Resistance distance and its generalizations

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Electrical networks

Definition

An electrical network $\mathcal{E}$ is a connected graph Γ with a distinguished subset of boundary vertices (nodes) $\partial\Gamma$ and a function on the edges $r : E(\Gamma) \to \mathbb{R}_+$. All vertices that are not boundary vertices are called interior vertices. The value of the function r on an edge ij will be called the electrical resistance of the edge ij and denoted by r_{ij}. The quantity inversely proportional to the resistance will be called the conductance and denoted by $c_{ij} = \frac{1}{r_{ij}}$.

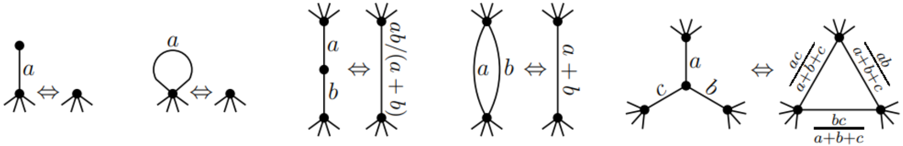
Fix an electrical network with boundary potentials $V = (V_i)$ and boundary currents $I = (I_i)$. Kirchhoff's law says that for each vertex j, $\sum_{i \to j} I_{ij} = I_j$. By Ohm's law, $I_{ij} = \frac{V_i - V_j}{r_{ij}}$. Therefore, the boundary potentials and the outgoing currents are related by a linear map $L_{\mathcal{E}}(\Gamma)V = I$.

Definition

The matrix $L_{\mathcal{E}}(\Gamma)$ is called the *response matrix* of the network.

Definition

Electrical networks will be called equivalent if their response matrices coincide. The response matrix of the equivalence class of the electrical network $\mathcal{E}$ will be denoted by $L(\mathcal{E})$.



Effective Resistance

Definition

Consider a connected graph Γ with a positive function on the edges, and distinguish two vertices $i, j \in V(\Gamma)$. We now regard the graph Γ as an electrical network with the set of boundary vertices $\partial\Gamma = \{i, j\}$. Apply boundary potentials (V_i, V_j) such that the boundary current is equal to $I = (-1, 1)$. Then the effective resistance between i and j is defined as the following quantity:

$$R_{ij} = |V_i - V_j|$$

The matrix $R(\mathcal{E}) = (R_{ij})$ will be called the matrix of effective resistances of the electrical network $\mathcal{E}$.

The effective resistance can be computed using the pseudoinverse of the response matrix $L^{\dagger}(\mathcal{E})$ by the following formula:

$$R_{ij} = L^{\dagger}(\mathcal{E})_{ii} + L^{\dagger}(\mathcal{E})_{jj} - 2L^{\dagger}(\mathcal{E})_{ij}$$

Spanning Trees

Definition

Consider a graph Γ. A subgraph $T \subset \Gamma$ is called a spanning tree if $V(T) = V(\Gamma)$ and T is a tree (a connected graph without cycles). The weight of a tree $w(T)$ is defined as the product of the conductances of all edges of the tree, that is,

$$w(T) = \prod_{e \in E(T)} c(e).$$

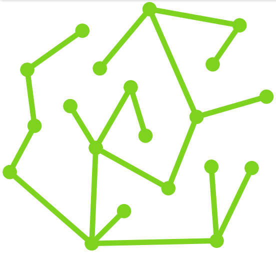
We denote by $W(\text{tree})$ the sum of the weights of all spanning trees of Γ:

$$W(\text{tree}) = \sum_{T \text{ is a spanning tree of } \Gamma} w(T)$$

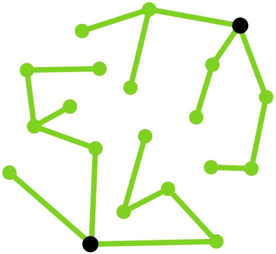
Groves

Definition

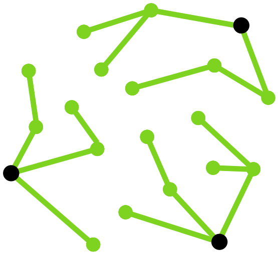
Consider the graph Γ, and let $T_1, \ldots, T_k$ be its subgraphs such that each subgraph T_i is a tree and $V(T_1) \sqcup \ldots \sqcup V(T_k) = V(\Gamma)$. Then the graph $F = T_1 \sqcup \ldots \sqcup T_k$ is called a spanning k-forest of the graph Γ. A spanning forest in which each tree contains at least one boundary vertex will be called a grove. The quantity $w(F) = \prod_{e \in E(F)} c(e)$ is called the weight of the forest (grove) F.



Spanning tree



2-grove



3-grove

Boundary Partitions and Groves

Definition

Let $V(\Gamma) = \{1, 2, \ldots, n\}$, and let $\sigma = \{A_1, \ldots, A_k\}$ be a partition of $V(\Gamma)$, that is, a collection of pairwise disjoint subsets $A_1, \ldots, A_k \subseteq V(\Gamma)$. We write this partition in the form $\sigma = A_1 | \cdots | A_k$. We denote by $W(\sigma)$ the total weight of all groves $F = T_1 \sqcup \cdots \sqcup T_k$ such that $A_i \subseteq V(T_i)$ for every i. Also we define $\Pr(\sigma)$:

$$\Pr(\sigma) = \frac{W(\sigma)}{\text{the total weight of all groves}}, \qquad W(\sigma) = \sum_{\substack{F = T_1 \sqcup \cdots \sqcup T_k \\ \forall i:\, A_i \subseteq V(T_i)}} w(F)$$

.

The partition $\sigma = 12\ldots n$, consisting of a single part, will be called tree, and the partition $\sigma = 1|2|\ldots|n$, consisting of n parts, will be called uncrossing. We also introduce the following notation:

$$\overline{\Pr}(\sigma) = \frac{\Pr(\sigma)}{\Pr(\text{tree})}, \qquad \ddot{\Pr}(\sigma) = \frac{\Pr(\sigma)}{\Pr(\text{uncrossing})}.$$

Grove polynomials

Theorem (Kenyon and Wilson)

For any planar partition σ,

$$\overline{\Pr}(\sigma) = \frac{\Pr(\sigma)}{\Pr(tree)} = \textit{integer-coefficient homogeneous polynomial in the } R_{i,j}/2\text{'s}$$

where the degree is $-1 + \#$*parts of* σ, *and*

$$\ddot{\ddot{\Pr}}(\sigma) = \frac{\Pr(\sigma)}{\Pr(uncrossing)} = \textit{integer-coefficient homogeneous polynomial in the } L_{i,j}\text{'s}$$

where the degree is $n - \#$*parts of* σ.

Polynomials for $n = 4$

$$\overline{\mathrm{Pr}}(1234) = 1$$

$$\overline{\mathrm{Pr}}(2|143) = \frac{1}{2}R_{23} + \frac{1}{2}R_{12} - \frac{1}{2}R_{13}$$

$$\overline{\mathrm{Pr}}(14|23) = \frac{1}{2}R_{13} + \frac{1}{2}R_{24} - \frac{1}{2}R_{14} - \frac{1}{2}R_{2,3}$$

$$\begin{aligned}\overline{\mathrm{Pr}}(1|2|34) =& \frac{1}{4}\big(R_{12}R_{13} + R_{12}R_{23} + R_{12}R_{14} + R_{12}R_{24} \\ & - 2R_{12}R_{34} - R_{12}^2 + R_{13}R_{2,4} + R_{14}R_{23} \\ & - R_{13}R_{14} - R_{23}R_{24}\big)\end{aligned}$$

Kirchhoff's Formula

Theorem (Kirchhoff's Formula)

The effective resistance between i and j can be computed by the following formula:

$$R_{ij} = \overline{\Pr}(i|j) = \frac{\Pr(i|j)}{\Pr(tree)} = \frac{W(i|j)}{W(tree)}$$

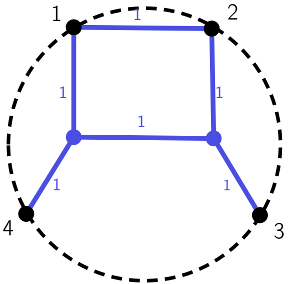
Resistance distance

From the definition of effective resistance, it is clear that $R_{ij} = R_{ji}$ and $R_{ii} = 0$. Moreover, the following theorem holds.

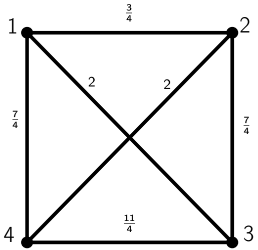
Theorem

For any $i, j, k \in \partial\Gamma$, we have

$$R_{ik} + R_{kj} - R_{ij} \geq 0.$$



Electrical network



Metric space

Split-decomposible metric

Let A be a subset of a set X consisting of n elements. Define the elementary dissimilarity function (split) $\delta(A) : X \times X \to \mathbb{R}$ by

$$\delta(A)_{ij} = \begin{cases} 1, & |A \cap \{i, j\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Definition

A metric d on a finite set X is called decomposable if it can be represented as a linear combination of splits with nonnegative coefficients:

$$d = \sum_{A \subset X} \alpha_A \delta(A), \;\; \alpha_A \geq 0$$

Chain of implications

Theorem

Let d be a metric on the set X. Then the following chain of implications holds:

d — a decomposable metric $d = \sum \alpha_A \delta_A$	$\Rightarrow$	*d — a hypermetric* $\sum b_i = 1 \Rightarrow \sum b_i b_j d_{ij} \leq 0$
		$\Downarrow$
$\left(X, \sqrt{d}\right)$ *is L_2-embeddable*	$\Leftrightarrow$	*d — a metric of negative type* $\sum b_i = 0 \Rightarrow \sum b_i b_j d_{ij} \leq 0$

Metric properties of Effective Resistance

Remark

Effective resistance is a decomposable metric. Moreover, the decomposition into splits is obtained directly from Kirchhoff's formula:

$$R = \sum_{A \subset \partial X} \overline{\Pr}(A|A^c)\delta(A)$$

- $\sqrt{R}$ — L_2-embeddable
- R - a hypermetric
- R - a metric of negative type

Resistant n-metric

Let X be the set of boundary vertices of an electrical network.

Definition

Define a function

$$R_n : X^n \to \mathbb{R}_{\geq 0}$$

by

$$R_n(x_1, \ldots, x_n) = \overline{\mathrm{Pr}}(x_1 | \ldots | x_n).$$

We call R_n the *resistant n-metric*.

- In words, $R_n(x_1, \ldots, x_n)$ is the probability that a random weighted grove separates the vertices $x_1, \ldots, x_n$ into different connected components.
- For $n = 2$, this construction recovers classical effective resistance:

$$R_2(x_1, x_2) = \overline{\mathrm{Pr}}(x_1 | x_2) = R_{x_1 x_2}.$$

What is an n-metric?

Let X be a set. A function
$$d_n : X^n \to \mathbb{R}_{\geq 0}$$
is called an *n-metric* if it satisfies the following axioms:

1. **Symmetry:**
$$d_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = d_n(x_1, \ldots, x_n) \qquad \forall \sigma \in S_n.$$
2. **Point separation:**
$$d_n(x_1, \ldots, x_n) = 0 \iff \exists\, i \neq j : \; x_i = x_j.$$
3. **Simplex inequality:**
$$d_n(x_1, \ldots, x_n) \leq \sum_{k=1}^{n} d_n(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n)$$
for all $x_1, \ldots, x_n, y \in X$.

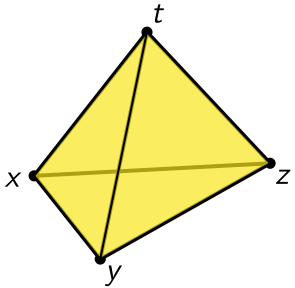
Example of n-metric

Remark

A classical example of an n-metric is the Euclidean $(n-1)$-dimensional volume. In particular, Euclidean area is a 3-metric.
For $n = 3$, the simplex inequality takes the form

$$d(x, y, z) \leq d(t, y, z) + d(x, t, z) + d(x, y, t),$$

which means that in a tetrahedron, the sum of the areas of three faces is greater than or equal to the area of the fourth face.



Resistance n-metric

Theorem

The function R_n is an n-metric.

More precisely, for all $x_1, \dots, x_n, y \in X$, it satisfies:

1. **Symmetry:**
$$R_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = R_n(x_1, \dots, x_n) \qquad \forall \sigma \in S_n.$$
2. **Point separation:**
$$R_n(x_1, \dots, x_n) = 0 \iff \exists\, i \neq j : \ x_i = x_j.$$
3. **Simplex inequality:**
$$R_n(x_1, \dots, x_n) \leq \sum_{k=1}^{n} R_n(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n).$$

Further properties

The resistant n-metric inherits important properties of classical effective resistance.

Theorem

The square root of R_n is L_2-embeddable.

That is, for an electrical network with $N+1$ boundary vertices, there exists a map

$$\Omega : X \to \mathbb{R}^N$$

such that

$$\sqrt{R_{x_1 \dots x_n}} = \mathrm{Vol}\big(\Omega(x_1), \dots, \Omega(x_n)\big).$$

- Thus the new multi-point quantity still admits a Euclidean interpretation.

Decomposability of the resistant n-metric

Definition

An n-metric $d : X^n \to \mathbb{R}$ is *decomposable* if

$$d = \sum_{A_1 \sqcup \cdots \sqcup A_n \subseteq X} \alpha_A \, \delta(A_1 | \cdots | A_n),$$

where the n-split $\delta(A_1 | \cdots | A_n)$ is given by

$$\delta(A_1 | \cdots | A_n)(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } \forall i : \ |A_i \cap \{x_1, \ldots, x_n\}| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The resistant n-metric R_n is decomposable.

From 2-metrics to n-metrics

A natural question is whether an n-metric is related to the classical 2-metric.

- Can we construct an n-metric from a given 2-metric?
- In the Euclidean case, the answer is well known.
- Euclidean distance and Euclidean area are connected by Heron's formula.

Let a triangle with vertices x, y, z have side lengths

$$a = d(x, y), \qquad b = d(y, z), \qquad c = d(x, z).$$

Then its area is

$$S = \sqrt{p(p-a)(p-b)(p-c)}, \qquad p = \frac{a+b+c}{2}.$$

Heron's formula in metric form

Heron's formula can be rewritten purely in terms of pairwise distances:

$$d_{xyz} = \frac{1}{4}\sqrt{2\left(d_{xy}^2 d_{yz}^2 + d_{yz}^2 d_{xz}^2 + d_{xz}^2 d_{xy}^2\right) - \left(d_{xy}^4 + d_{yz}^4 + d_{xz}^4\right)}.$$

- Here d_{xyz} denotes the Euclidean area of the triangle (x, y, z).
- So in the Euclidean setting, the 3-metric can be expressed through the underlying 2-metric.

How can we express the Euclidean n-metric through pairwise distances?

Cayley–Menger matrix

The generalization of Heron's formula is given by the Cayley–Menger construction.

Definition

Let (X, d) be a metric space, and let

$$D_{ij}^2 = d^2(x_i, x_j)$$

be the matrix of squared pairwise distances. The Cayley–Menger matrix is defined by

$$CM(X, d) = \begin{pmatrix} 0 & \mathrm{e}^T \\ \mathrm{e} & D^2 \end{pmatrix},$$

where e is the all-ones column vector.

- This matrix is built only from pairwise distances.
- It provides a way to recover higher-dimensional Euclidean volume from a 2-metric.

Cayley–Menger volume formula

For a Euclidean simplex, the determinant of the Cayley–Menger matrix gives its volume.

Theorem

If $\mathcal{S} \subset \mathbb{R}^n$ is a simplex with vertices $s_0, \dots, s_n$, then

$$\mathrm{Vol}^2(\mathcal{S}) = \frac{(-1)^{n+1}}{2^n (n!)^2} \det CM(\mathcal{S}).$$

Equivalently,

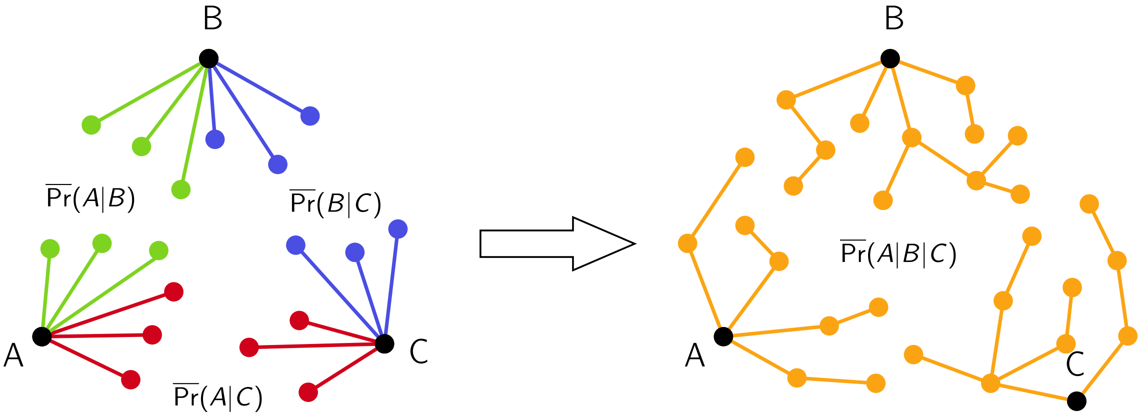
$$\mathrm{Vol}^2(\mathcal{S}) = \frac{(-1)^{n+1}}{2^n (n!)^2} \det \begin{pmatrix} 0 & \mathrm{e}^T \\ \mathrm{e} & D^2(\mathcal{S}) \end{pmatrix},$$

where

$$D^2(\mathcal{S})_{ij} = \|s_i - s_j\|^2.$$

- Thus Cayley–Menger extends Heron's formula from triangles to arbitrary simplices.

What about resistance n-metric?



Cayley–Menger formula

A central result of the work is an explicit formula relating R_n to the matrix of pairwise effective resistances.

Theorem

Let $x_1, \ldots, x_n$ be boundary vertices of an electrical network with effective resistance matrix $R(\mathcal{E})$. Then

$$R_{x_1 \ldots x_n} = \frac{(-1)^n}{2^{\,n-1}} \det \mathrm{CM}_R,$$

where CM_R is the Cayley–Menger matrix built from pairwise effective resistances.

$$\mathrm{CM}_R = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & R_{x_1x_2} & \cdots & R_{x_1x_n} \\ 1 & R_{x_2x_1} & 0 & \cdots & R_{x_2x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & R_{x_nx_1} & R_{x_nx_2} & \cdots & 0 \end{pmatrix}.$$

Cayley–Menger formula

For comparison, let us write these two formulas side by side.

$$R_{x_0 \dots x_n} = \frac{(-1)^{n+1}}{2^n} \det CM_R \qquad \mathrm{Vol}(\mathcal{S})^2 = \frac{(-1)^{n+1}}{2^n (n!)^2} \det CM(\mathcal{S})$$

$$CM_R = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & R_{x_0 x_1} & \cdots & R_{x_0 x_n} \\ 1 & R_{x_1 x_0} & 0 & \cdots & R_{x_1 x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & R_{x_n x_0} & R_{x_n x_1} & \cdots & 0 \end{pmatrix} \quad CM(\mathcal{S}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 & \cdots & d_{0n}^2 \\ 1 & d_{10}^2 & 0 & \cdots & d_{1n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n0}^2 & d_{n1}^2 & \cdots & 0 \end{pmatrix}$$

Minors of Cayley-Menger matrix

Theorem

Let $\mathcal{E}$ be an electrical network on n vertices with boundary vertex set X, and let CM_R be the Cayley–Menger matrix of this electrical network, with rows and columns numbered by $0, 1, \ldots, n$. Suppose that $A, B, C, D \subset \{1, \ldots, n\}$ are pairwise disjoint index sets such that $|A| = |B|$ and $A \sqcup B \sqcup C \sqcup D = X$. Then the determinant of the matrix $CM_R{}^{(A \cup C)^c}_{(B \cup C)^c}$ of size $m \times m$ is given by

$$\det CM_R{}^{(A \sqcup C)^c}_{(B \sqcup C)^c} = K \cdot \frac{\sum_\pi (-1)^\pi \Pr(a_1, b_{\pi(1)} | \cdots | a_{|A|}, b_{\pi(|A|)} | \cdots | d_1 | \cdots | d_{|D|})}{\Pr(tree)},$$

where $K = (-1)^{\sum A + \sum B + m + 1} 2^{m-2}$.

Fiedler's bijection

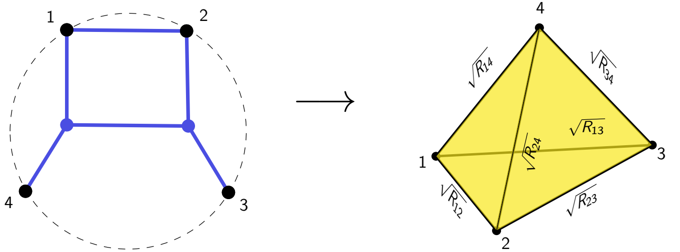
Theorem (Fiedler)

There is a bijection ω between the set of electrical networks on n vertices and the set of hyperacute $(n-1)$-dimensional simplices. Moreover, the Laplacian matrix of the electrical network $\mathcal{E}$ coincides with the Gram matrix $N(\mathcal{S}_{\mathcal{E}})$ of facet normals of the simplex $\mathcal{S}_{\mathcal{E}} \overset{def}{=} \omega(\mathcal{E})$:

$$L(\mathcal{E}) = N(\mathcal{S}_{\mathcal{E}}).$$

Corollary

The squared distance d_{ij}^2 between the vertices s_i and s_j of the simplex $\mathcal{S}_{\mathcal{E}}$ is equal to the effective resistance R_{ij} between the nodes i and j. In other words, the matrices $D^2(\mathcal{S}_{\mathcal{E}})$ and $R(\mathcal{E})$ coincide.



Electrical network $\mathcal{E}$ Simplex $\mathcal{S}_{\mathcal{E}}$

Theorem

Let E be an electrical network with boundary vertex set $X = \{1, \ldots, n\}$, and let

$$I = \{x_1, \ldots, x_k\} \subset X.$$

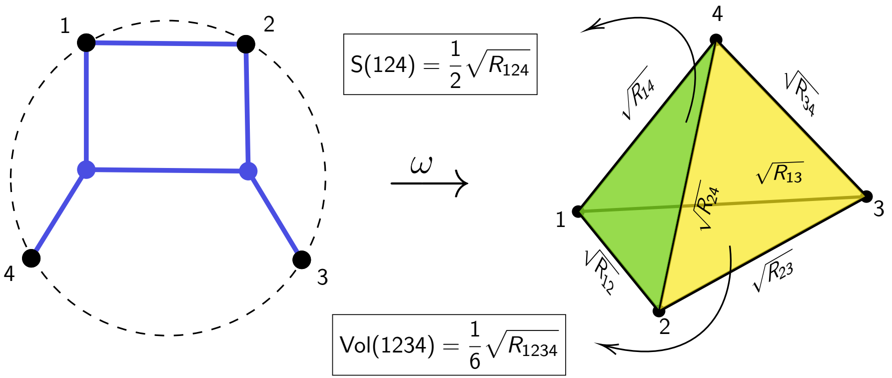
Then

$$\mathrm{Vol}^2(S_I) = \frac{\overline{\mathrm{Pr}}(x_1 | \ldots | x_k)}{((k-1)!)^2}.$$

Simplex interpretation for $n = 4$

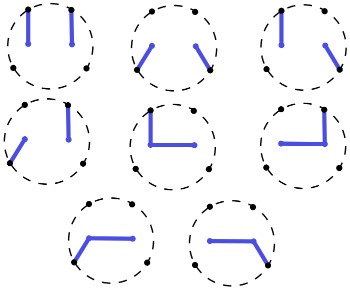
Consider a network $\mathcal{E}$ with boundary vertices $\{1, 2, 3, 4\}$. It determines a tetrahedron in $\mathbb{R}^3$ such that

$$\mathrm{Vol}(1234) = \frac{1}{6}\sqrt{R_{1234}}, \qquad \mathrm{Vol}(ijk) = \frac{1}{2}\sqrt{R_{ijk}}.$$

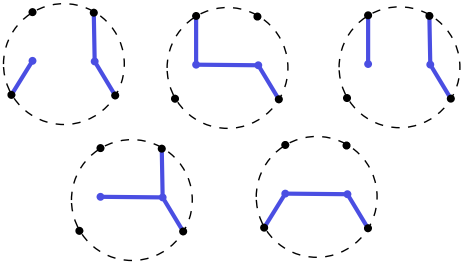


Simplex interpretation for $n = 4$

Thus, in order to compute the volume of $\mathcal{S}_{\mathcal{E}}$, we need to determine the quantity $W(1|2|3|4)$ of all 4-groves separating the vertices $1, 2, 3, 4$, as well as the number of spanning trees $W(\text{tree})$. Since $W(1|2|3|4) = 8$ and $W(\text{tree}) = 5$, we obtain $\mathrm{Vol}(1234) = \frac{1}{6}\sqrt{\frac{8}{4}} = \frac{\sqrt{2}}{6}$. Similarly, we compute $W(1|2|4) = 5$ and obtain the area of the face $S(124)$: $S(124) = \frac{1}{2}\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$.



All 4-groves separating the vertices 1,2,3,4



All 3-groves separating the vertices 1,2,4

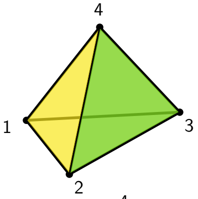
Theorem

Let $A, B, C, D \subseteq X$ be pairwise disjoint and assume that $A \sqcup B \sqcup C \sqcup D = X$ with $|A| = |B|$. Then

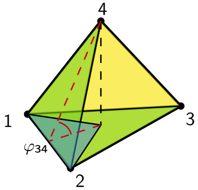
$$\frac{\sum_\pi (-1)^\pi \operatorname{Pr}(a_1, b_{\pi(1)}|\cdots|a_{|A|}, b_{\pi(|A|)}|\cdots|d_1|\cdots|d_{|D|})}{\operatorname{Pr}(\mathrm{tree})} = ((n-1)!)^2 \operatorname{Vol}^2(S_\mathcal{E}) \det N(S_\mathcal{E})^{A\sqcup C}_{B\sqcup C}.$$

Here $N(S_\mathcal{E})$ is the Gram matrix of the facet normals of the simplex $S_\mathcal{E}$ associated with the electrical network $\mathcal{E}$.

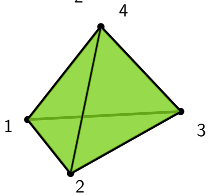
Groves and simplexes



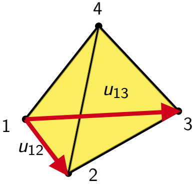
$\overline{\Pr}(2|3|4) =$
$= 4\,\mathrm{Vol}^2(234)$



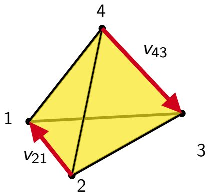
$\overline{\Pr}(1|2|34) =$
$= \mathrm{Vol}(124)\,\mathrm{Vol}(123)\cos\varphi_{34}$



$\overline{\Pr}(1|2|3|4) =$
$= 36\,\mathrm{Vol}^2(1234)$



$\overline{\Pr}(1|23) = \langle u_{12}, u_{13}\rangle$



$\overline{\Pr}(14|24) - \overline{\Pr}(13|24) =$
$= R_{13} + R_{24} - R_{14} - R_{23} = 2\langle v_{21}, v_{43}\rangle$

Conclusion

Thank you for your attention!