The prediction of Manin-Batyrev-Peyre on the number of rational points of algebraic varieties

Ratko Darda

University of Paris

March 3, 2021

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1. Geometric invariants and rational points

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2. Heights

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When $\dim(X) > 1$ the condition $g = 0$ is replaced by $K_X^{-1} = \wedge^{\text{top}} T_X$ is ample.

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- Application: Probability that two numbers are coprime $\frac{6}{\pi^2} = \frac{1}{\zeta(2)}$.

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Northcott property for ample line bundles.

Statement of the conjecture

Conjecture

Let X be a Fano variety (the anti-canonical line bundle K_X^{-1} is ample). There exists an open subvariety $U \subset X$ such that:

$$|\{U(\mathbb{Q})|H_{K_X^{-1}}(x) \leq B\}| \sim CB(\log(B))^{\mathrm{rk}(\mathrm{Pic}(X))-1}.$$

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For p prime, we define $|\cdot|_p$ by $|0|_p = 0$ and by

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$\mathbb{Z}_p := \{x \in \mathbb{Q}_p |\ |x|_p \leq 1\}$.

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Example: $\mathbb{P}^1(\mathbb{Q}_p)$.
If $L \to X$ is a line bundle, then $L(\mathbb{Q}_p) \to X(\mathbb{Q}_p)$ is a topological $\mathbb{Q}_p$-line bundle.

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Peyre defines a measure ω on $\prod_p X(\mathbb{Q}_p)$.

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This version admits counterexamples. There exists version without known counterexamples.

Some of the known cases

1. Toric varieties
2. Equivariant compactifications of vector groups
3. Some Del-Pezzo surfaces

An approach with abstract harmonic analysis.

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We apply "Tauberian theorems" to obtain the asymptotic behaviour for the wanted asymptotic.

Idelès

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We deduce a Haar measure μ on $\mathbb{G}_m(\mathbb{A}_\mathbb{Q})$.

Idelès

We consider subgroup

$$\mathbb{G}_m(\mathbb{A}_\mathbb{Q}) := \prod_p{}' \mathbb{G}_m(\mathbb{Q}_p)[\mathbb{G}_m(\mathbb{Z}_p)] \subset \prod_p \mathbb{G}_m(\mathbb{Q}_p).$$

It is locally compact.
For every p one has inclusion $\mathbb{G}_m(\mathbb{Q}) \subset \mathbb{G}_m(\mathbb{Q}_p)$.
The image of $\mathbb{G}_m(\mathbb{Q}) \to \prod_p \mathbb{G}_m(\mathbb{Q}_p)$ lies in $\mathbb{G}_m(\mathbb{A}_\mathbb{Q})$ and is a discrete and closed subgroup.
Measure $\mu_p := \frac{dx_p}{(1-|\pi_v|_v)|x|_p}$ is a Haar measure on $\mathbb{G}_m(\mathbb{Q}_p)$.
We deduce a Haar measure μ on $\mathbb{G}_m(\mathbb{A}_\mathbb{Q})$.

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Abstract harmonic analysis

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Theorem (Poisson formula)

Let $f : G \to \mathbb{C}$ be an L^1-function on G and $\hat{f}$ its Fourier transform with respect to dg. Suppose that $\hat{f}$ is an L^1-function on $(G/H)^*$. One has that

$$\int_H f dh = \int_{(G/H)^*} \hat{f}(\chi) d\chi,$$

where $d\chi = (dg/dh)^*$.

Idea of the proof for toric varieties

Count rational points on the torus

$$\mathbb{G}_m^1 \subset \mathbb{P}^1.$$

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We extend the definition of H on $\mathbb{G}_m(\mathbb{Q})$ to $H = \prod_p H_p$ on $\mathbb{G}_m(\mathbb{A}_{\mathbb{Q}})$. We calculate Fourier transform of $x \mapsto H(x)^{-s}$ with the respect to a character χ of $\mathbb{G}_m(\mathbb{A}_{\mathbb{Q}})$ which is trivial on $\mathbb{G}_m(\mathbb{Q})$.

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One has

$$\widehat{H}(s, \chi) = \prod_p \int_p H_{\mathbb{Q}_p}(x)^{-s} \chi_p(x) \frac{dx_p}{|x|_p}.$$

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We will apply Poisson formula for the function $x \mapsto H(x)^{-s}$ which is

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Thank you!