Lecture 2. Count of simple closed geodesics on Riemann surfaces (after Maryam Mirzakhani). Random square-tiled surfaces of large genus and random multicurves of surfaces of large genus (after a joint work with V. Delecroix, E. Goujard and P. Zograf)

Anton Zorich

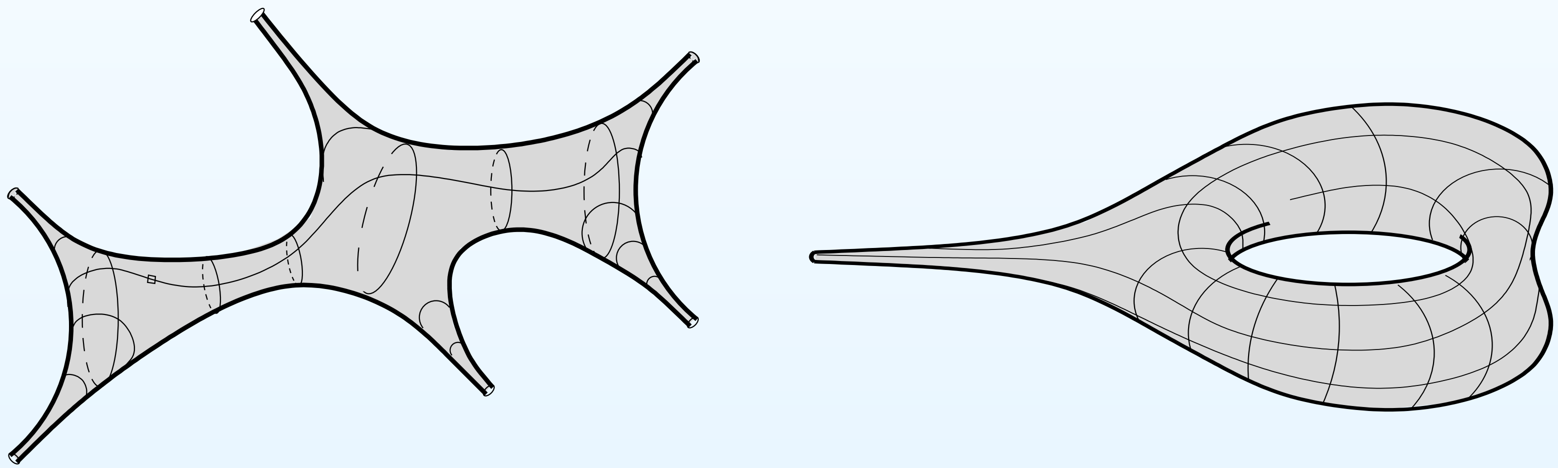
(Reference: M. Mirzakhani, *“Growth of the number of simple closed geodesics on hyperbolic surfaces”*, Annals of Math. (2) **168** (2008), no. 1, 97–125;
V. Delecroix, E. Goujard and P. Zograf arXiv.2007.04740)

HSE, February 17, 2022

Hyperbolic geometry of surfaces

Hyperbolic surfaces

Any smooth orientable surface of genus $g \geq 2$ admits a metric of constant negative curvature (usually chosen to be -1), called *hyperbolic* metric.
Allowing to metric to have several singularities (cusps), one can construct a hyperbolic metric also on a sphere and on a torus.



Simple closed curves and simple closed geodesics

A smooth closed curve on a surface is called *simple* if it does not have self-intersections.

Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp.

Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

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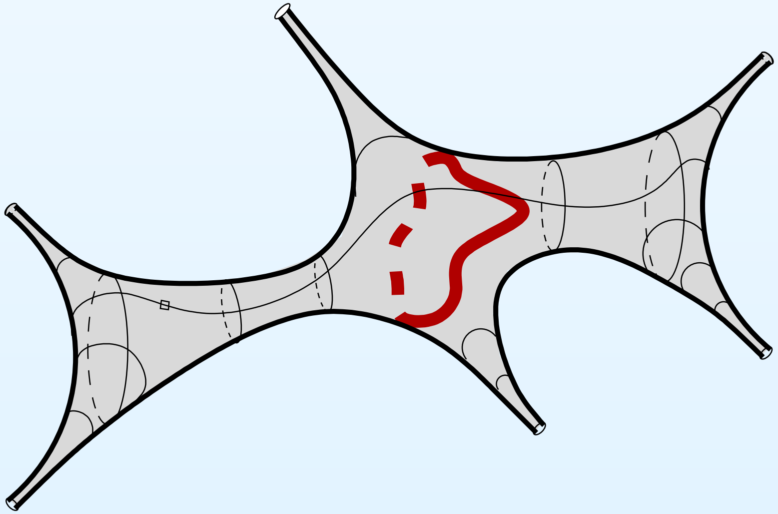

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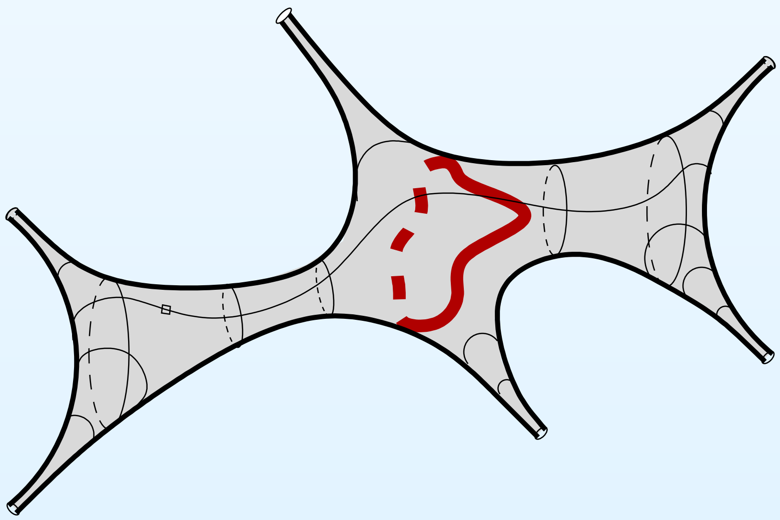

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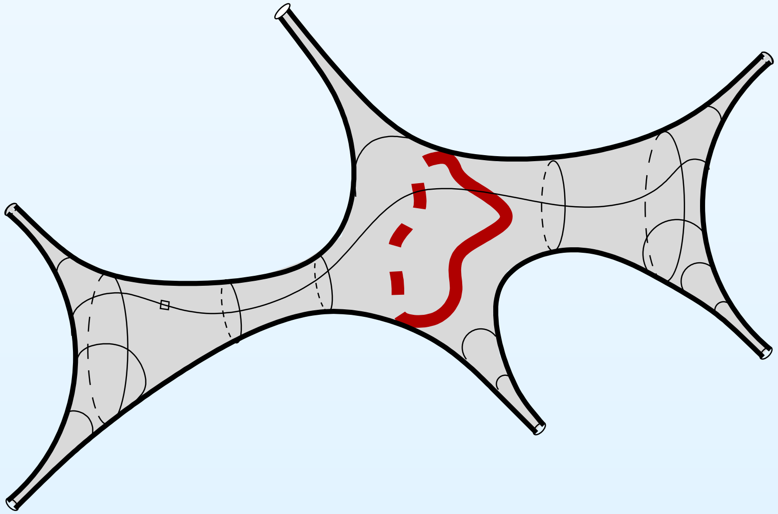

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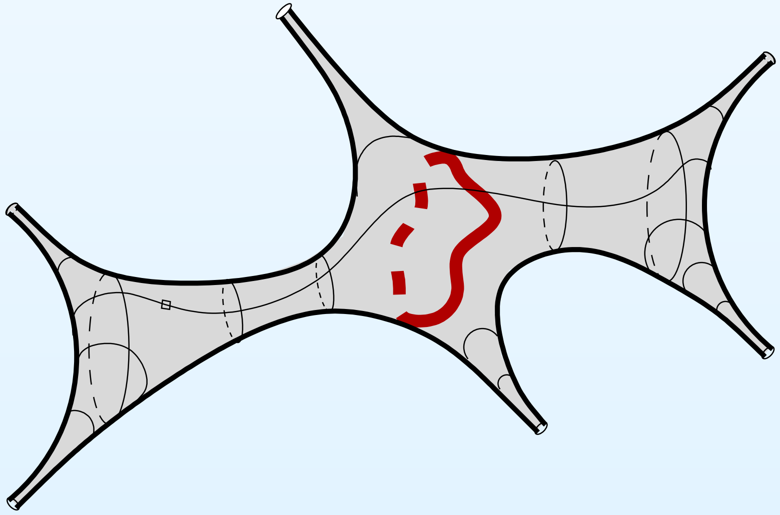

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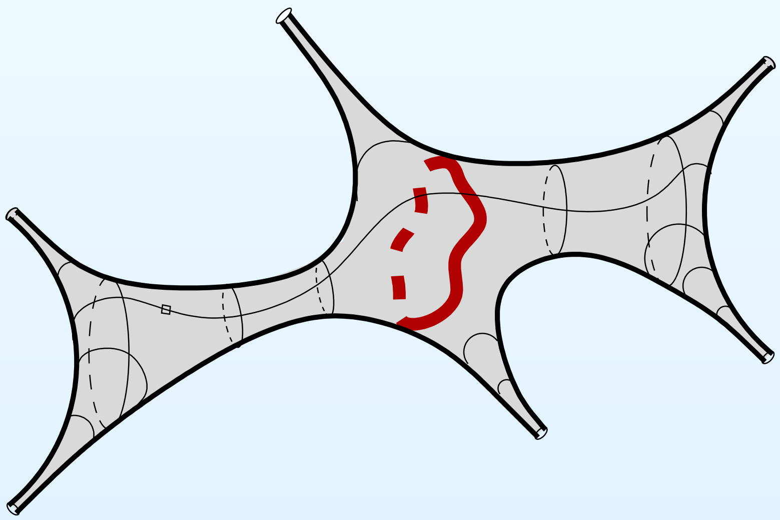

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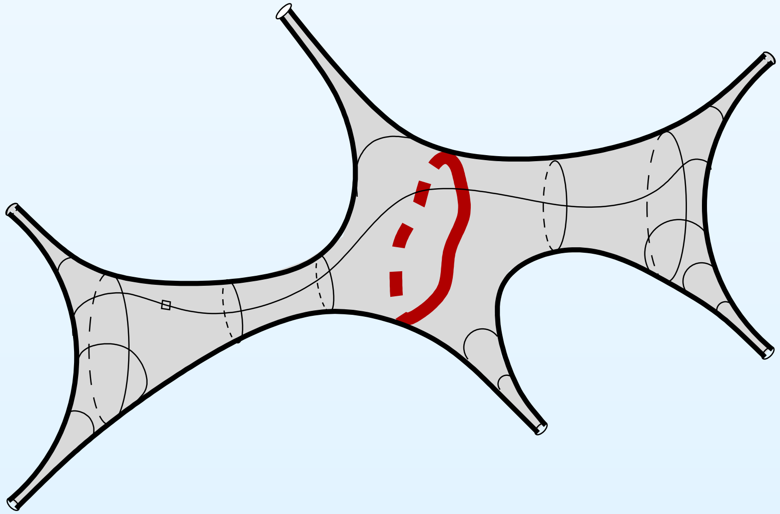

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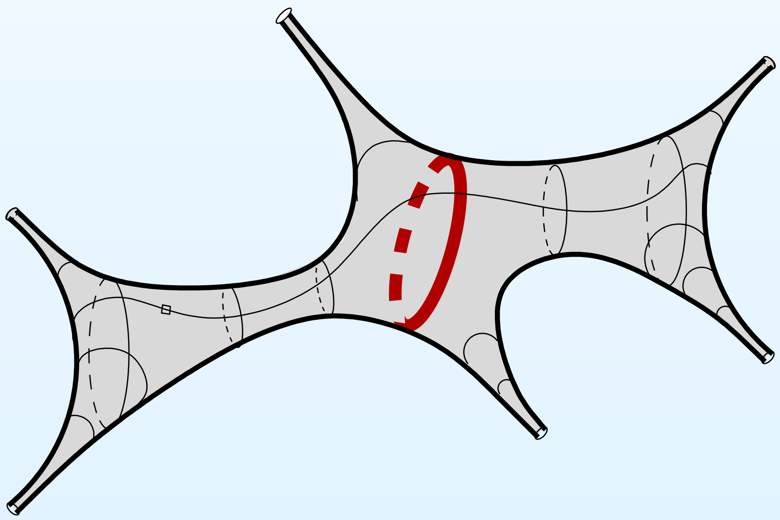

Families of hyperbolic surfaces

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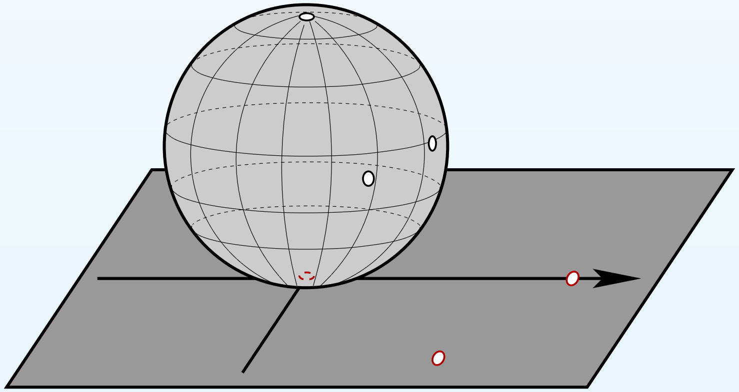

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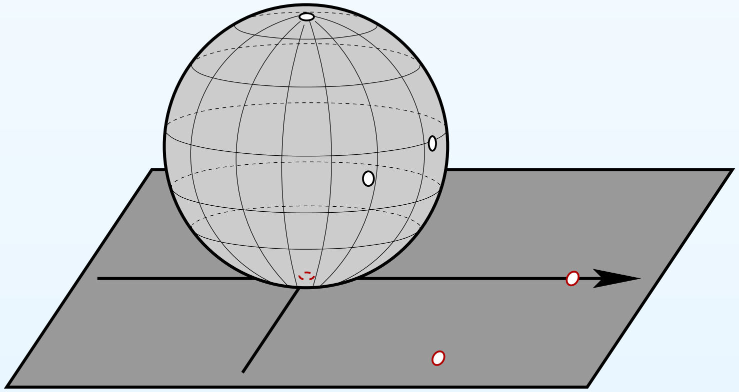

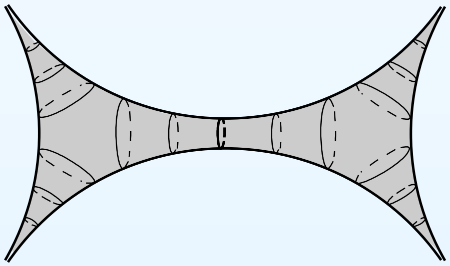

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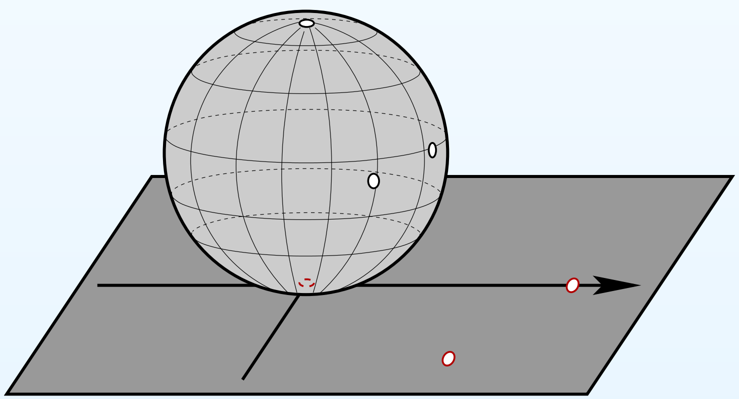

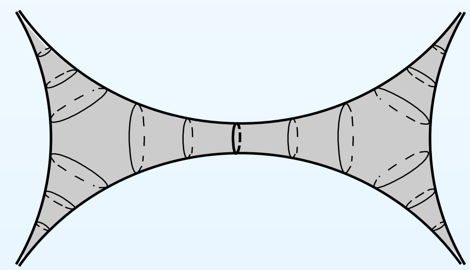

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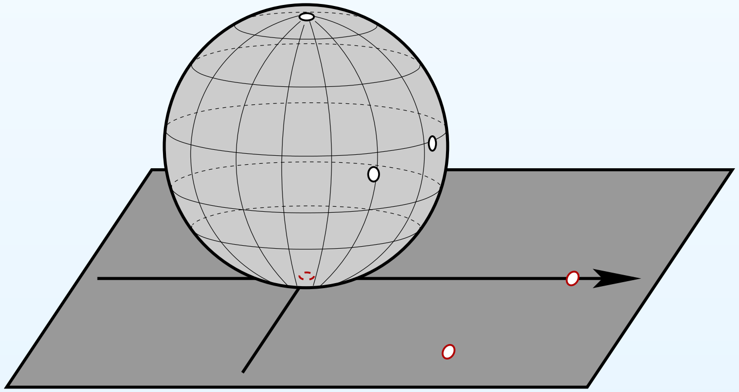

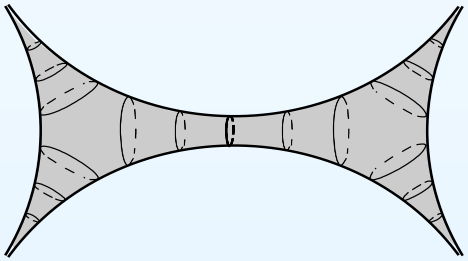

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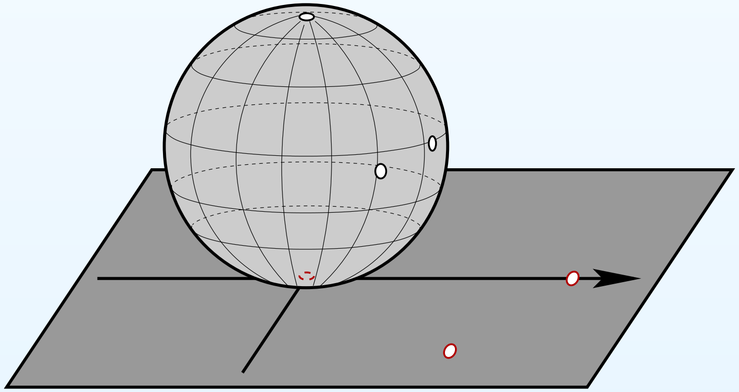

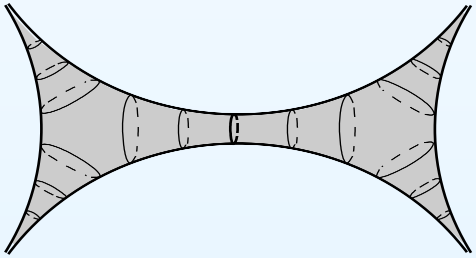

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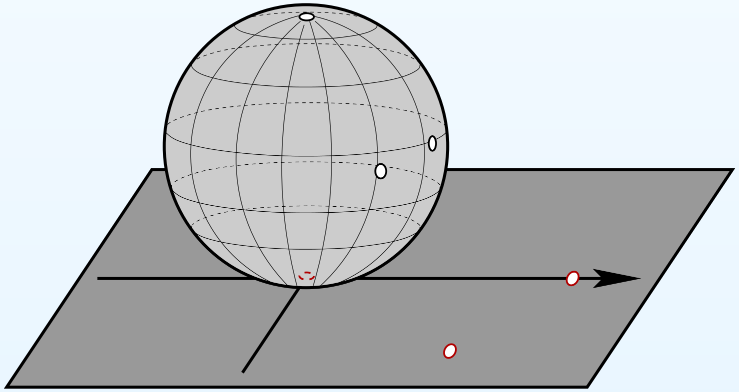

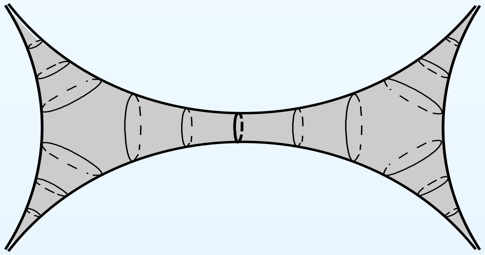

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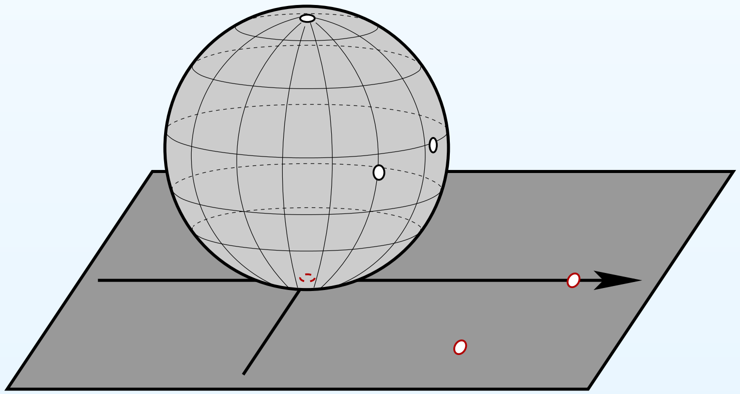

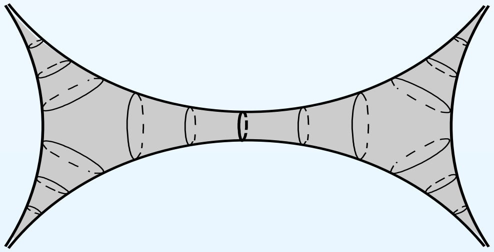

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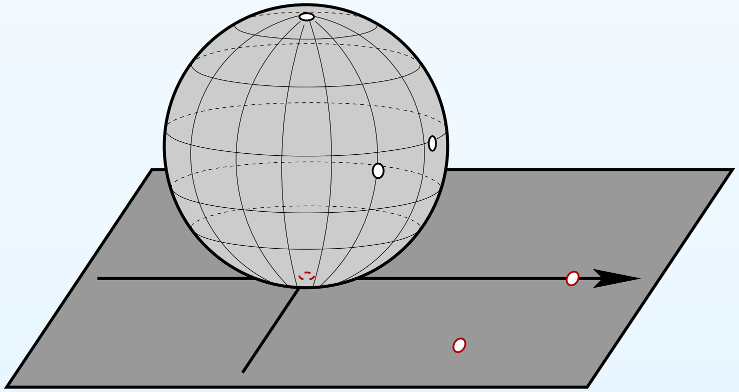

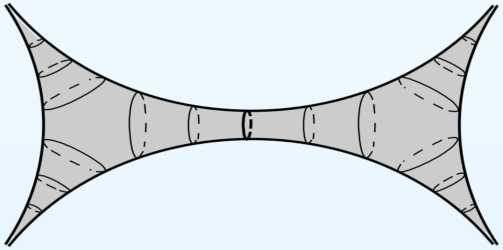

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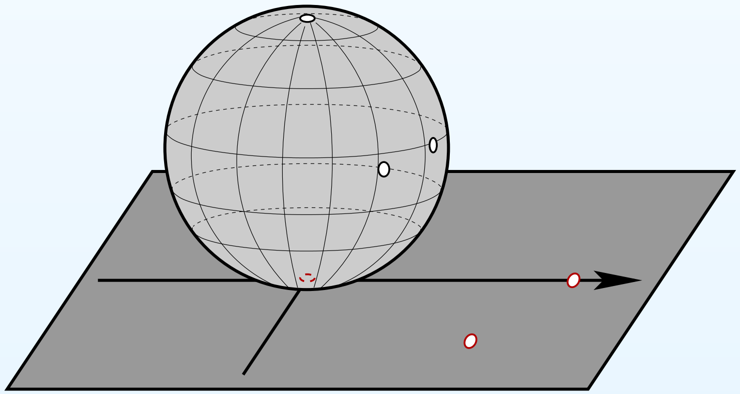

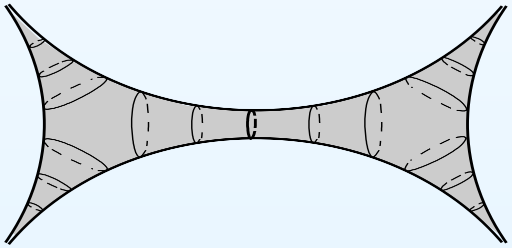

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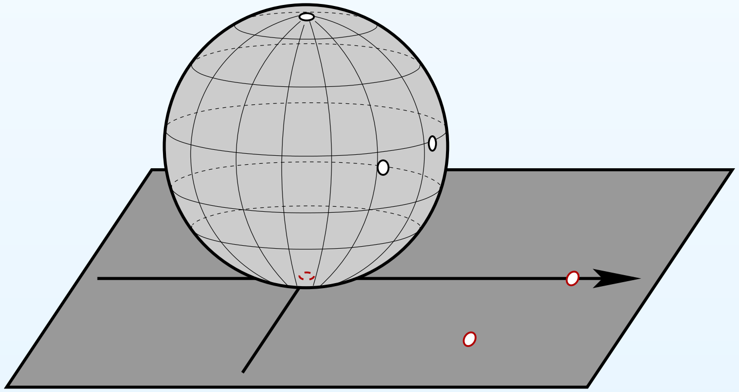

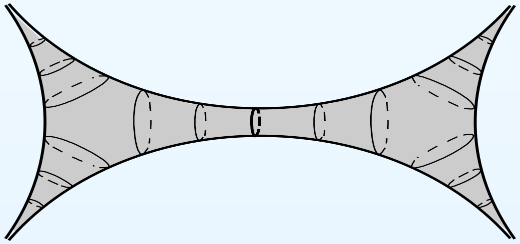

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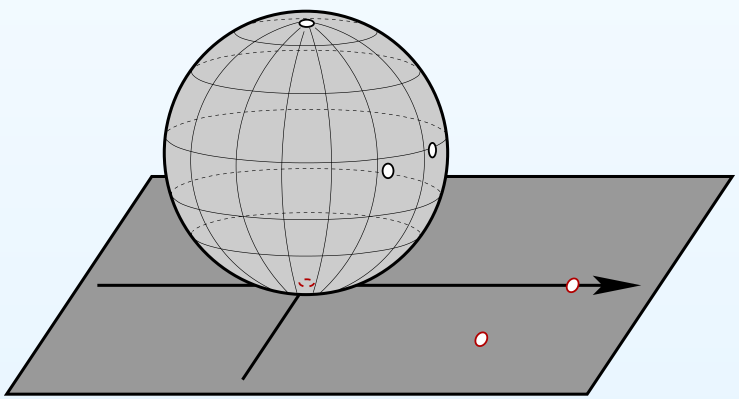

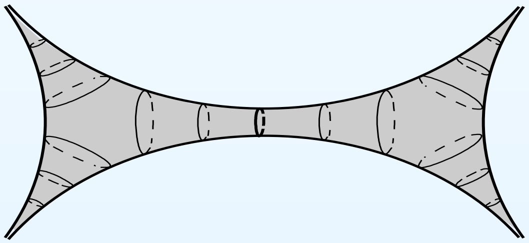

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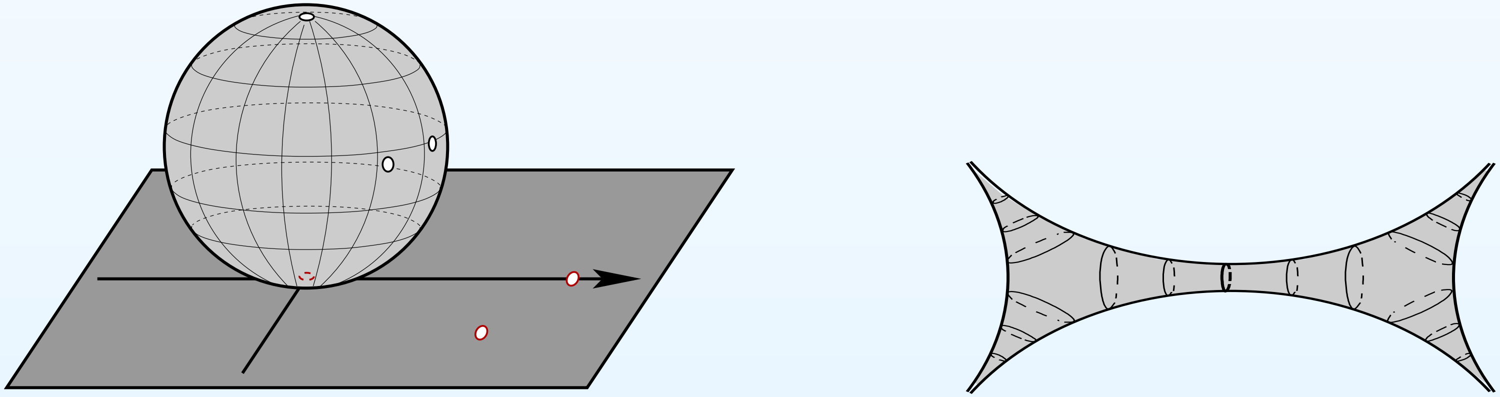

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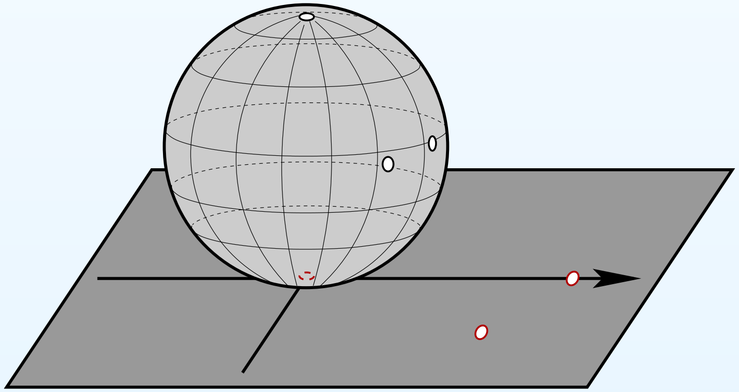

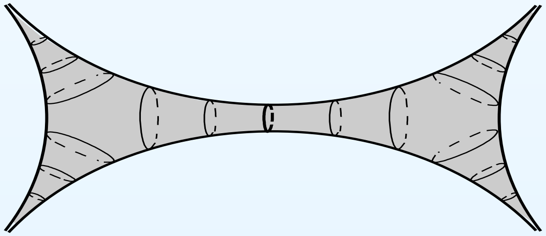

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Space of multicurves

Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another.

It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class.

One can consider more general *primitive multicurves*: collections of pairwise disjoint non-homotopic simple closed curves. For any fixed pair (g, n) the number of topological types of primitive multicurves on a surface of genus g with n punctures is also finite.

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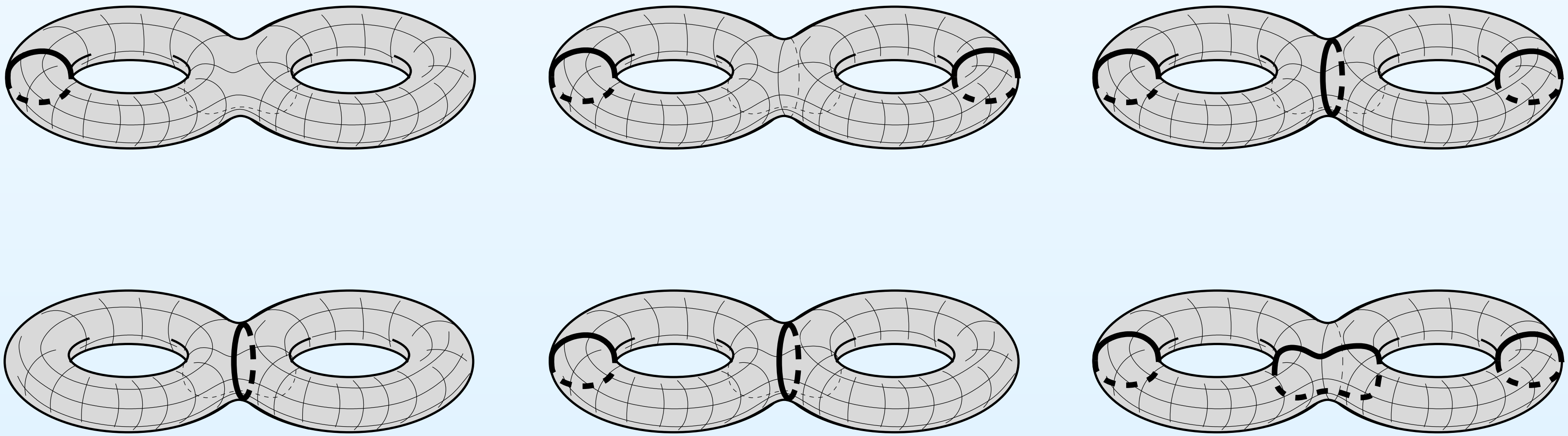
Example: primitive multicurves on a surface of genus two

The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures.

Note that contracting all components of a multicurve we get a “stable curve” — a Riemann surface degenerated in one of the several regular ways. In this way the “topological types of primitive multicurves” on a smooth surface $S_{g,n}$ of genus g with n punctures are in the natural bijective correspondence with boundary classes of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of pointed complex curves.



Mapping class group

The group of all diffeomorphisms of a closed smooth orientable surface of genus g quotient over diffeomorphisms homotopic to identity is called the *mapping class group* and is denoted by Mod_g.

When the surface has n marked points (punctures) we require that diffeomorphism sends marked points to marked points; the corresponding mapping class group is denoted $\mathrm{Mod}_{g,n}$.

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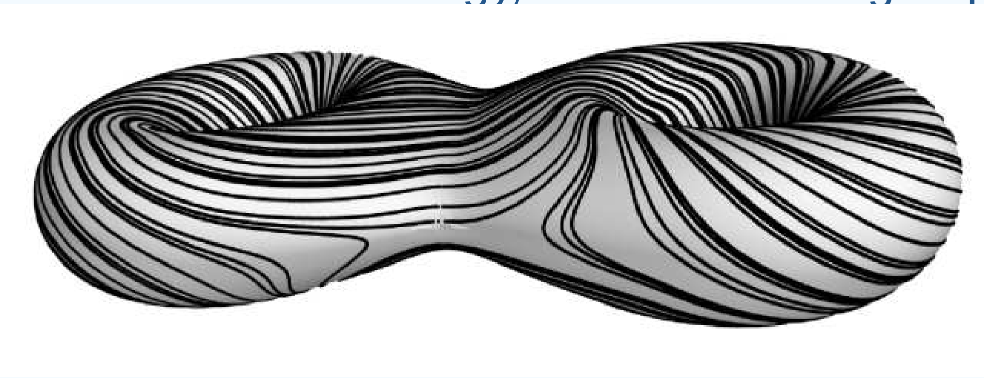
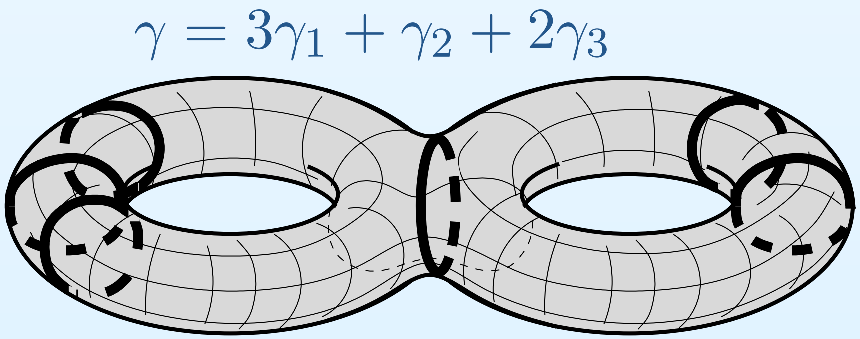
Simple closed multicurve, its topological type and underlying primitive multicurve

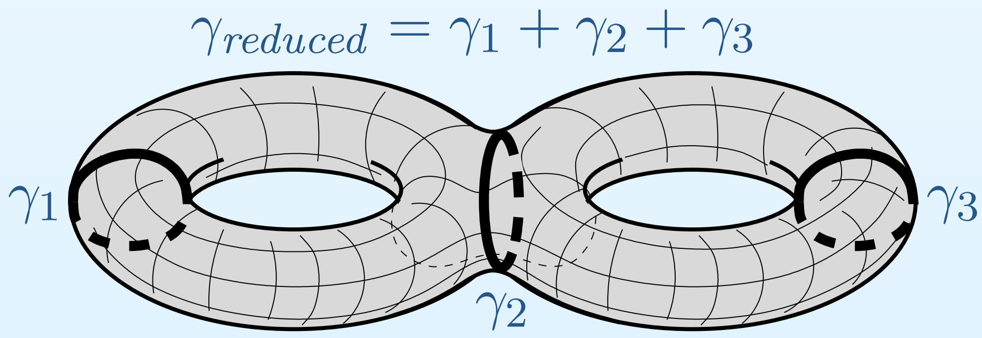
The first homology $H_1(M^2;\mathbb{Z})$ of the surface is great to study closed curves, but it ignores some interesting curves. The fundamental group $\pi_1(M^2)$ is also wonderful, but it is mainly designed to work with self-intersecting cycles. Thurston invented yet another structure to work with simple closed multicurves; in many aspects it resembles the first homology, but there is no group structure.

A general multicurve ρ:



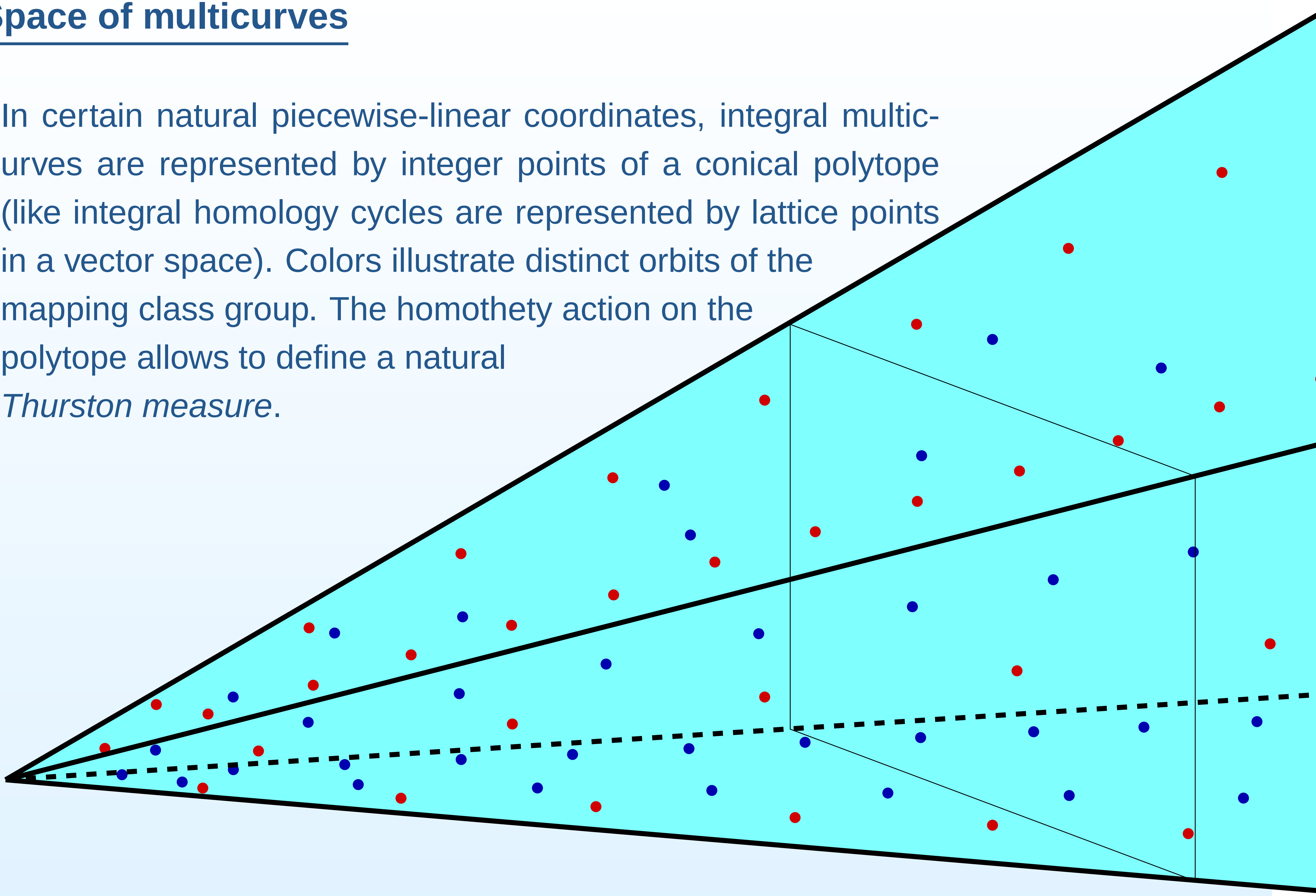
the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.





Space of multicurves

In certain natural piecewise-linear coordinates, integral multicurves are represented by integer points of a conical polytope (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. The homothety action on the polytope allows to define a natural *Thurston measure*.



Space of measured laminations $\mathcal{ML}_{g,n}$. Ergodicity of the Thurston measure

In the presence of a hyperbolic metric the integral multicurves take the shape of simple closed geodesic multicurves. Moreover, every (not necessary integral) point of the conical polytope defines a *measured geodesic lamination*. The “natural coordiantes” are, for example, the *train tracks* coordinates.

Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the “integral lattice” $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the Thurston measure μ_{Th}.

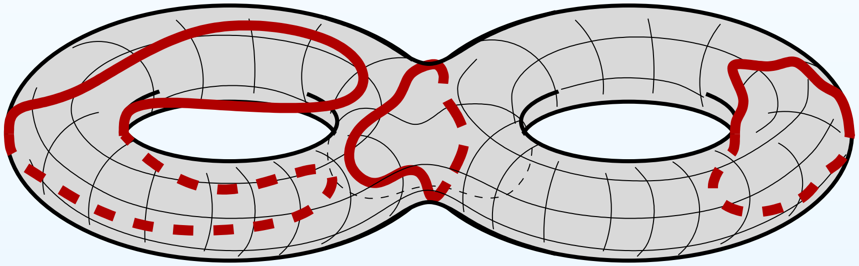
Theorem (H. Masur, 1985). *The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of $\mathcal{ML}_{g,n}$ invariant under $\mathrm{Mod}_{g,n}$ has measure zero or its complement has measure zero). Any $\mathrm{Mod}_{g,n}$-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

Mirzakhani's count of simple closed geodesics: statement of results

Picture by François Labourie taken at CIRM

Geodesic representatives of multicurves

Consider now several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. We have seen that in the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves become simple closed geodesics.



Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

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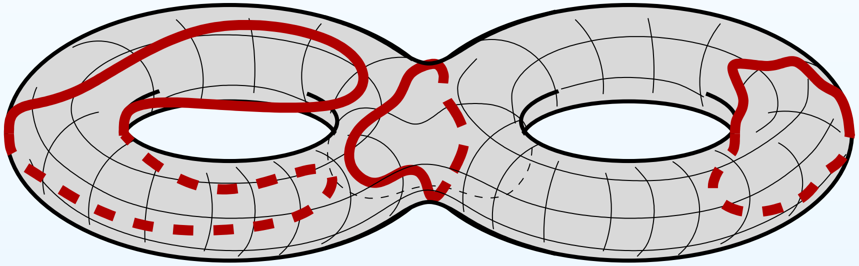

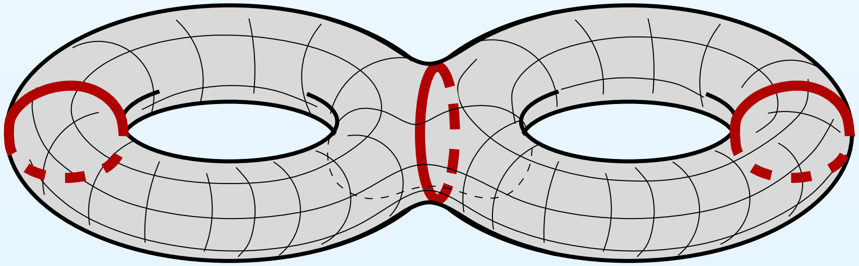

Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

Hyperbolic length of a multicurve

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i\gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i\ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L,\gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

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Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

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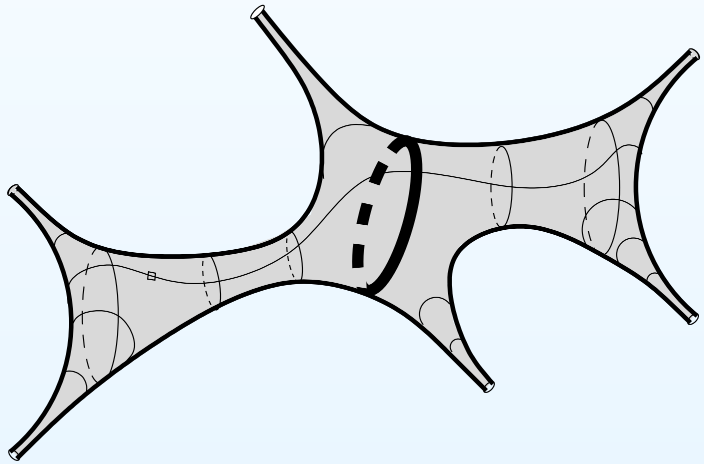
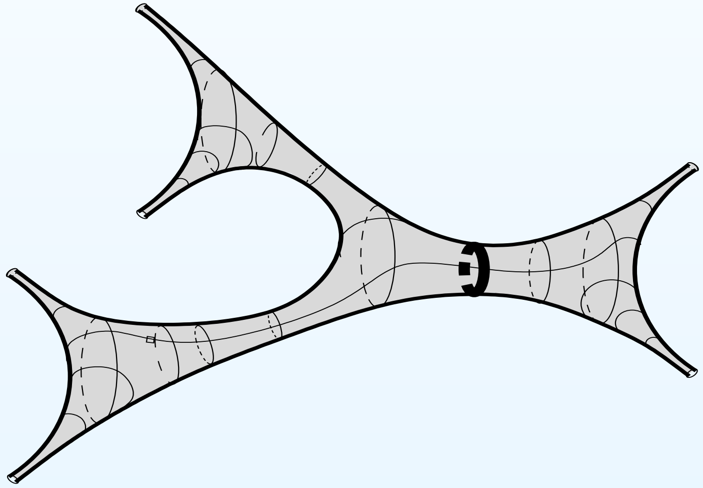
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$*, and any two rational multicurves* γ_1, γ_2 *on a smooth surface* $S_{g,n}$ *considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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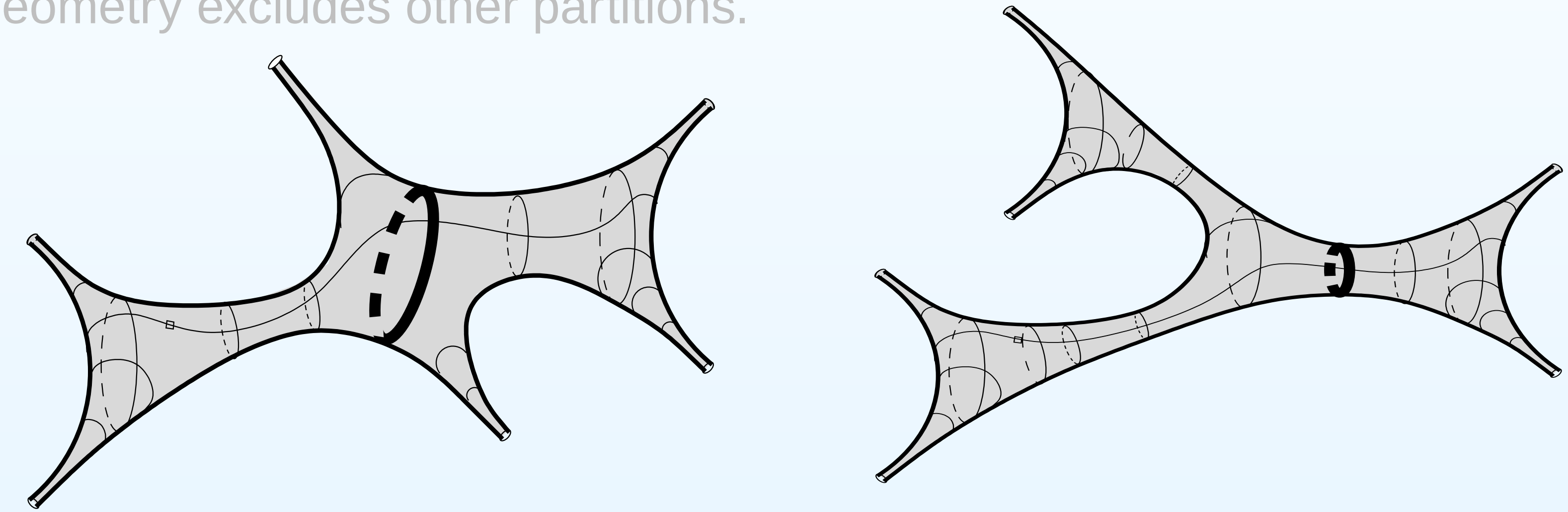
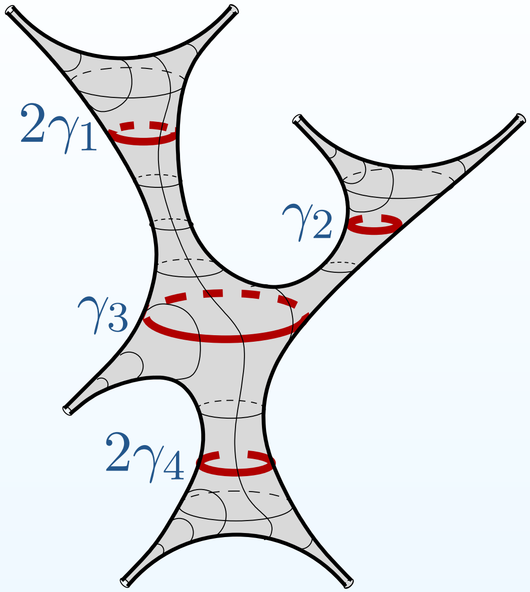
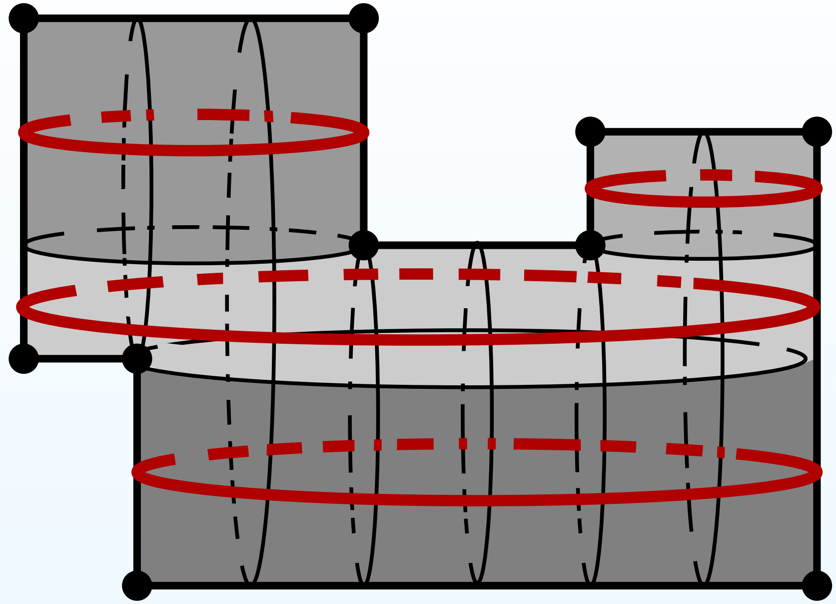


Example. (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

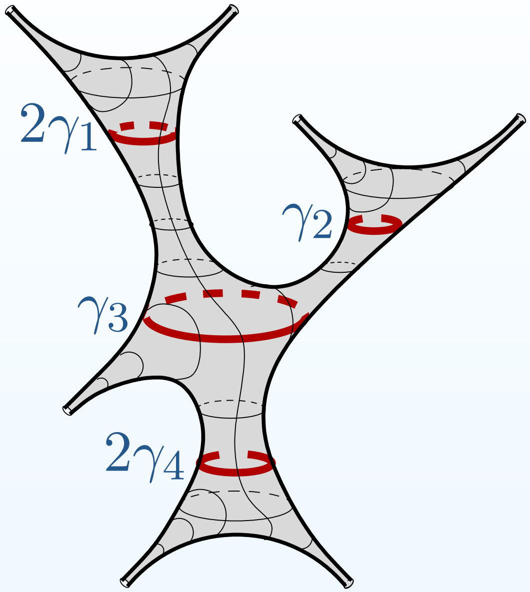
Hyperbolic and flat geodesic multicurves

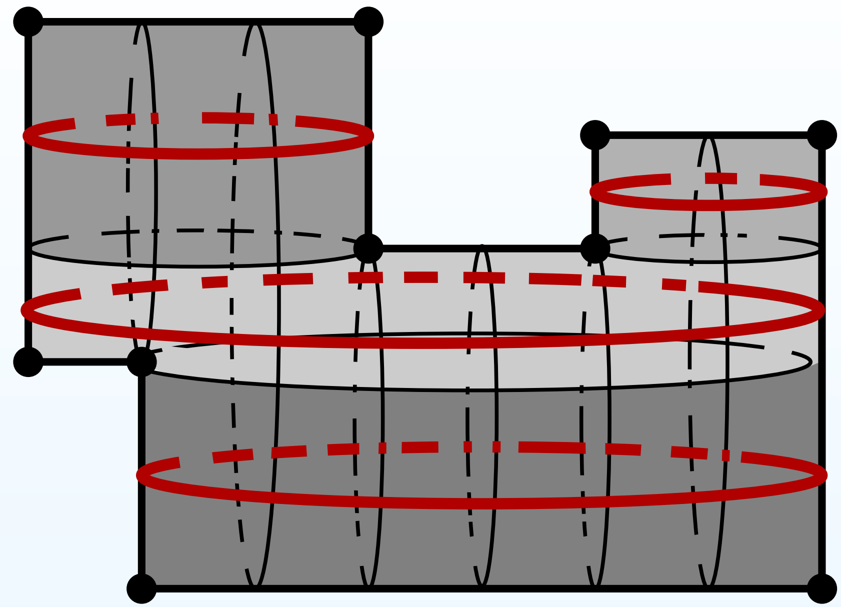




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves



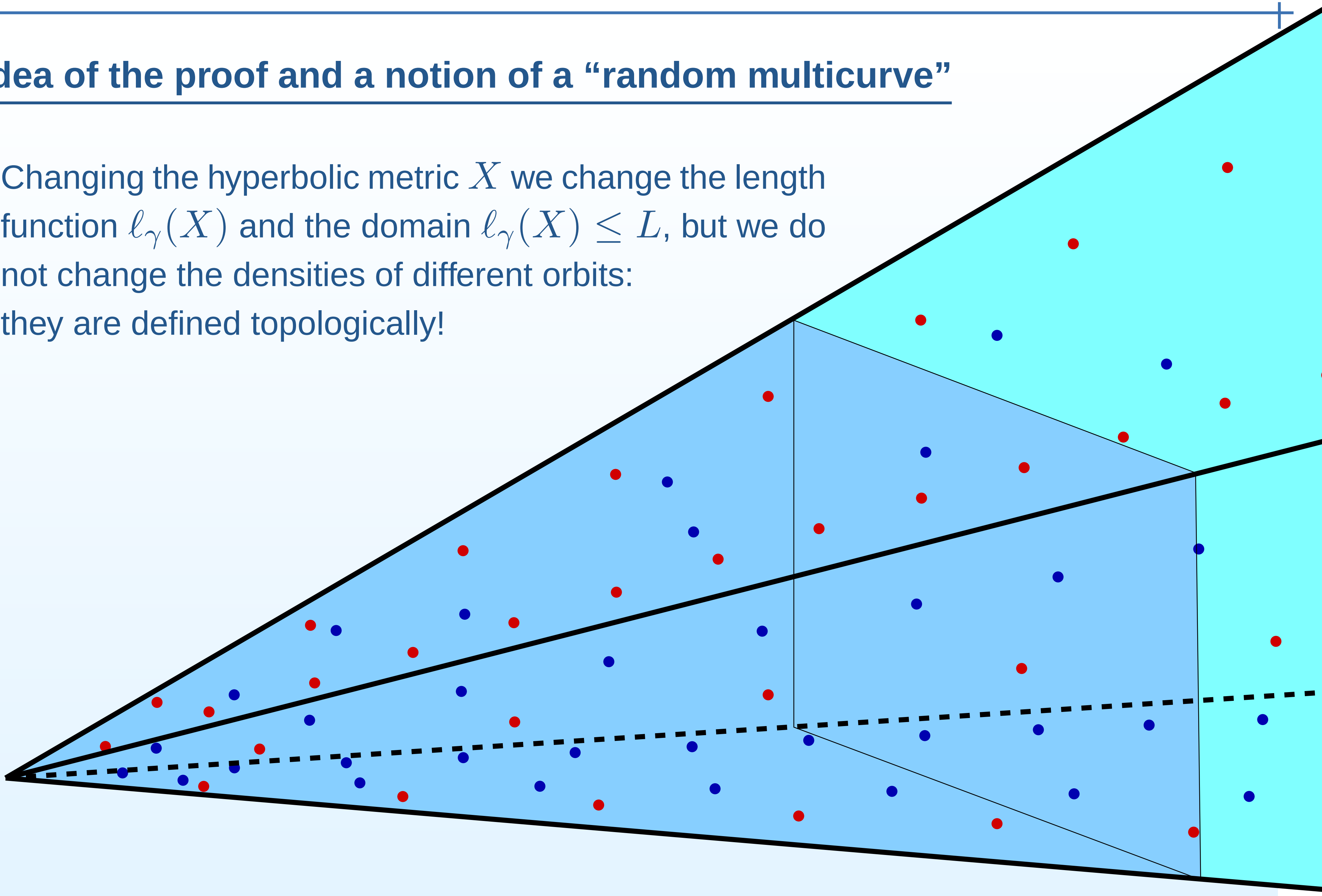


Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Idea of the proof and a notion of a “random multicurve”

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



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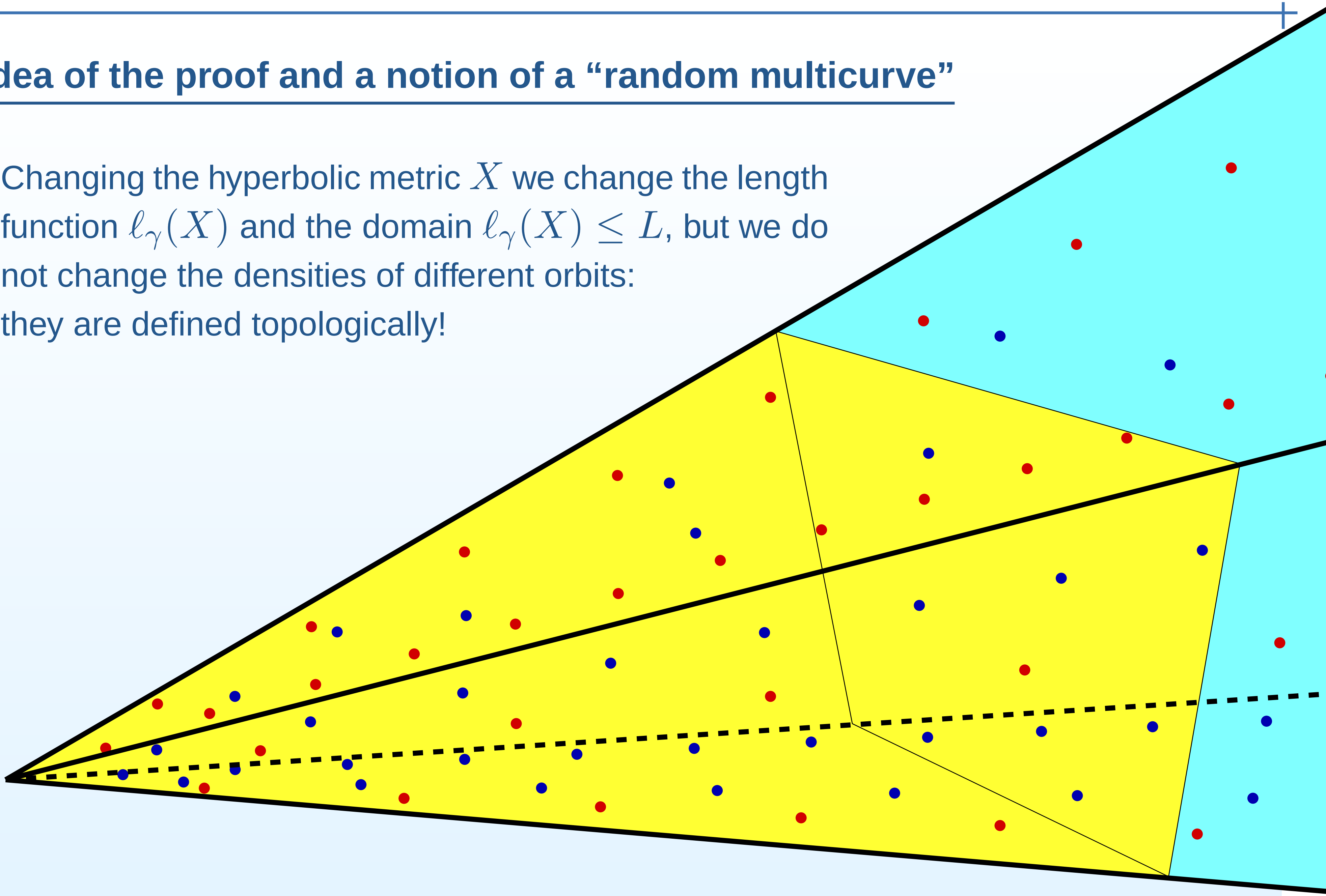

More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\,dX =$$

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$$= \lim_{L\to+\infty} \frac{1}{L^{6g-6+2n}} \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\,dX = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^{6g-6+2n}}\,dX = c(\gamma)\,,$$

so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

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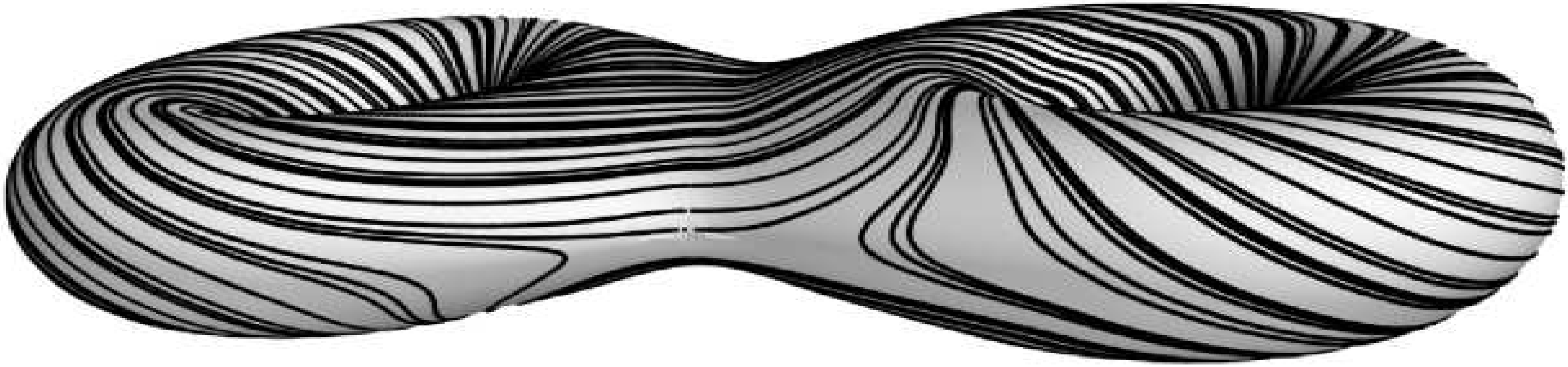
Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

- *Which simple closed geodesics are more frequent: separating or non-separating?*

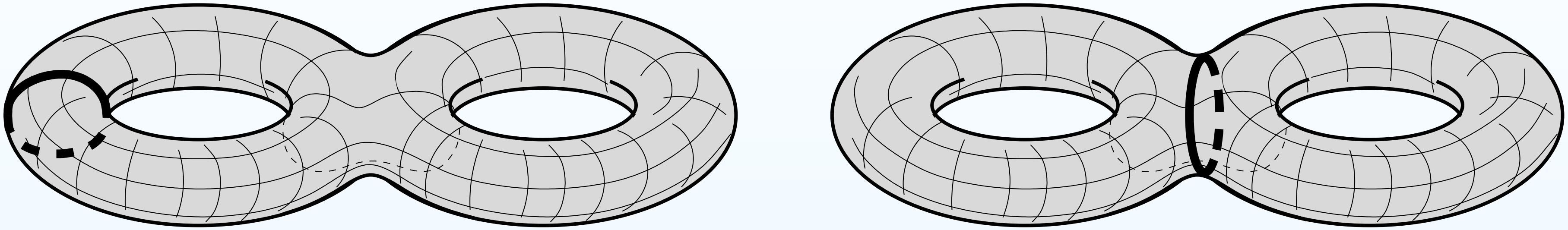
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

- *What is the probability that* $\gamma_{reduced}$ *separates* S *into distinct connected components?*

- *What are the probabilities that* $\gamma_{reduced}$ *has* $k = 1, 2, 3$ *primitive connected components* $\gamma_1, \ldots, \gamma_k$*?*

Separating versus non-separating simple closed curves in $g = 2$

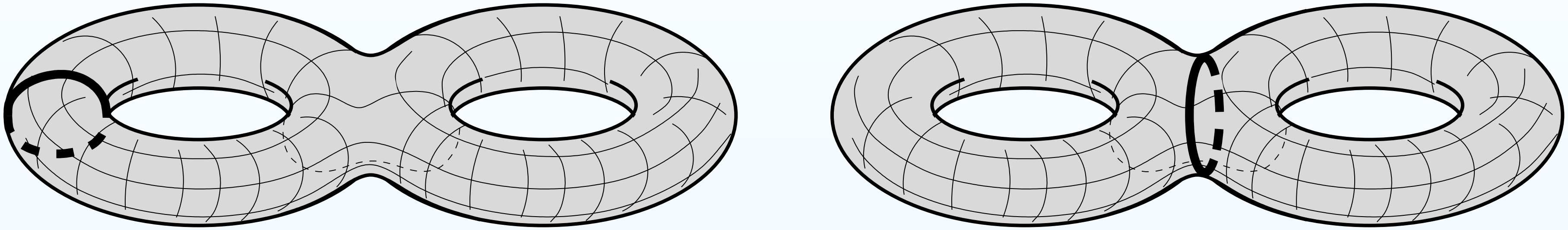
Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$



$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

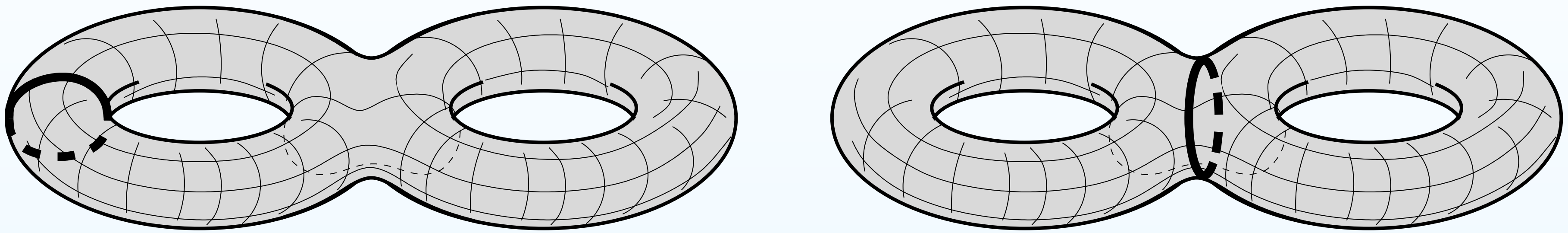


$$\lim_{L \to +\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

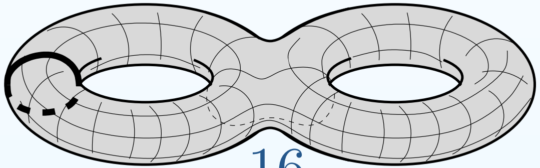


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

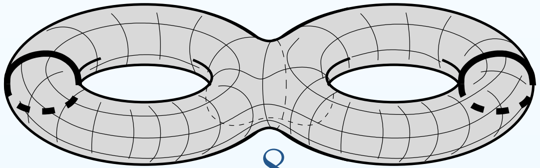
after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

Multicurves on a surface of genus two and their frequencies

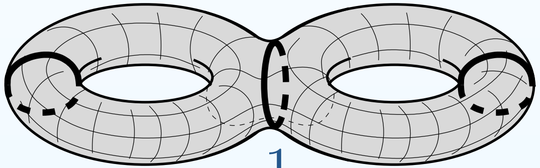
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



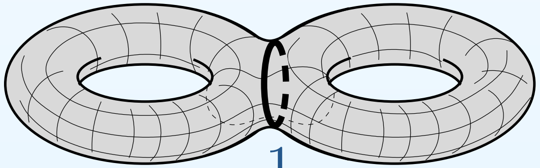
$\frac{16}{63}$



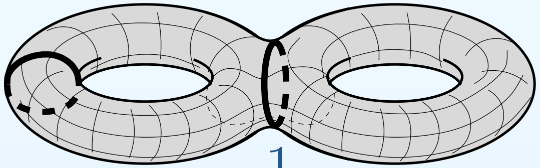
$\frac{8}{15}$



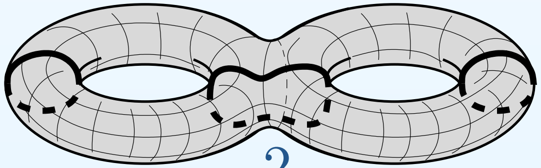
$\frac{1}{9}$



$\frac{1}{189}$



$\frac{1}{45}$



$\frac{2}{27}$

In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Statistics of prime decompositions: random integer numbers

The Prime Number Theorem states that an integer number n taken randomly in a large interval $[1, N]$ is prime with asymptotic probability $\frac{\log N}{N}$.

Actually, one can tell much more about prime decomposition of a large random integer. Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities. In other words, if n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, let $\omega(n) = k$. By the Erdős–Kac theorem, the centered and rescaled distribution prescribed by the counting function $\omega(n)$ tends to the normal distribution:

Erdős–Kac Theorem (1939)

$$\lim_{N\to+\infty} \frac{1}{N} \operatorname{card}\left\{ n \le N \,\middle|\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$

The subsequent results of of A. Selberg (1954) and of A. Rényi and P. Turán (1958) describe the rate of convergence.

Statistics of prime decompositions: random permutations

Denote by $\mathrm{K}_n(\sigma)$ the number of disjoint cycles in the cycle decomposition of a permutation σ in the symmetric group S_n. Consider the uniform probability measure on S_n. A random permutation σ of n elements has exactly k cycles in its cyclic decomposition with probability $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. It is immediate to see that $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$. V. L. Goncharov computed the expected value and the variance of K_n as $n \to +\infty$:

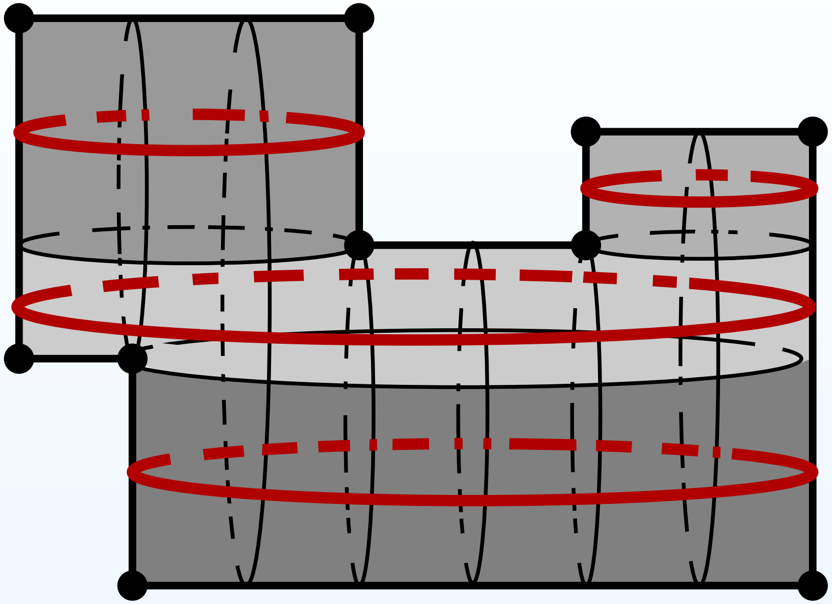
$$\mathbb{E}(\mathrm{K}_n) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n) = \log n + \gamma - \zeta(2) + o(1)\,,$$

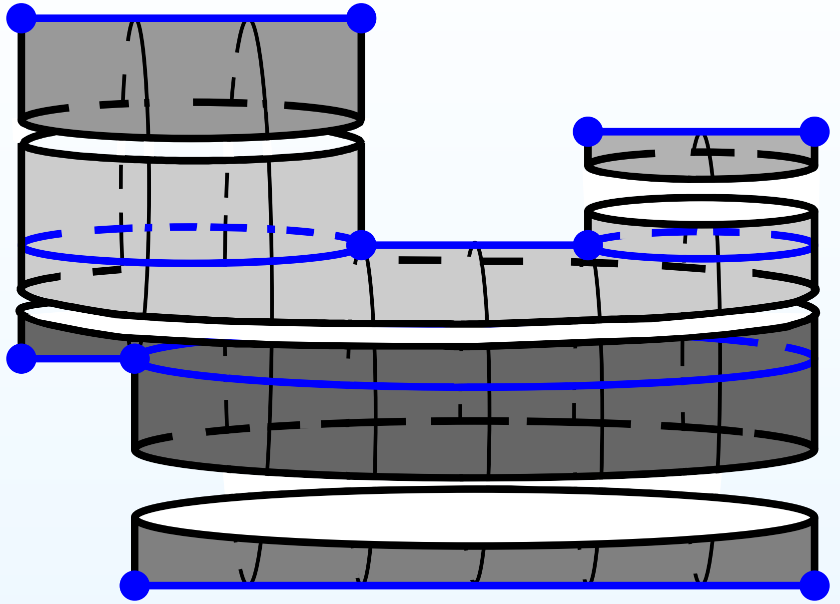
and proved the following central limit theorem:

Theorem (V. L. Goncharov, 1944)

$$\lim_{n\to+\infty} \frac{1}{n!}\,\mathrm{card}\left\{\sigma \in S_n \,\middle|\, \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \le x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt\,.$$

Shape of a random square-tiled surface of large genus?





Questions.

- *How many singular horizontal leaves (in blue on the right picture) has a random square-tiled surface of genus g?*
- *Find the probability distribution for the number $K_g(S) = 1, 2, 3, \ldots, 3g-3$ of maximal horizontal cylinders (represented by red waist curves on the left picture)*
- *What are the typical heights $h_1, \ldots, h_k$ of the cylinders?*
- *What is the shape of a random square-tiled surface of large genus?*

Random multicurves and random square-tiled surfaces

Denote by $K_{g,n}(\gamma)$ the number k of components of a multicurve $\gamma = \sum_{i=1}^{k} m_i \gamma_i$ (counted *without* multiplicities m_i) on a surface of genus g with n cusps.

Denote by $K_{g,n}(S)$ the number of maximal horizontal cylinders in the cylinder decomposition of a square-tiled surface S of genus g with n cone-angles π

Theorem (Delecroix–Goujard–Zograf–Zorich'21.). *For any genus $g \geq 2$ and for any $k \in \mathbb{N}$, the probability $p_g(k)$ that a random multicurve γ on a surface of genus g has exactly k components counted without multiplicities coincides with the probability that a random square-tiled surface S of genus g has exactly k maximal horizontal cylinders:*

$$\mathbb{P}\big(K_{g,n}(\gamma) = k\big) = \mathbb{P}\big(K_{g,n}(S) = k\big)\,.$$

In other words, $K_{g,n}(\gamma)$ and $K_{g,n}(S)$, considered as random variables, determine the same probability distribution for any g, n, $3g + n \geq 4$.

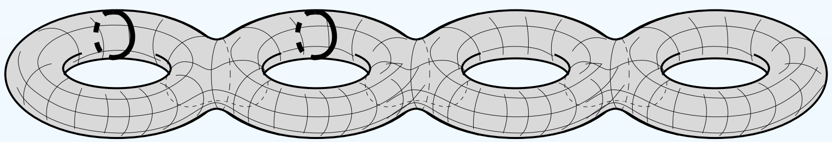
From now on we consider only hyperbolic surfaces without cusps and only square-tiled surfaces without cone-angles π (i.e. the ones corresponding to *holomorphic* quadratic differentials).

Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

Theorem (Delecroix–Goujard–Zograf–Zorich'20.). *With probability which tends to 1 as $g \to \infty$,*

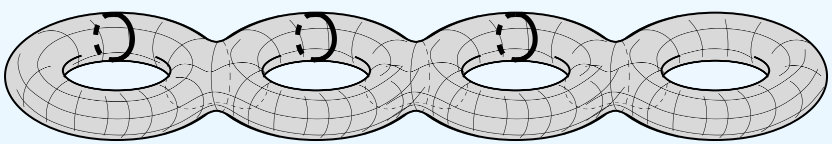
• The reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ associated to a random integral multicurve $\gamma = m_1\gamma_1 + \ldots m_k\gamma_k$ does not separate the surface;

• $\gamma_{reduced}$ has about $(\log g)/2$ components and has one of the following types:



$0.09\log(g)$ components

$\dots$ $\dots$ $\dots$ $\dots$



$0.62\log(g)$ components

$$\mathbb{P}\Big(0.09\log g < K_g(\gamma) < 0.62\log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right) .$$

A random square-tiled surface (without conical points of angle π) of large genus has about $\frac{\log(g)}{2}$ cylinders, and all its conical points sit at the same horizontal and at the same vertical level with probability which tends to 1 as $g \to \infty$.

Weights of a random multicurve (heights of cylinders of a ransom square-tiled surface)

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *A random integer multicurve $m_1\gamma_1 + \dots + m_k\gamma_k$ with bounded number k of primitive components is reduced (i.e., $m_1 = \dots = m_k = 1$) with probability which tends to 1 as $g \to +\infty$. In other terms, if we consider a random square-tiled surface with at most K cylinders, the heights of all cylinders would be very likely equal to 1 for $g \gg 1$.*

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *A general random integer multicurve $m_1\gamma_1 + \dots + m_k\gamma_k$ is reduced (i.e., $m_1 = \dots = m_k = 1$) with probability which tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, all weights $m_1, \dots m_k$ of a random multicurve are bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

In other words, for more 70% of square-tiled surfaces of large genus, the heights of all cylinders are equal to 1.

However, the mean value of $m_1 + \dots + m_k$ is infinite in any genus g.

Number of cycles in a random permutation

Given a permutation $\sigma \in S_n$ of cycle type $(1^{\mu_1} 2^{\mu_2} \dots n^{\mu_n})$ define its *weight* as

$$w_\theta(\sigma) := \theta_1^{\mu_1} \theta_2^{\mu_2} \cdots \theta_n^{\mu_n} \,,$$

where $\theta_k = \frac{\zeta(2k)}{2}$, $k \in \mathbb{N}$. Define a probability measure on S_n by setting

$$\mathbb{P}_\theta(\sigma) := \frac{w_\theta(\sigma)}{W_\theta} \,, \qquad \text{where} \qquad W_\theta := \sum_{\sigma \in S_n} w_\theta(\sigma).$$

Measures with $\theta_k = const$, $k \in \mathbb{N}$, are called *Ewens measures*; for $const = 1$ we get the uniform measure on S_n.

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The random variable $K(\sigma)$ counting the number of disjoint cycles in the cyclic decomposition of a random permutation is very well studied (Goncharov'44, ... Hwang'94–95, ... Kowalski–Nikeghbali'10,...). The corresponding probability distribution is given by the Poisson distribution with parameter depending on n, corrected by a convolution with certain explicit function independent of n.

Using this *Mod-Poisson convergence* technique we also get a very precise description of the law for the number of cycles $K(\sigma)$ in a random permutation for our nonuniform Ewens-like measure $\mathbb{P}_\theta$.

Main Theorem (informally)

Main Theorem (Delecroix–Goujard–Zograf–Zorich'20). *As g grows, the probability distribution $\mathbb{P}(K_g = k)$ rapidly becomes, basically, indistinguishable from the distribution of the number $K_{3g-3}(\sigma)$ of disjoint cycles in a $\mathbb{P}_\theta$-random permutation σ of $3g-3$ elements. In particular, for any $j \in \mathbb{N}$ the difference of the j-th moments of the two distributions is of the order $O(g^{-1})$.*

We have an explicit asymptotic formula for all cumulants. It gives

$$\mathbb{E}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,$$

$$\mathbb{V}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,$$

where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

In practice, already for $g = 12$ the match of the graphs of the distributions is such that they are visually indistinguishable.

Mod-Poisson convergence (Hwang'94–95) . For any $x > 0$ the distribution of the number of cycles of a uniformly random permutation $\sigma \in S_n$ of n elements is uniformly well-approximated in a neighborhood of $x \log n$ by the Poisson distribution with parameter $\log n + a(x)$ with an explicit correction $a(x)$.

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Let $\lambda_{3g-3} = \log(6g-6)/2$. We have uniformly in $0 \le k \le 1.233 \cdot \lambda_{3g-3}$

$$\mathbb{P}\big(K_g(\gamma) = k{+}1\big) = e^{-\lambda_{3g-3}} \cdot \frac{\lambda_{3g-3}^k}{k!} \cdot \left(\frac{\sqrt{\pi}}{2\Gamma\left(1 + \frac{k}{2\lambda_{3g-3}}\right)} + O\left(\frac{k}{(\log g)^2}\right)\right)\,.$$

Keystone underlying results and further conjectures

Our results use the Delecroix–Goujard–Zograf–Zorich'19 conjecture proved in

Theorem (Aggarwal'21). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol} \mathcal{Q}_g \sim \frac{4}{\pi} \cdot \left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

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The similar conjecture of Eskin–Zorich'03 on the large genus asymptotics of Masur–Veech volumes of individual strata of *Abelian* differentials is recently proved by Aggarwal'19 and by Chen–Möller–Sauvaget–Zagier'20. The analogous conjecture for *quadratic* differentials still resists:

Conjecture (ADGZZ'20). *The Masur–Veech volume of any stratum of meromorphic quadratic differentials with at most simple poles has the following large genus asymptotics (with the error term uniformly small for all partitions $\boldsymbol{d}$):*

$$\operatorname{Vol}\mathcal{Q}(d_1,\dots,d_n) \overset{?}{\sim} \frac{4}{\pi}\cdot\prod_{i=1}^{n}\frac{2^{d_i+2}}{d_i+2} \quad \text{as } g \to +\infty\,,$$

under assumption that the number of simple poles is bounded or grows much slower than the genus.

Another Keystone result and one more conjecture

We also strongly use the uniform large genus asymptotics of ψ-classes which we conjectured in 2019 and which was proved by Aggarwal:

Theorem (Aggarwal'21). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1 + \cdots + d_n = 3g-3+n$*, as* $g \to +\infty$*.*

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Conjecture* (Delecroix–Goujard–Zograf–Zorich). *The distribution of the number of maximal horizontal cylinders in a random Abelian square-tiled surfaces of genus* g *gets very well approximated by the distribution of the number of disjoint cycles in a uniformly random permutation of* $4g-3$ *elements as* $g \to \infty$.

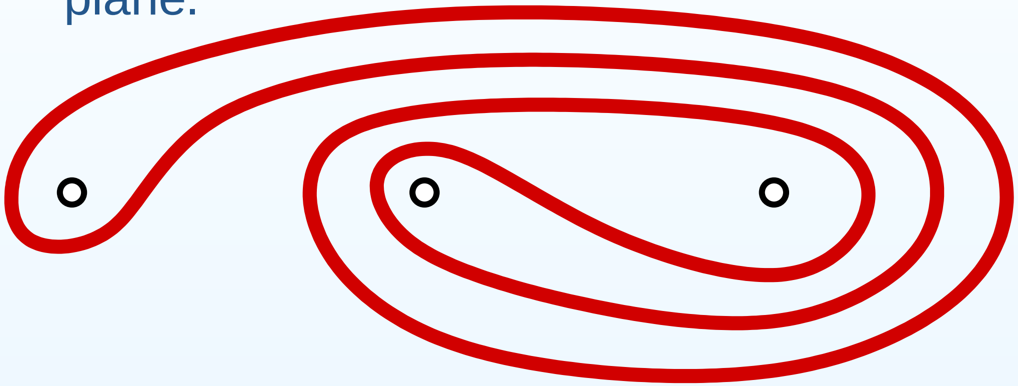
We already proved that a random square-tiled surface in a stratum $\mathcal{H}$ has a single cylinder with probability close to $\frac{1}{\dim \mathcal{H}}$.

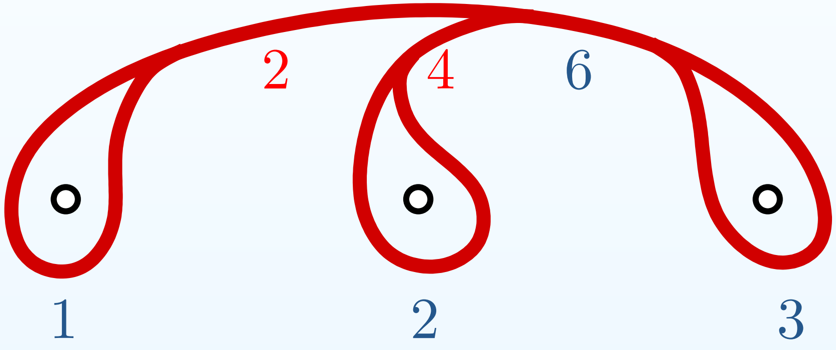
* About 2 years of CPU-time of two independent computer experiments with strata of genera from 40 to 10 000.

Train track coordinates (after section 15.1 of the book of B. Farb and D. Margalit "A Primer on Mapping Class Groups")

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.





We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

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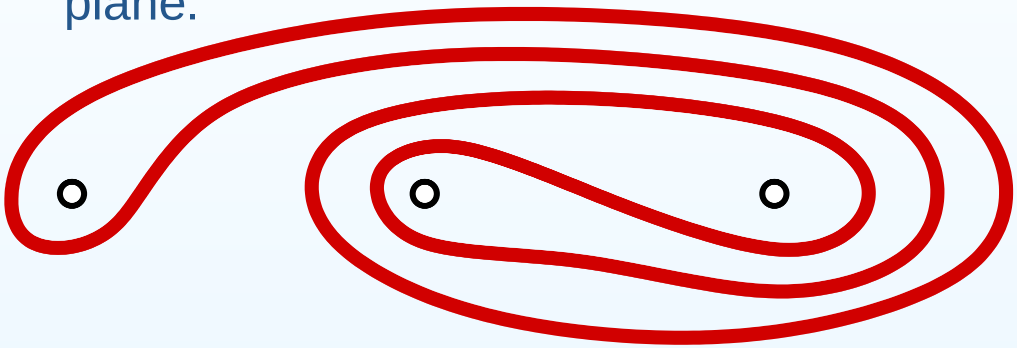

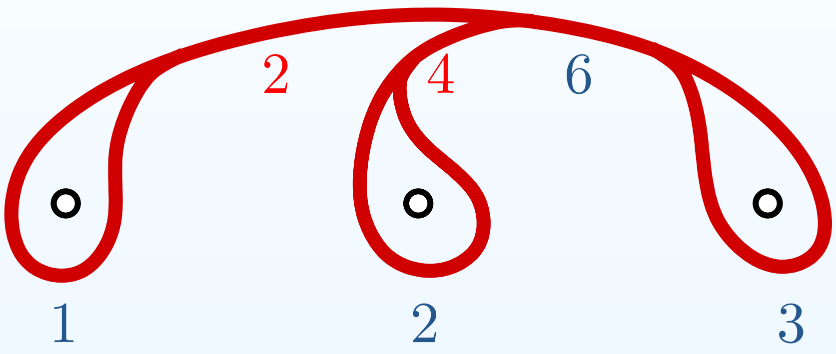

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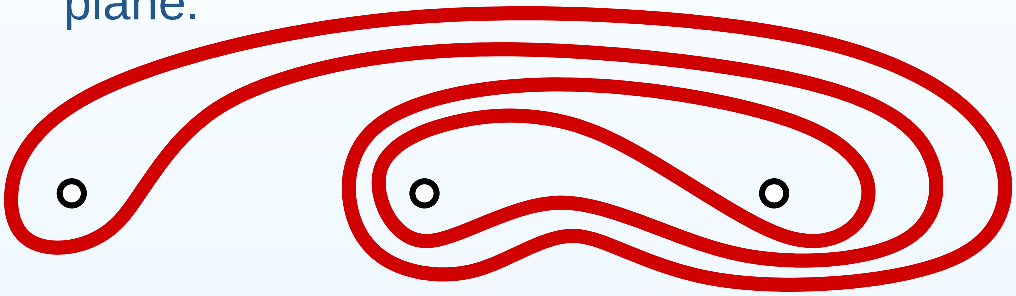

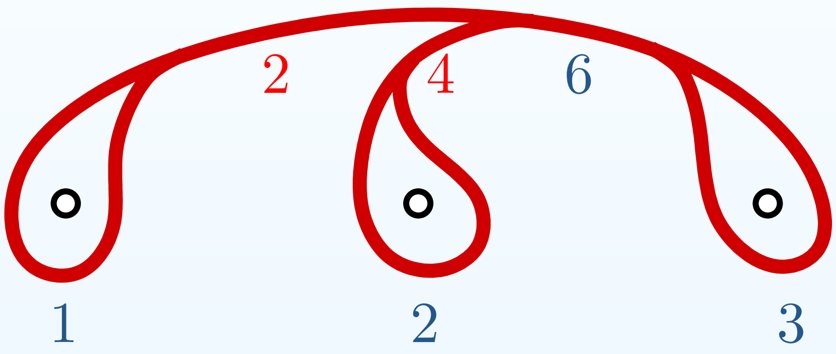

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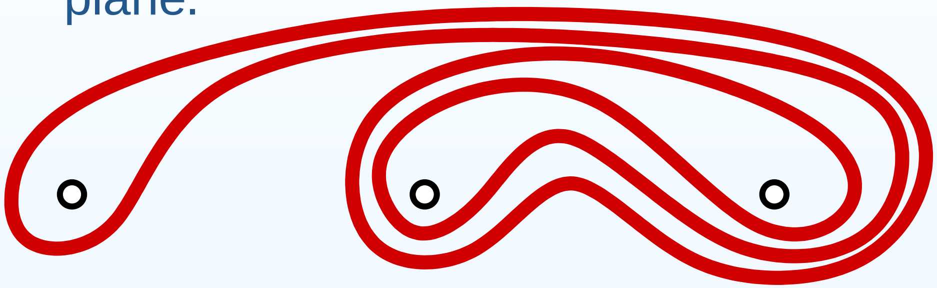

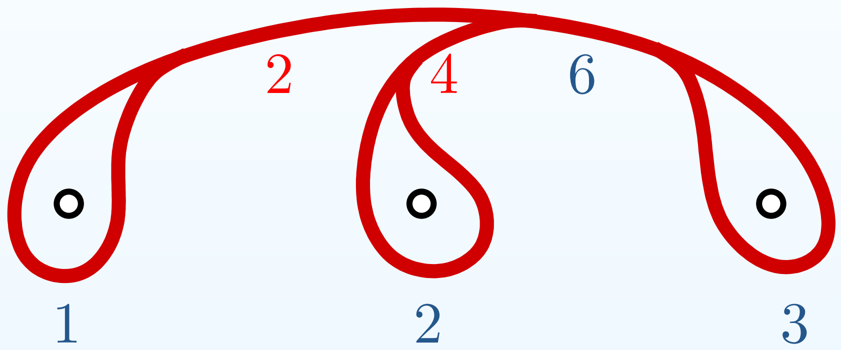

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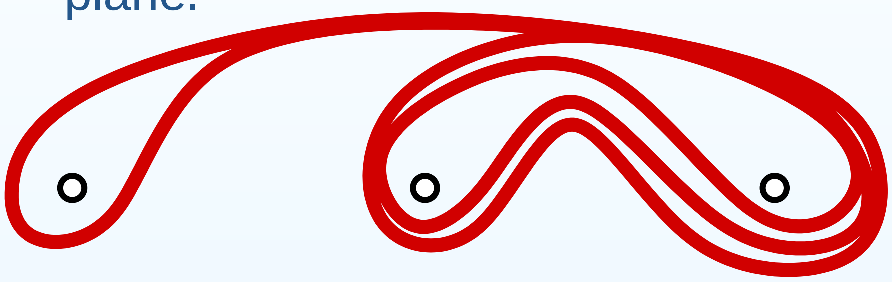

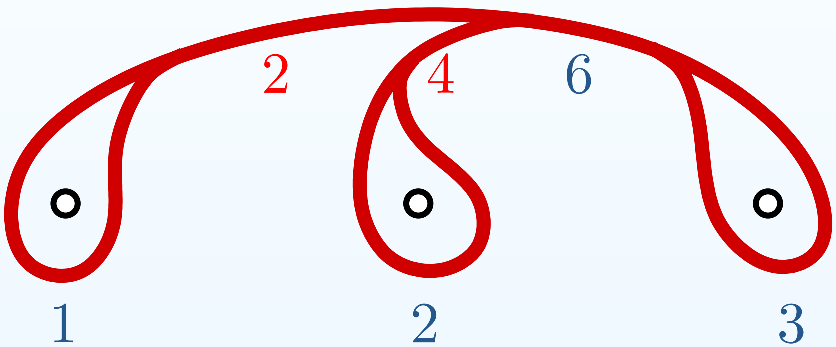

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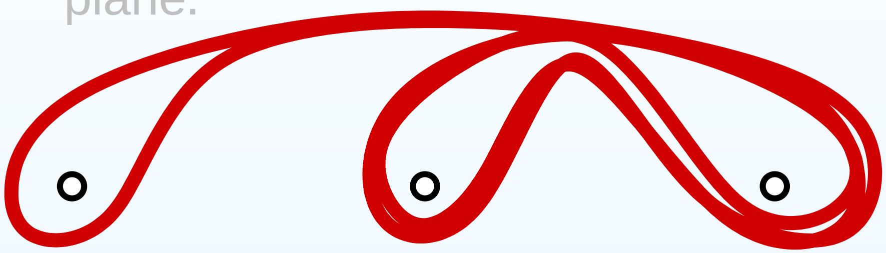

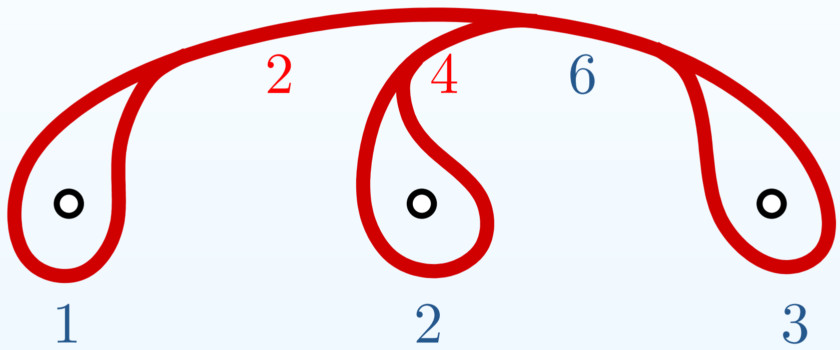

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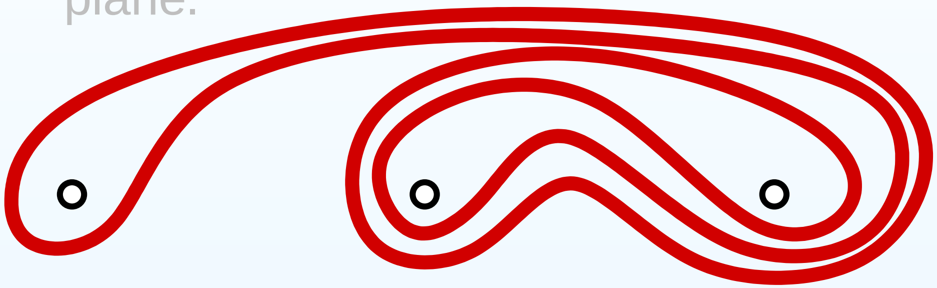

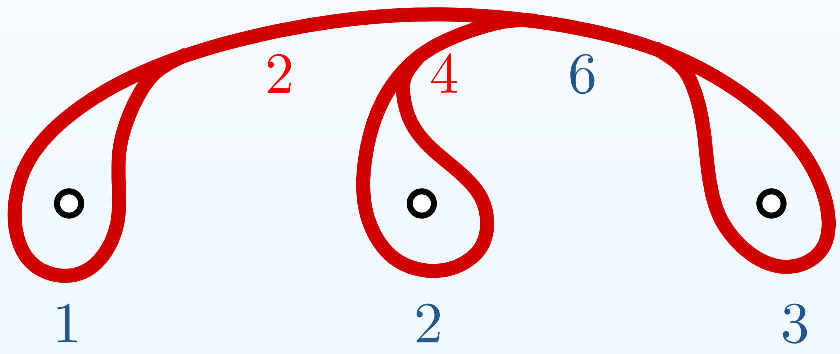

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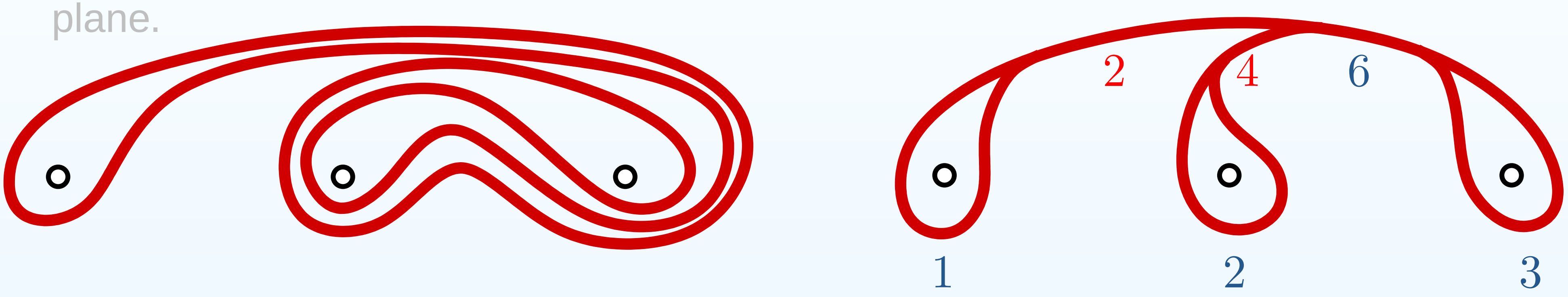

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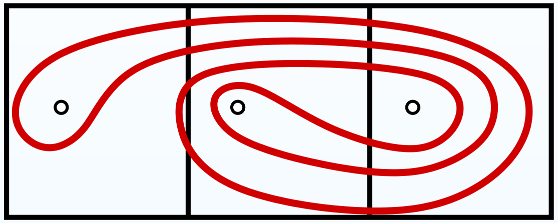
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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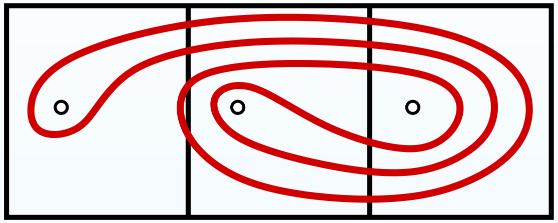

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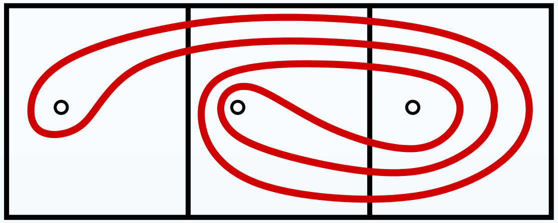

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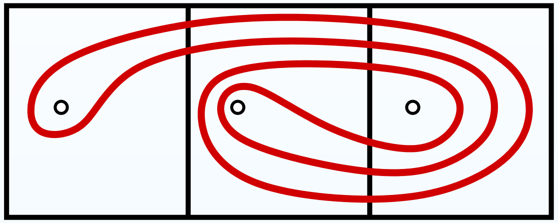
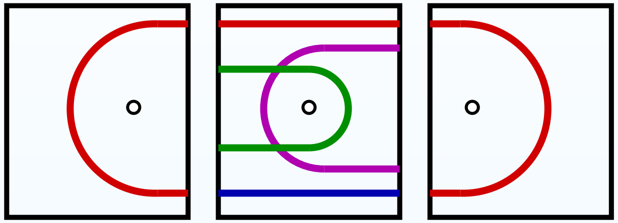
Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:





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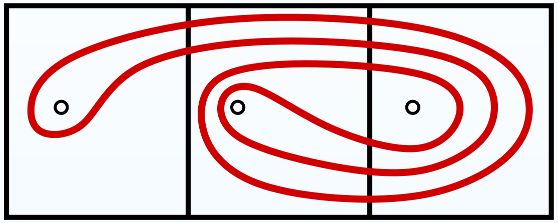

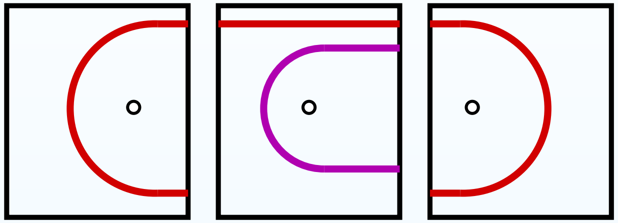

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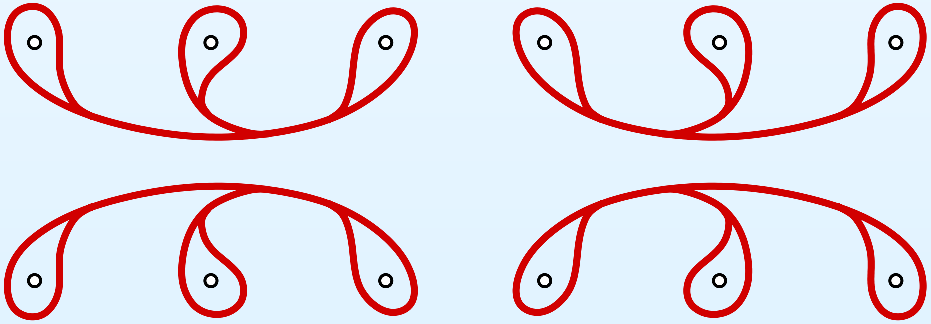
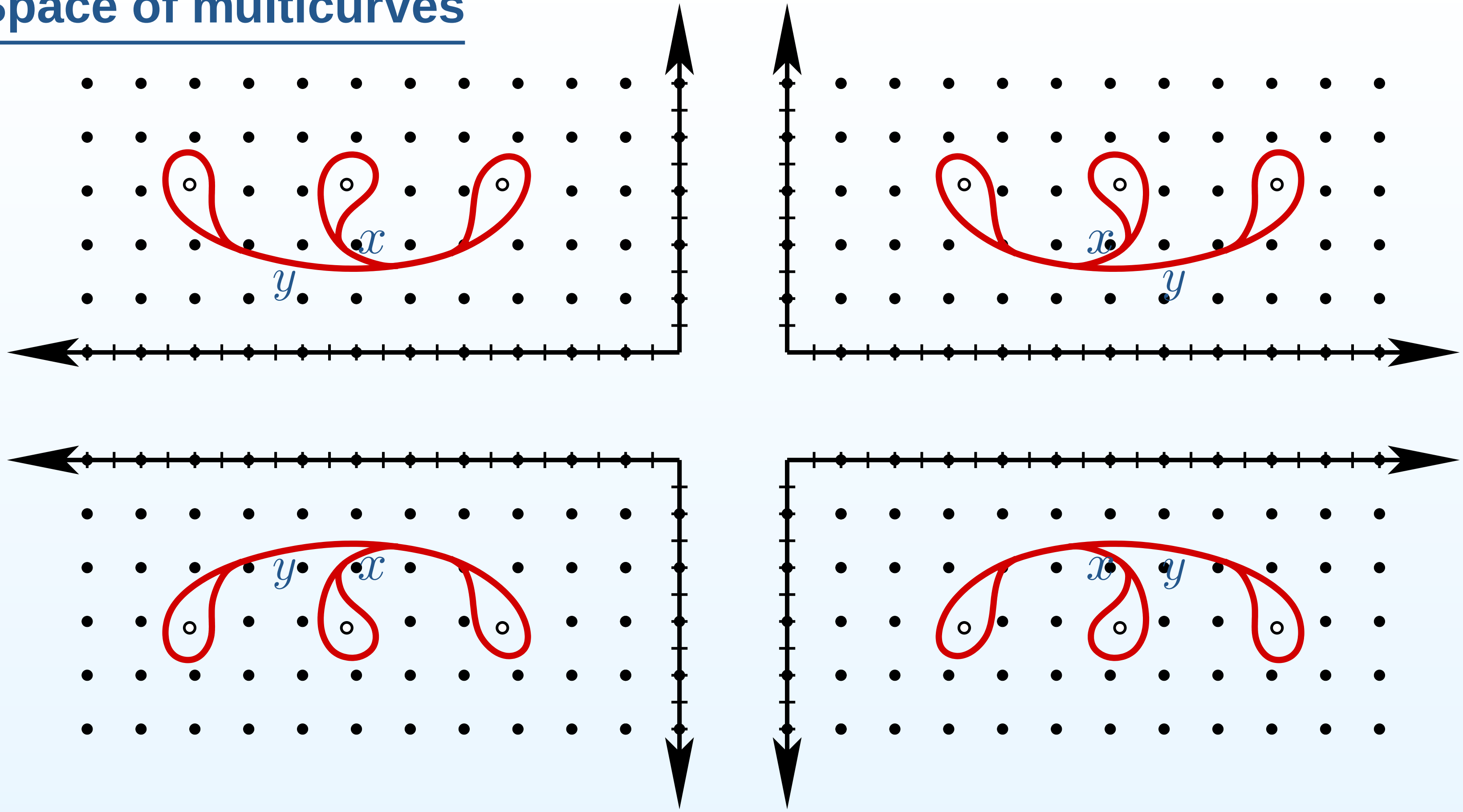
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve in is carried by one of the following four train tracks:

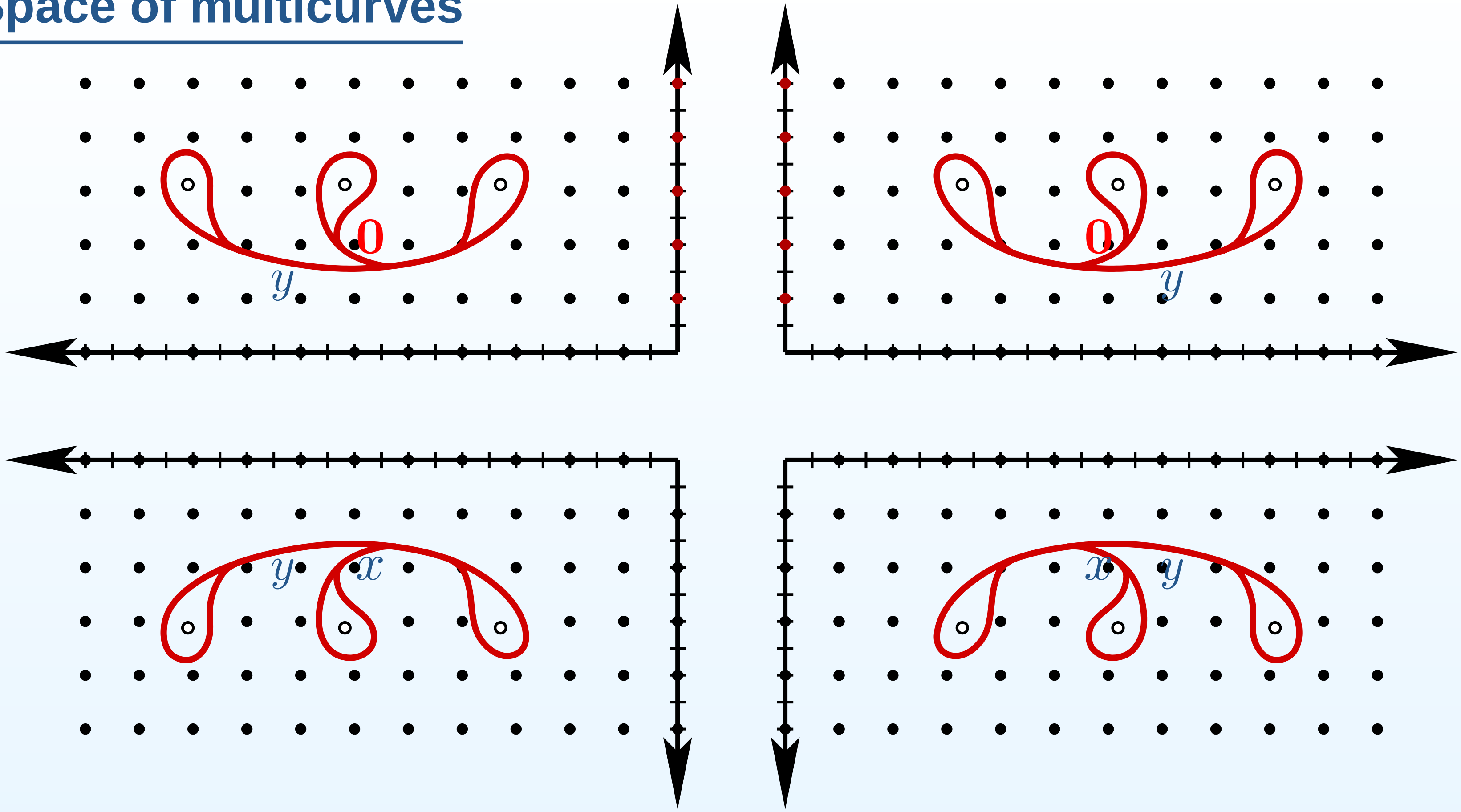


Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

Space of multicurves



Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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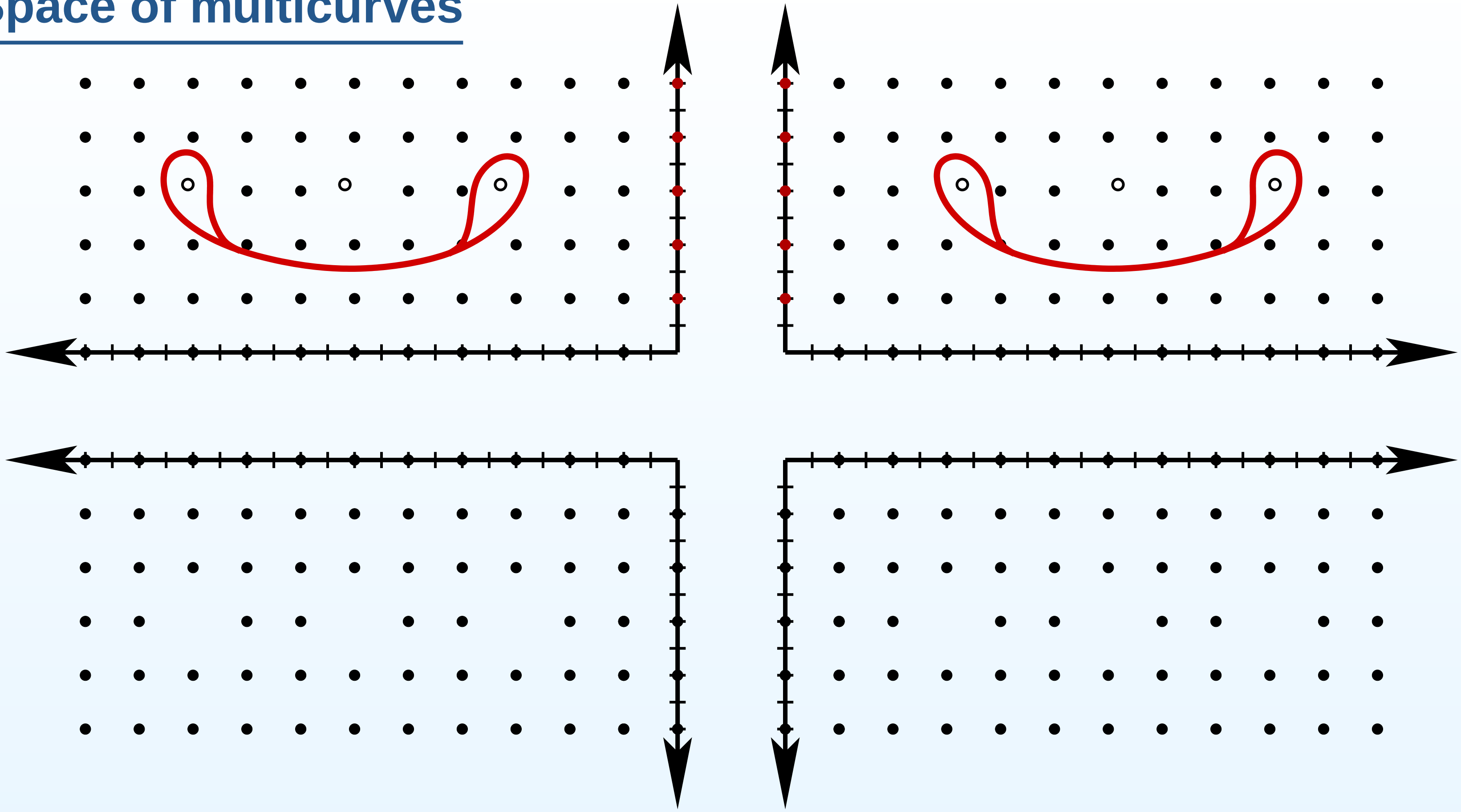

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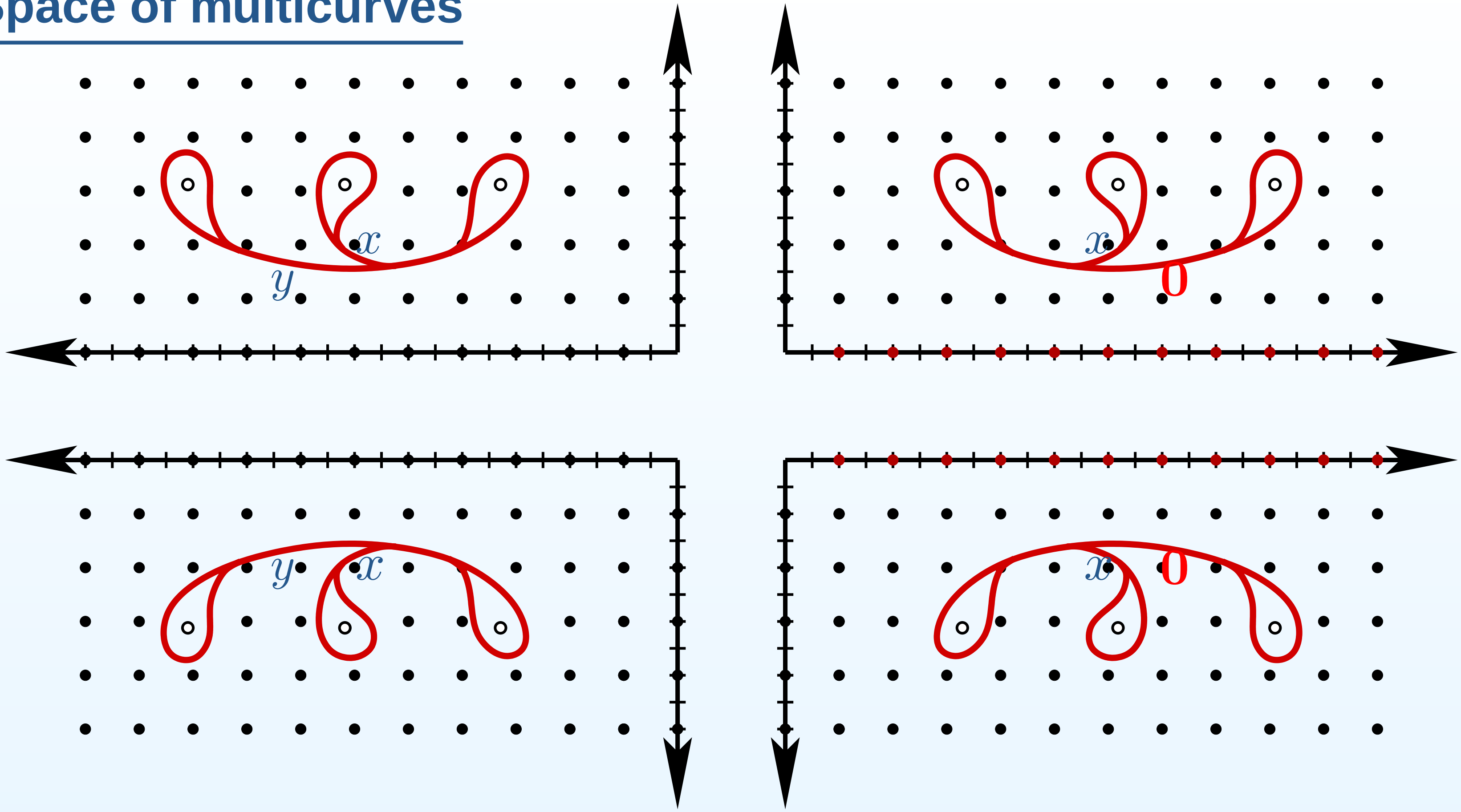

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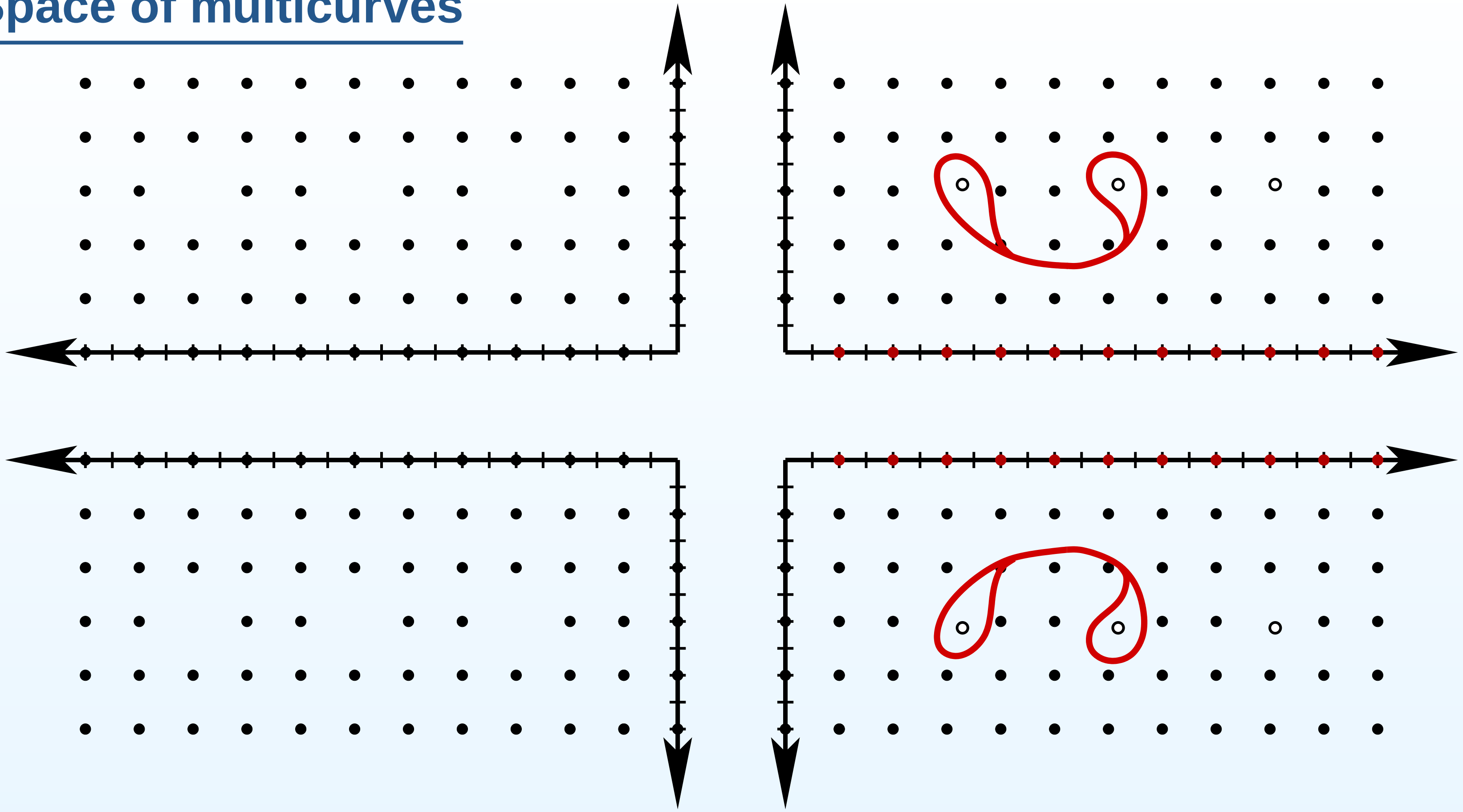

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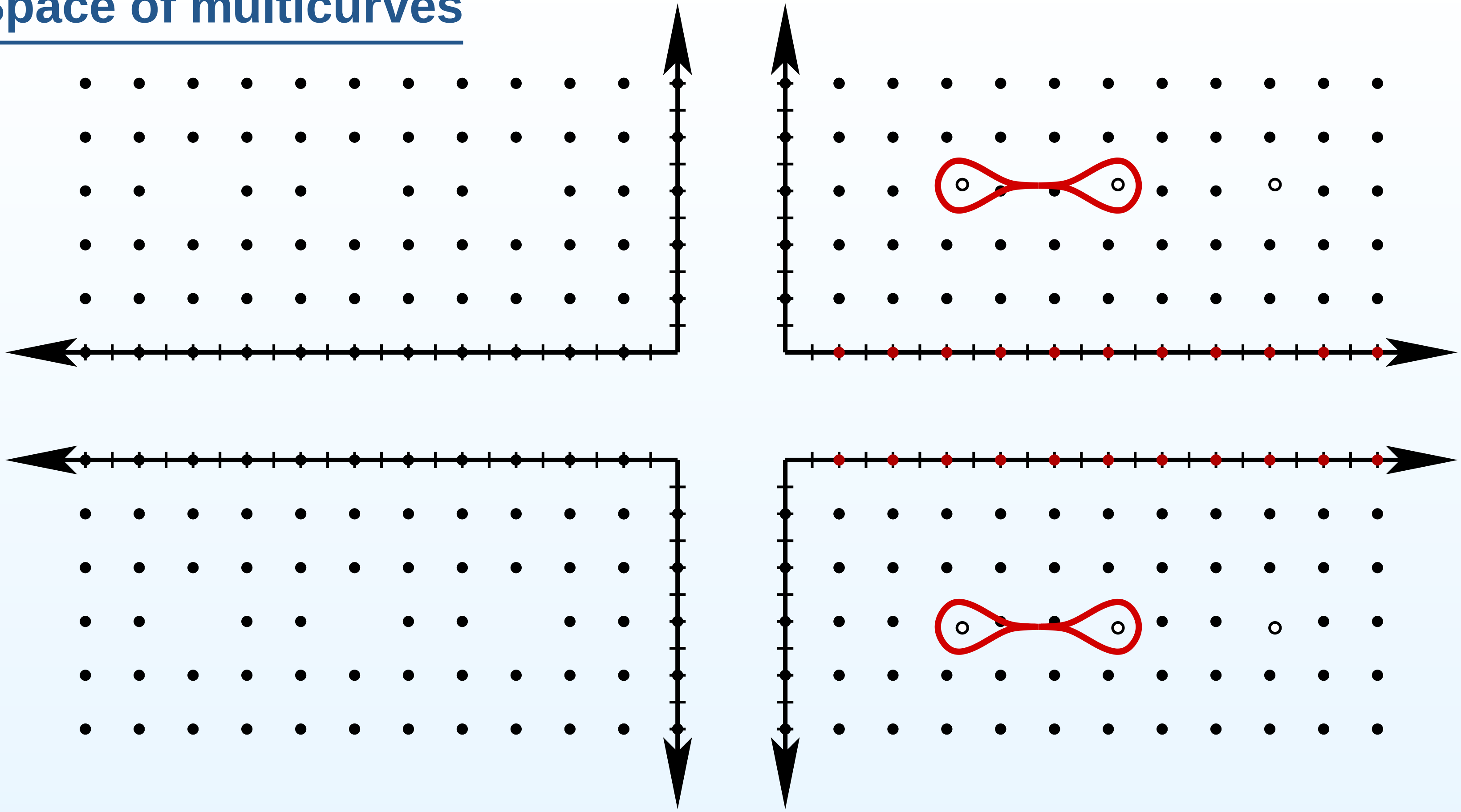

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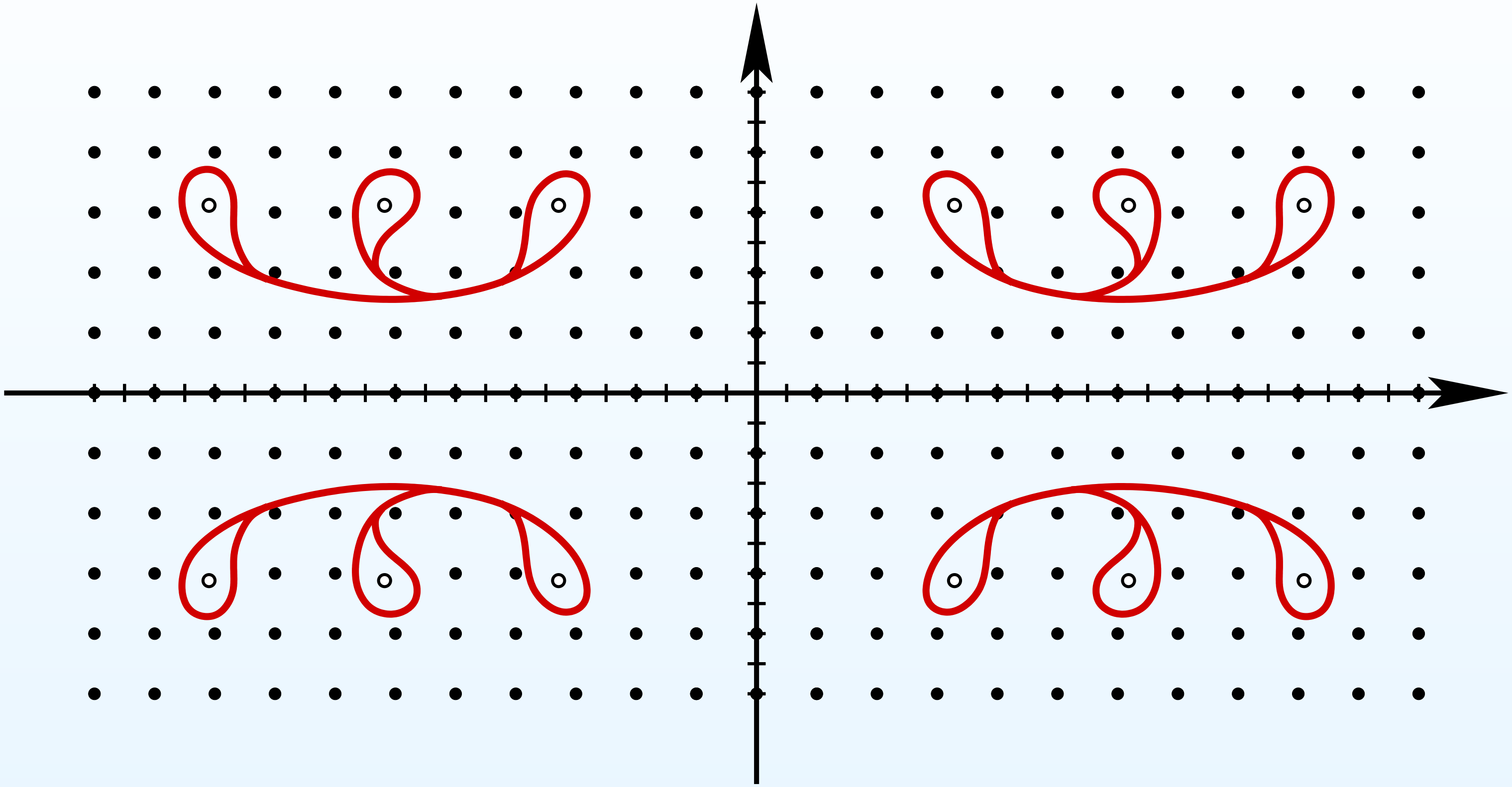

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