^*(X) \cong H^*(X) = \cong \mathbb{Z}[v_1, \ldots, v_n]/\langle I + J \rangle$$
- $\bar{L}_{0,5}$ is a toric variety over hexagon.

Corollary

$H^*(\bar{L}_{0,5})$ is generated by 2-degree classes $v_1, \ldots, v_6$ and

$$v_1 + v_3 + v_4 + v_6 = 0, \quad v_1 + v_2 + v_4 + v_5 = 0,$$

$$v_1 v_3 = v_1 v_4 = v_1 v_5 = v_2 v_4 = v_2 v_5 = v_2 v_6 = v_3 v_5 = v_3 v_6 = v_4 v_6 = 0.$$

Lemma

The relation between the two generators for $H^*(\bar{L}_{0,5})$ are

$$v_1 = u_2 - u_3, \; v_3 = u_2 - u_4, \; v_5 = u_2 - u_5, \; v_2 = -u_2 - u_3 - u_4 - 2u_5,$$

$$v_4 = -u_2 - 2u_3 - u_4 - u_5, \; v_6 = -u_2 - u_3 - 2u_4 - u_5.$$

Recall:

- $\overline{\mathcal{M}}_{0,5}$ is a blowop of

$$\bar{L}_{0,5} \subset (\mathbb{C}P^1)^3 : c_{ij}c_{ik}^{'}c_{jk} = c_{ij}^{'}c_{ik}c_{jk}^{'}$$

  at one point $((1:1),(1:1),(1:1))$.

- It is the same as doing blow up of $\mathbb{C}P^2$ at four points $(1:0:0:),(0:1:0),(0:0:1),(1:1:1)$:

$$\overline{\mathcal{M}}_{0,5} \cong \mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$$

- $\bar{L}_{0,5}$ – toric over hexagon - blowing up - X_7 toric over heptagon.

$$\overline{\mathcal{M}}_{0,5} \cong_{\text{diff}} dP_5 \cong_{\text{diff}} X_7.$$

This does not give a toric structure on $\overline{\mathcal{M}}_{0,5}$

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