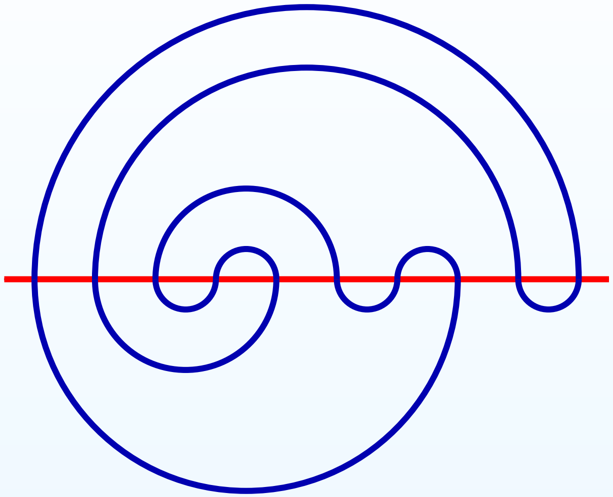
# Enumeration of meanders and volumes of moduli spaces of quadratic differentials

Anton Zorich
after a joint work with V. Delecroix, E. Goujard and P. Zograf
(partly published in Forum of Mathematics Pi, **8:4** (2020)).

HSE, February 24, 2022

# Meanders

## Meanders and arc systems

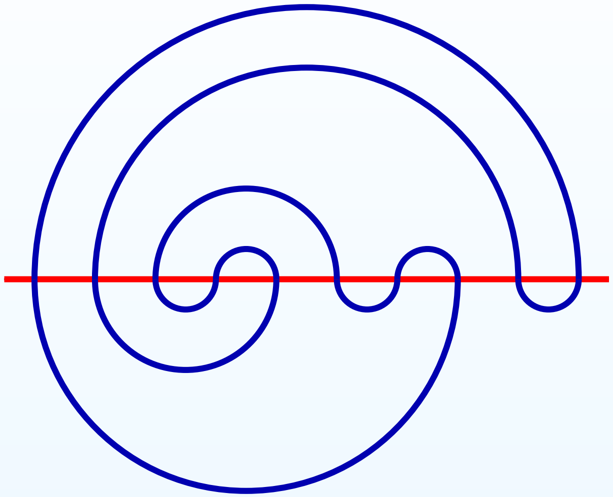


A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in mathematics, physics and biology.

## Meanders and arc systems

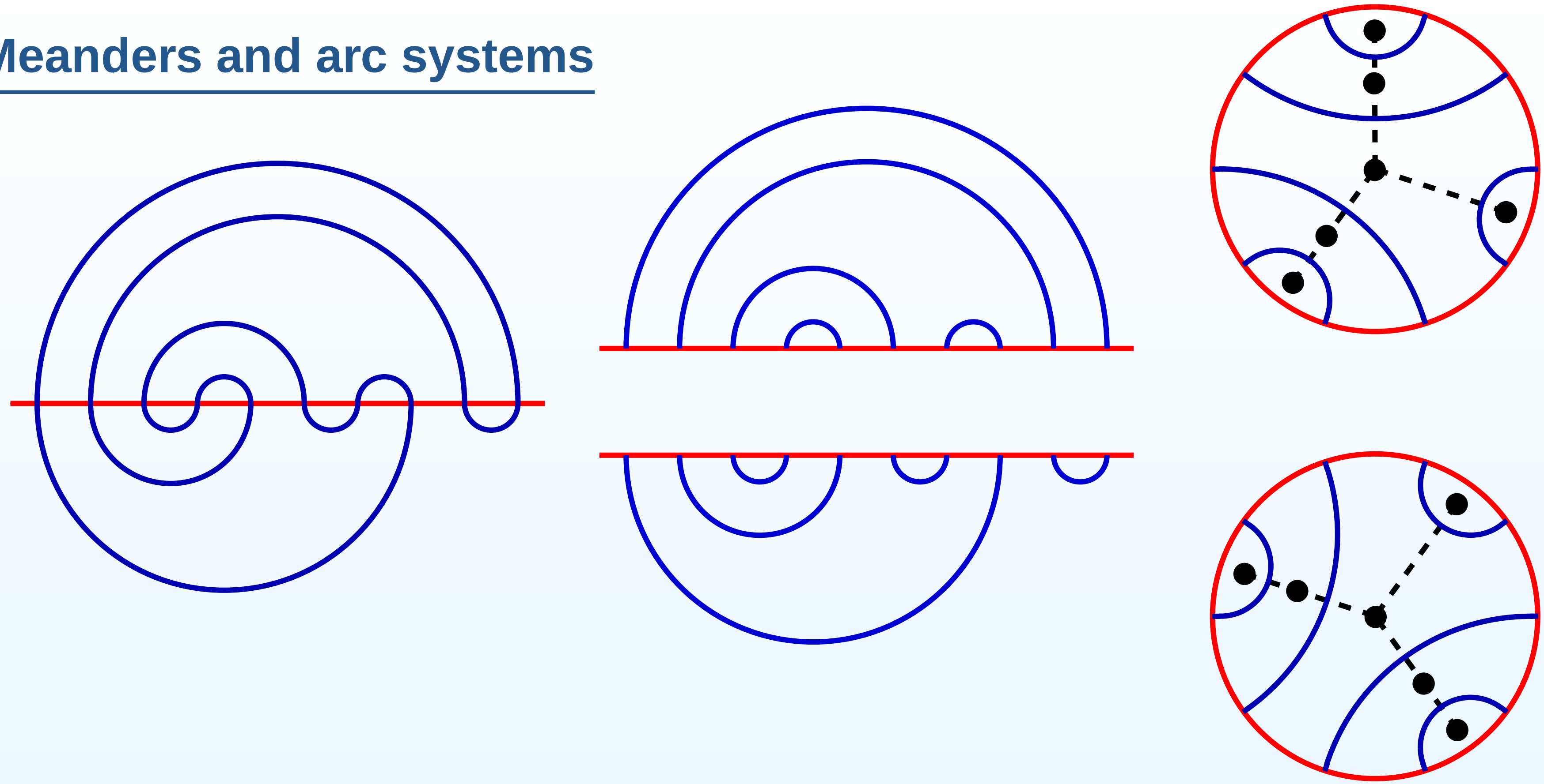


**Conjecture (S. Lando and A. Zvonkin, 1993).** *The number of meanders with $2N$ crossings is asymptotic to*

$$const \cdot R^{2N} \cdot N^{\alpha} \quad \text{for} \quad N \to \infty\,,$$

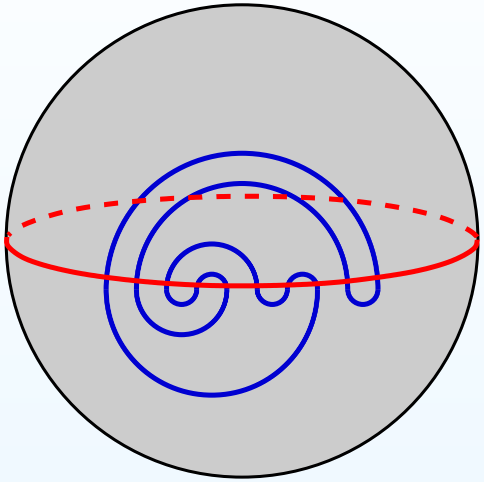
*where $R^2 \approx 12.26$ (value is due to I. Jensen) and $\alpha = -\frac{29+\sqrt{145}}{12}$ (conjectural value due to P. Di Francesco, O. Golinelli, E. Guitter, 1997).*

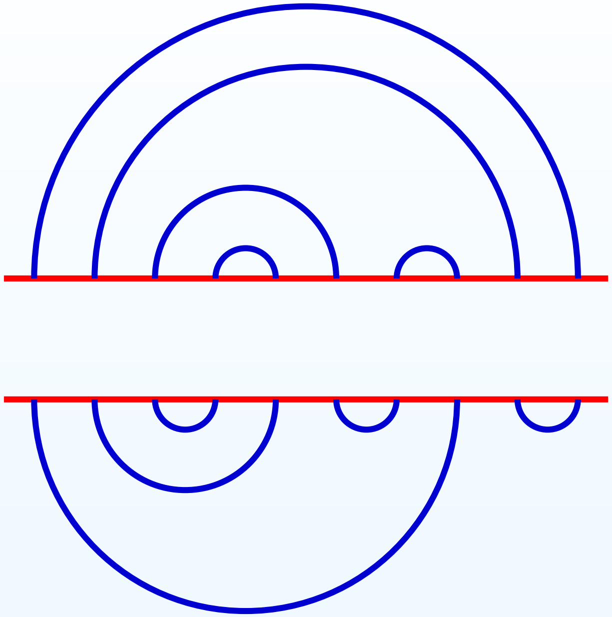
## Meanders and arc systems

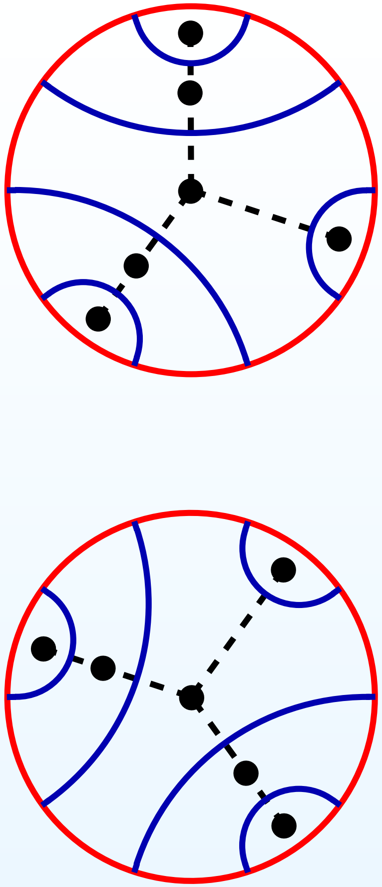


A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence $2$ from dual trees.

## Meanders and arc systems





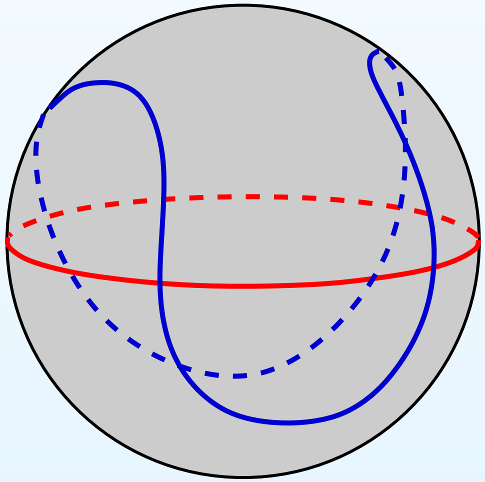


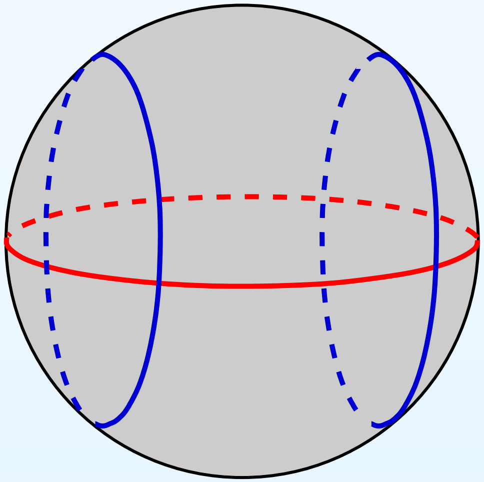
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence $2$ from dual trees.

Compactifying the plane (left picture) with one point at infinity, or gluing together arc systems on the two discs (right picture) we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

## Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.

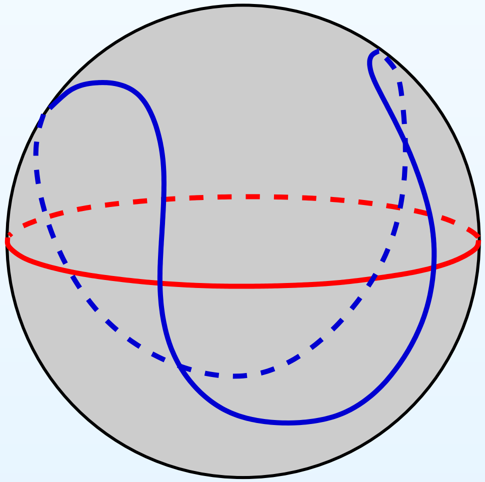


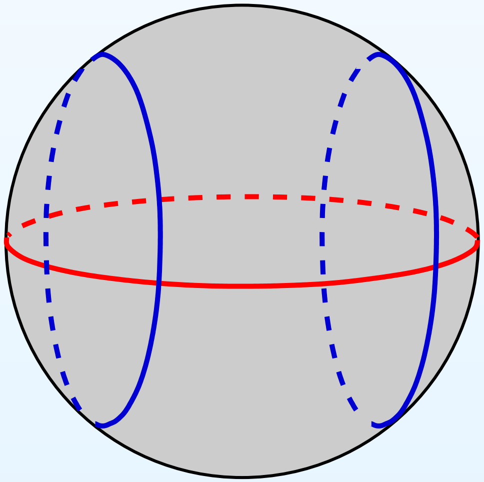


Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

## Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

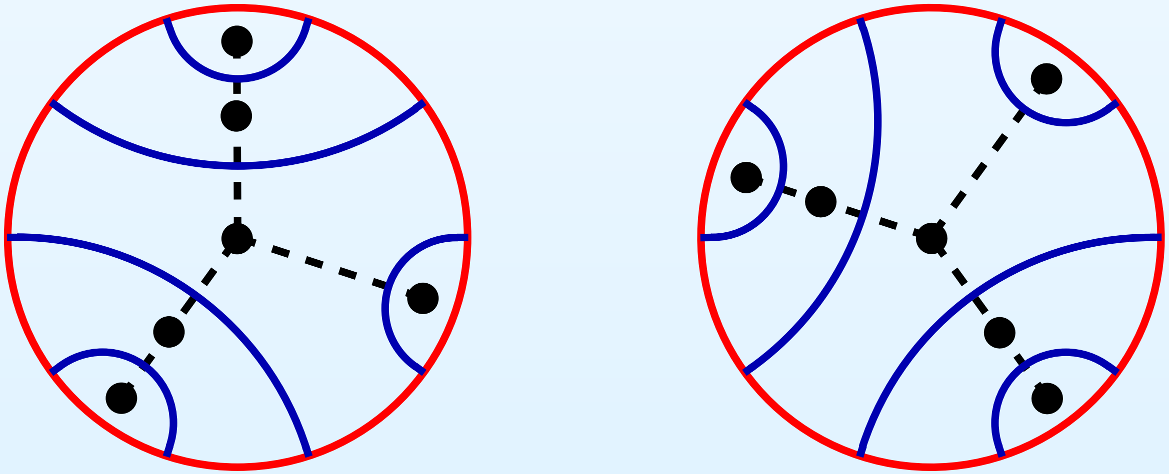
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



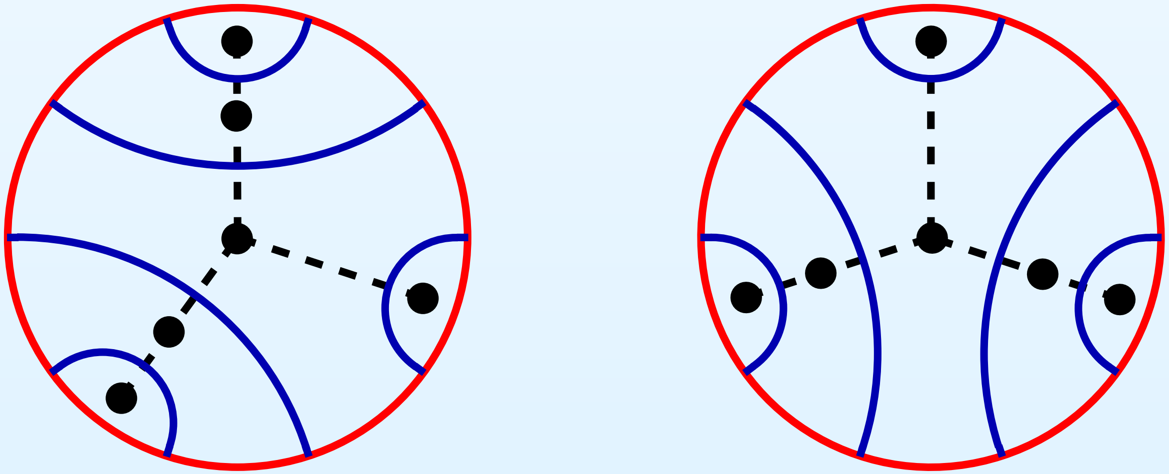
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



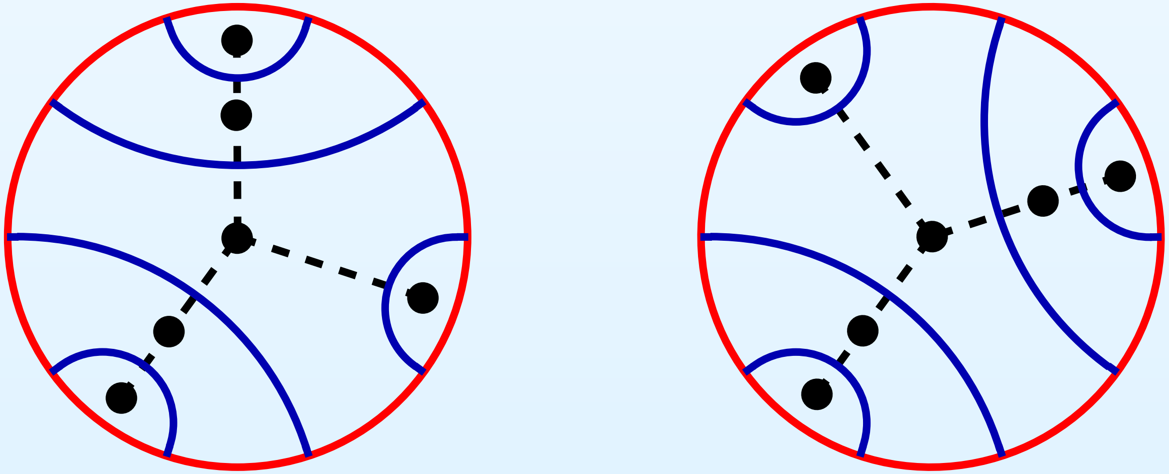
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



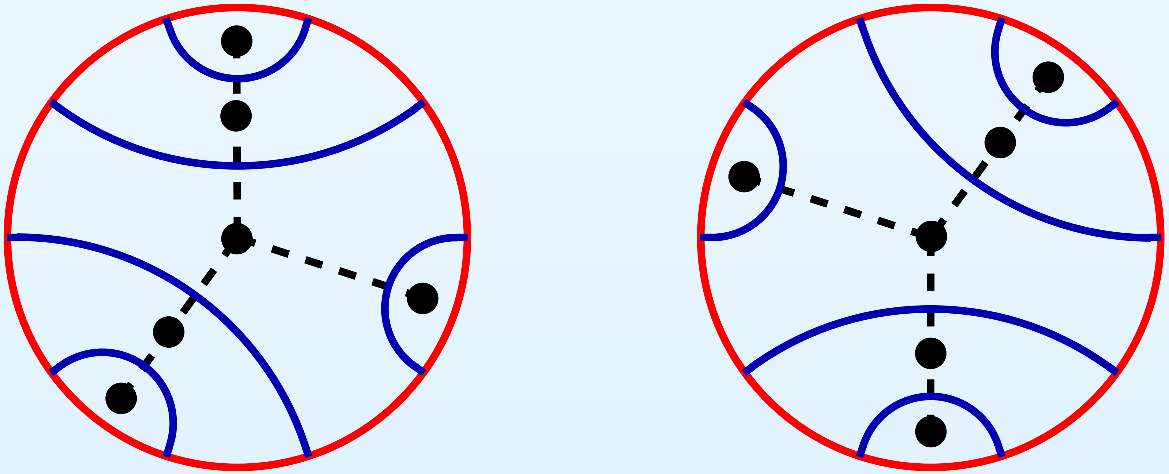
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



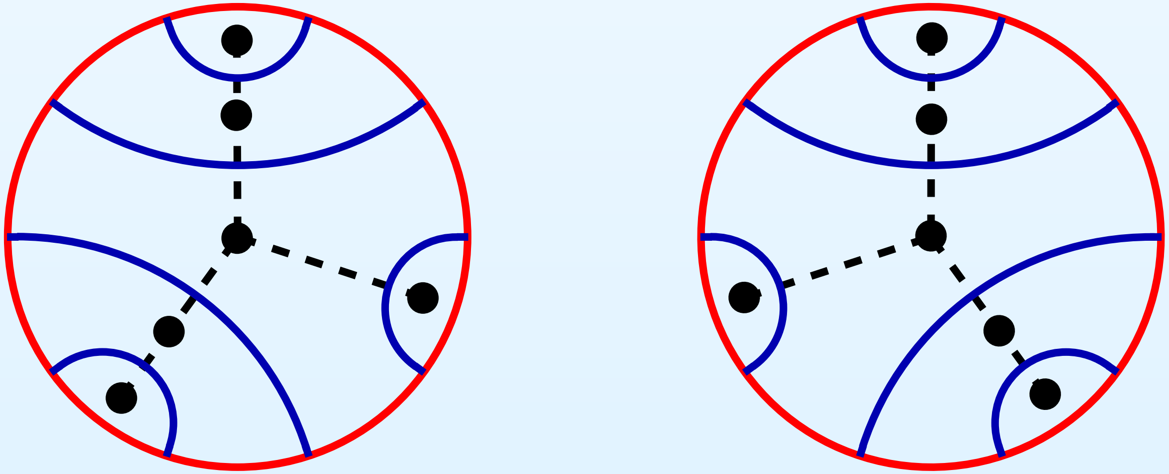
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



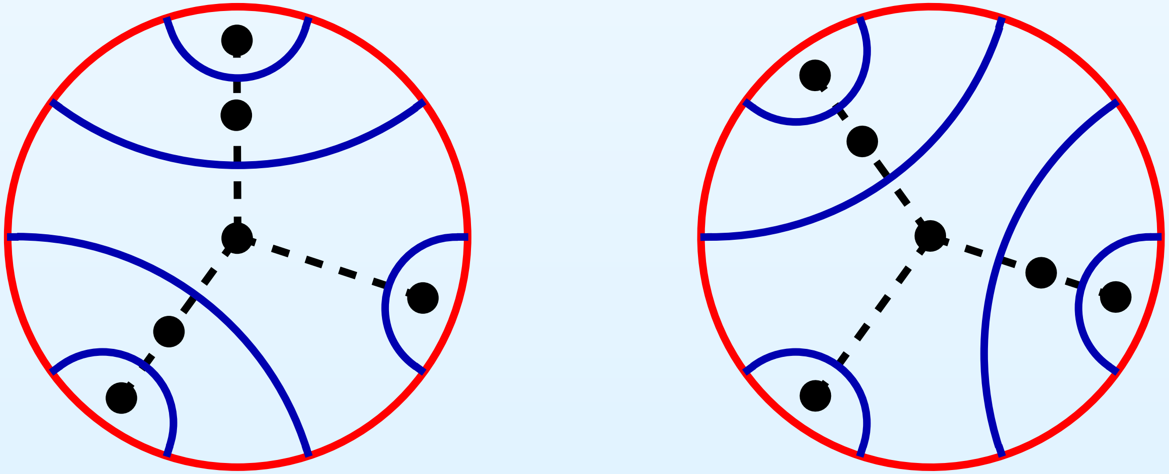
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



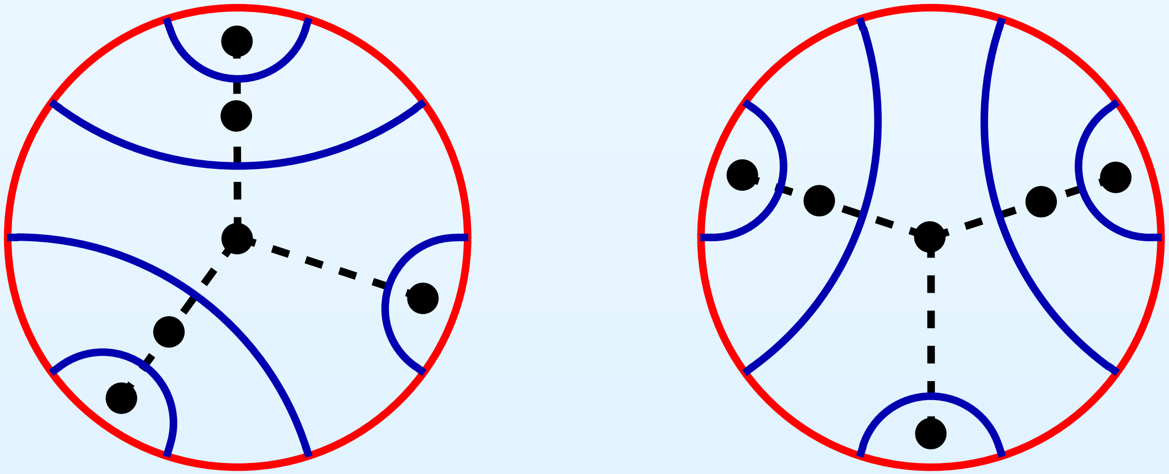
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



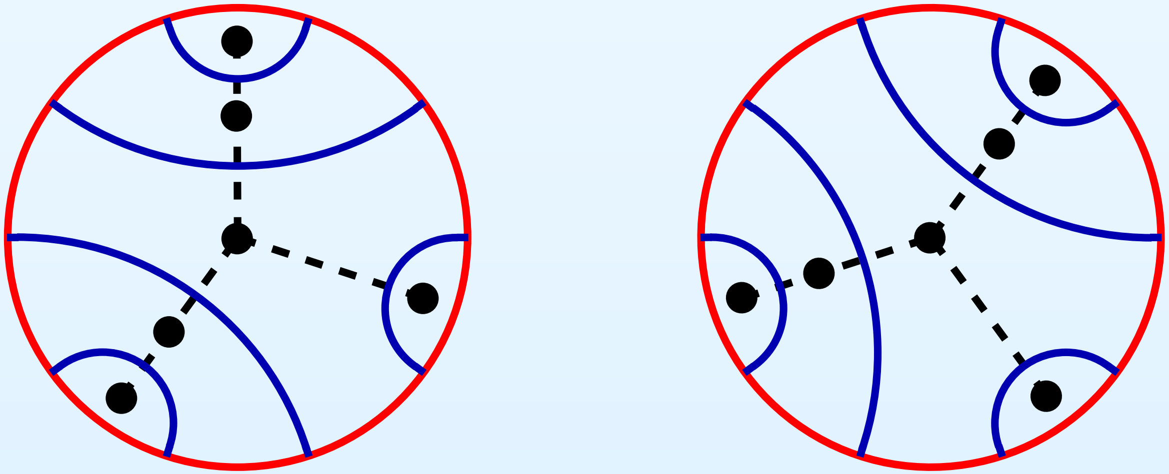
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



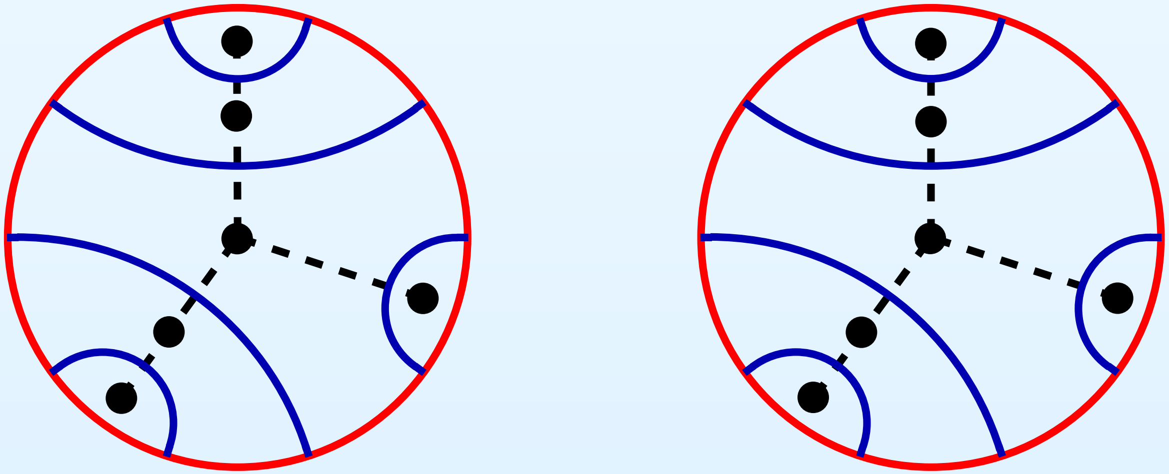
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\ n\text{-arc system of type } \mathcal{T}_{South};\ \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



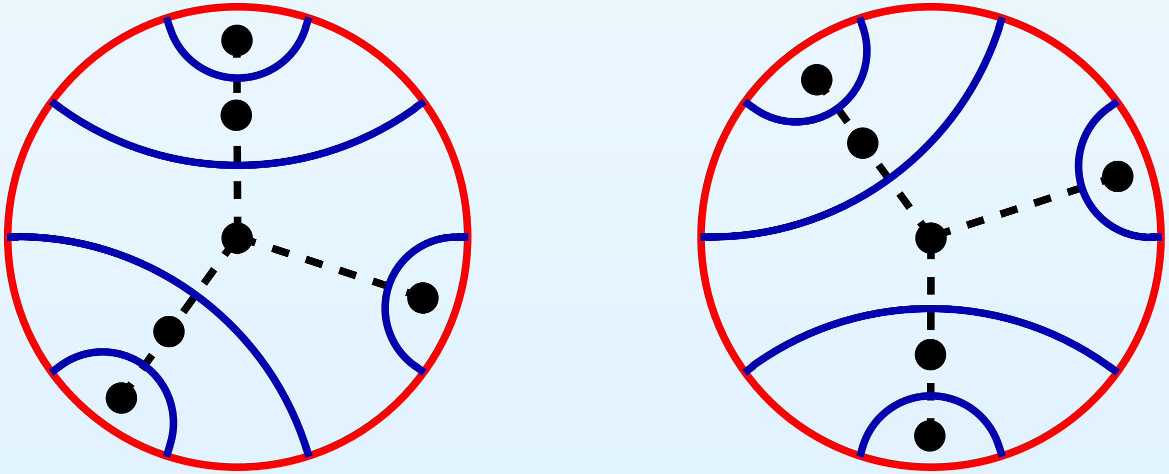
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



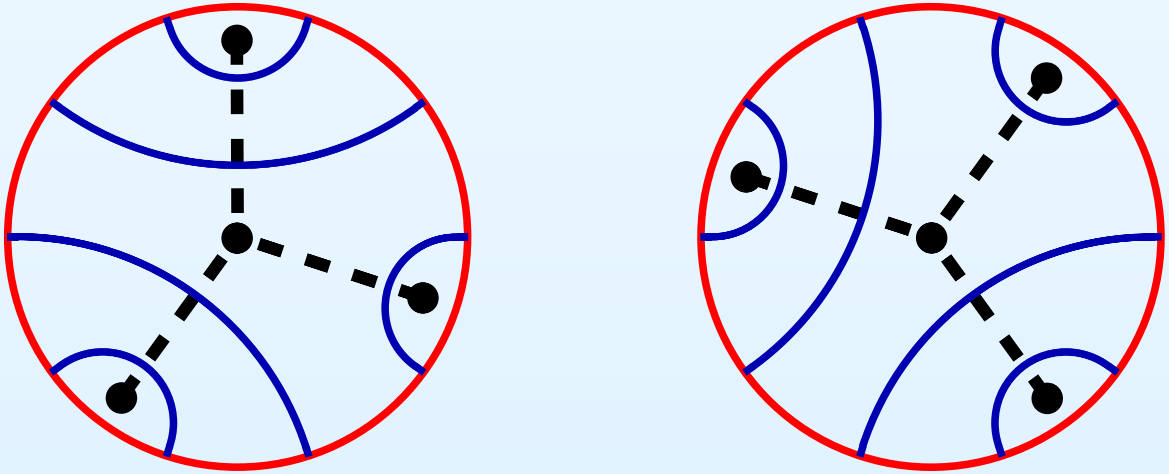
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}} \,.$$



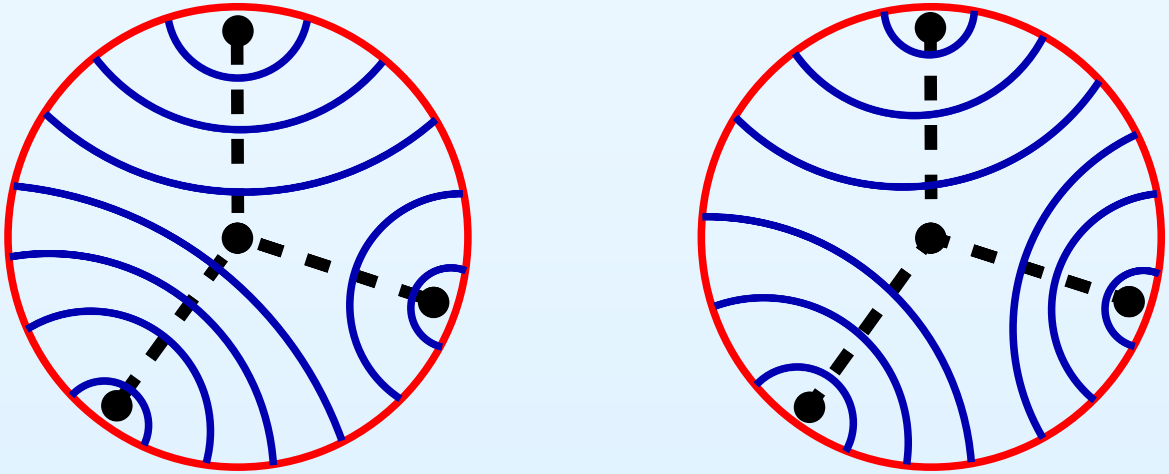
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



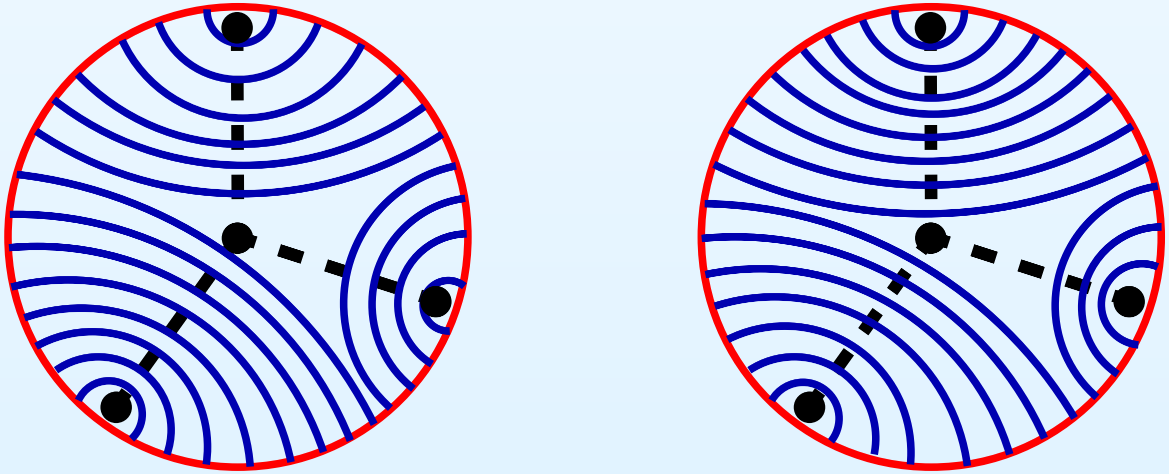
## Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence $2$. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

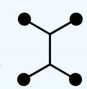
$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$

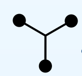
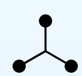


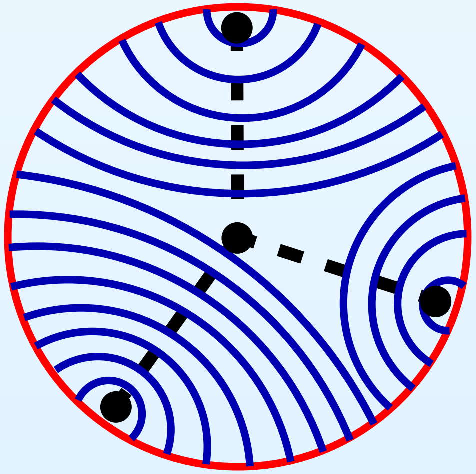
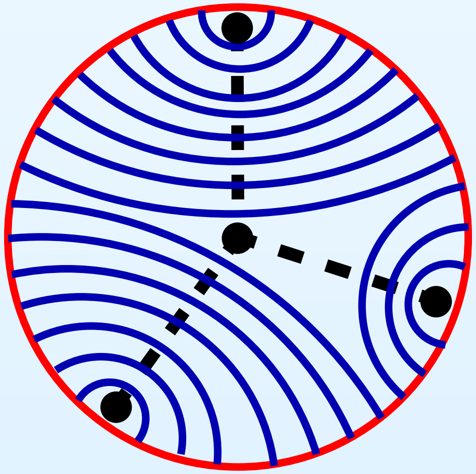
## Asymptotic frequency of meanders

**Theorem.** *Let $p_{North}, p_{South} \geq 2$. Let $p = p_{North} + p_{South}$. The frequency $\mathrm{P}_{connected}(p_{North}, p_{South}; N)$ of meanders obtained by all possible identifications of all arc systems with at most $N$ arcs represented by all possible pairs of plane trees having $p_{North}, p_{South}$ of leaves (vertices of valence one) has the following limit:*

$$\lim_{N\to+\infty} \mathrm{P}_{connected}(p_{North}, p_{South}; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}\,.$$

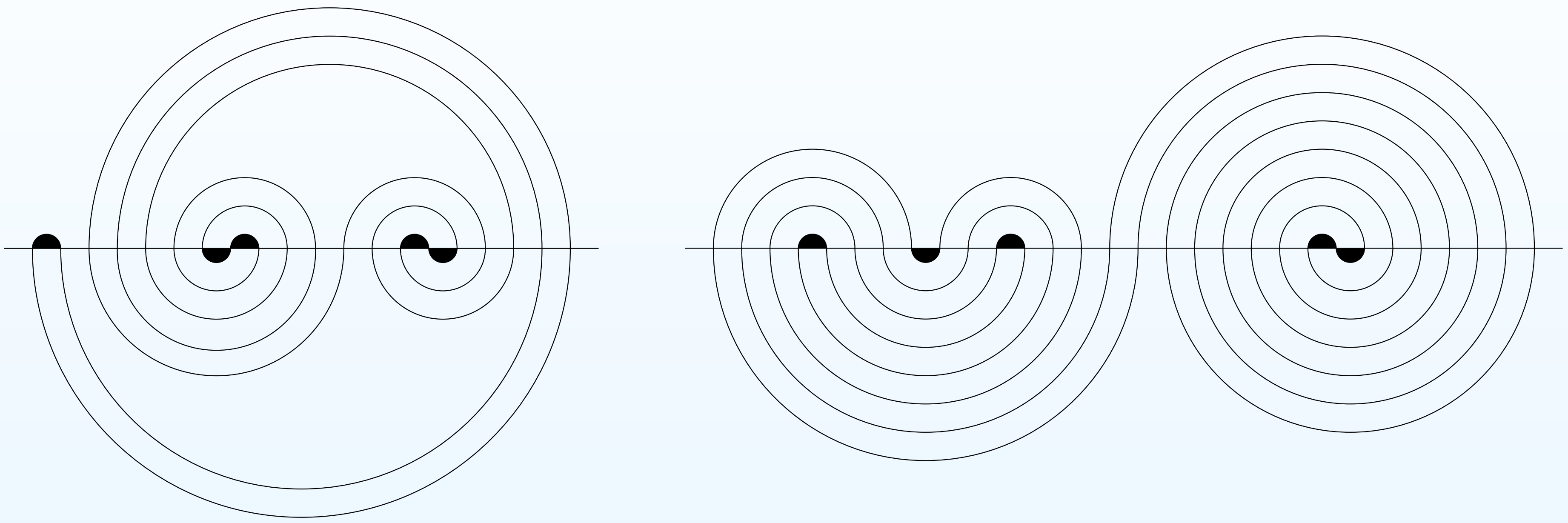
**Example.** $\lim_{N\to+\infty} \mathrm{P}_{connected}($  ,  $, N) =$

$= \lim_{N\to+\infty} \mathrm{P}_{connected}($  ,  $, N) = \frac{280}{\pi^6} \approx 0.291245\,.$

## Meanders with and without maximal arc

These two meanders have $5$ minimal arcs (“pimples”) each.



Meander with a maximal arc (“rainbow”) contributes to $\mathcal{M}_5^+(N)$

Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc (“rainbow”) and having at most $2N$ crossings with the horizontal line and exactly $p$ minimal arcs (“pimples”). We consider $p$ as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

## Counting formulae for meanders

**Theorem.** *For any fixed $p$ the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with $p$ minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

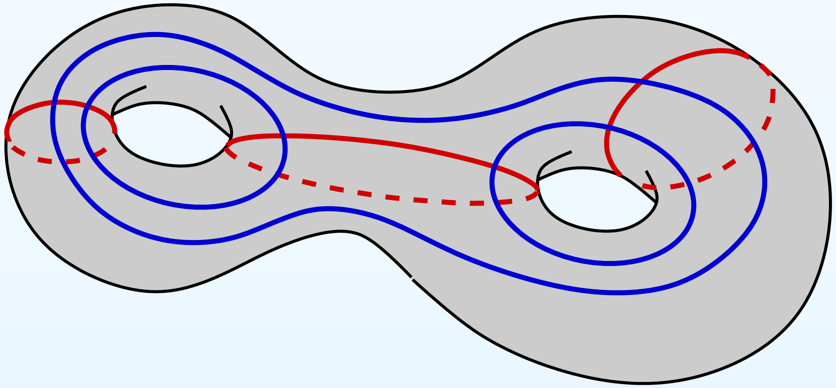
$$\mathcal{M}_p^+(N) = \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

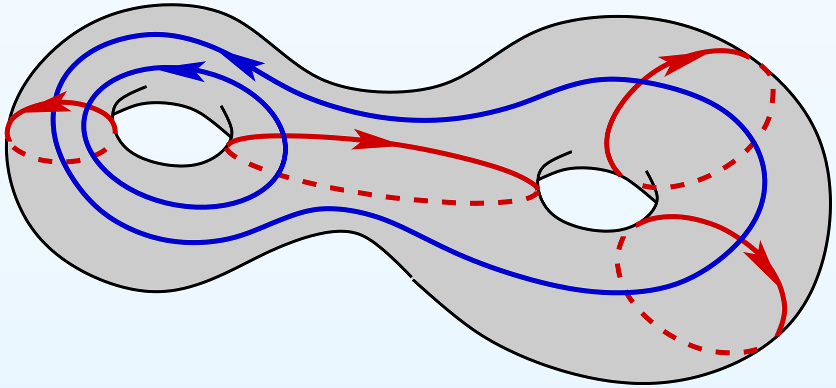
$$\mathcal{M}_p^-(N) = \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as $N^{2p-4}$ while $\mathcal{M}_p^-(N)$ grows as $N^{2p-5}$.

## Meanders in higher genera

A pair of smooth simple closed transverse oriented multicurves is called *positively intersecting* if each connected component of each multicurve is oriented in such way that all intersections match the orientation of the surface. A pair of transverse multicurves is called *orientable* if it admits an orientation with positive intersection and *non-orientable* otherwise.





**Exercise.** Verify that the pair of transverse multicurves on the right is positively intersecting and that the pair of multicurves on the left is non-orientable.

**Definition.** A *meander* on a surface of genus $g$ is an ordered pair of smooth transverse simple closed curves considered up to a diffeomorphism of the surface. Similarly we define *positive meanders*.

## Results: general (non-orientable) case

Fix the genus $g$ of the surface. Fix a nonnegative integer $p$ denoting the number of bigons produced by intersections of pairs of multicurves.

**Observation.** *The following quantities have polynomial asymptotics:*

- *Number of pairs of transverse multicurves with at most $N$ intersections and with exactly $p$ bigons $= c(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of pairs (simple closed curve, transverse multicurve) with at most $N$ intersections and $p$ bigons $= c_1(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of meanders with at most $N$ intersections and with exactly $p$ bigons $= c_{1,1}(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*

**Theorem.** *The coefficients $c(g,p), c_1(g,p), c_{1,1}(g,p)$ satisfy the following relation:*

$$\frac{c_1(g,p)}{c(g,p)} = \frac{c_{1,1}(g,p)}{c_1(g,p)}\,.$$

## Results: general (non-orientable) case

The coefficients in these asymptotics have the following arithmetic nature:

$$c(g,p) = r(g,p) \cdot \pi^{6g-6+2n}, \quad c_1(g,p) = r_1(g,p); \quad r(g,p),\, r_1(g,p) \in \mathbb{Q}\,.$$

For small $p$ and for $g$, say, up to $500$, we can compute the rational numbers $r(g,p)$ explicitly (based on results of D. Chen–M. Möller–A. Sauvaget combined with results of M. Kazarian or D. Yang, D. Zagier, and Y. Zhang).

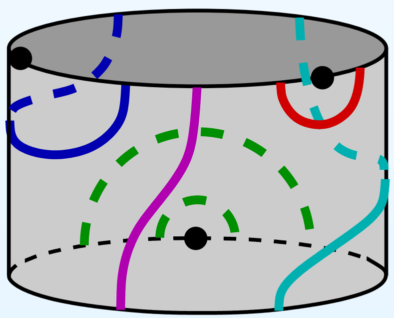
For the same range of $g$ and $p$ we can compute the rational numbers $r_1(g,p)$ explicitly (based on our own results).

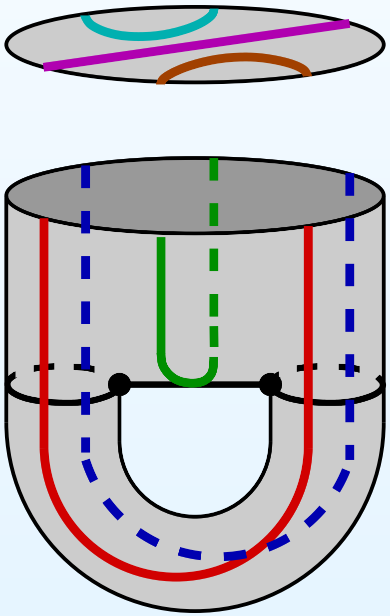
For any fixed $p$ we also have a simple asymptotic formulae for $r(g,p)$ and $r_1(g,p)$ as $g \to +\infty$ (our results combined with results of A. Aggarwal).

Since $c_{1,1}(g,p) = \dfrac{c_1^2(g,p)}{c(g,p)}$, in all these cases we get absolutely explicit asymptotic formulae for the count of meanders in genus $g$.

## Asymptotic frequency of meanders

As in genus $0$, we can construct multicurves from systems of arcs on a surface of genus $g-1$ with two boundary components or on a pair of surfaces of genera $g_1$ and $g_2$, where $g_1+g_2=g$, each with a single boundary component. As before we fix the total number $p$ of bigons. We assume that there are exactly $n$ arcs landing to each of the two boundary components, and that $n \leq N$.





**Theorem.** *The asymptotic probability to get a meander after a random gluing of a random system of arcs as above is* $\frac{c_1(g,p)}{c(g,p)}$.

## Results on positively intersecting pairs of multicurves

The case of positively intersecting pairs of multicurves is analogous. However, the power of $N$ and all the coefficients in the polynomial asymptotics do change. Fix the genus $g$ of the surface.

**Observation.** *The following quantities have polynomial asymptotics:*

- *Number of pairs of transverse positively intersecting multicurves with at most $N$ intersections $= c^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positively intersecting pairs (simple closed curve, transverse multicurve) with at most $N$ intersections $= c_1^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positive meanders with at most $N$ intersections $= c_{1,1}^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*

**Theorem.** *The coefficients $c^+(g), c_1^+(g), c_{1,1}^+(g)$ satisfy the relation:*

$$\frac{c_1^+(g)}{c^+(g)} = \frac{c_{1,1}^+(g)}{c_1^+(g)} \,.$$

## Results on positively intersecting pairs of multicurves

The coefficients in these asymptotics have the following arithmetic nature:

$$c^+(g) = r^+(g)\cdot\pi^{2g}, \quad c_1^+(g) = r_1^+(g), \text{ where } r^+(g),\, r_1^+(g) \in \mathbb{Q}\,.$$

(the first result was proved by A. Eskin and A. Okounkov; the second one is simple.) For $g$, say, up to $2000$, we can compute the rational numbers $r^+(g)$ explicitly (small $g$ — due to A. Eskin and A. Okounkov, 2003; larger $g$ — D. Chen–M. Möller–A. Sauvaget–D. Zagier, 2020).

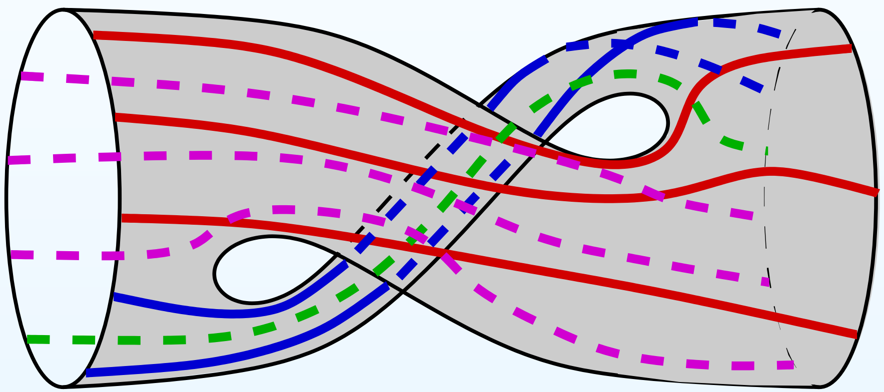
For any $g$ we have a simple formula for $r_1^+(g)$ (our own results, 2020).

We also have simple asymptotic formulae for $r^+(g)$ and $r_1^+(g)$ as $g \to +\infty$ (results of D. Chen–M. Möller–D. Zagier, 2018, for $r^+(g)$; independent result of A. Aggarwal, 2019; our result based on the result of D. Zagier, 1995, for $r_1^+(g)$).

Since $c_{1,1}^+(g) = \dfrac{\left(c_1^+(g)\right)^2}{c^+(g)}$, we can count positive meanders in genus $g$.

## Asymptotic frequency of positive meanders

As before we can glue systems of arcs on a surface of genus $g-1$ with two boundary components. This time we assume that each of $n$ arc goes from one boundary component to the other, and that $n \le N$.

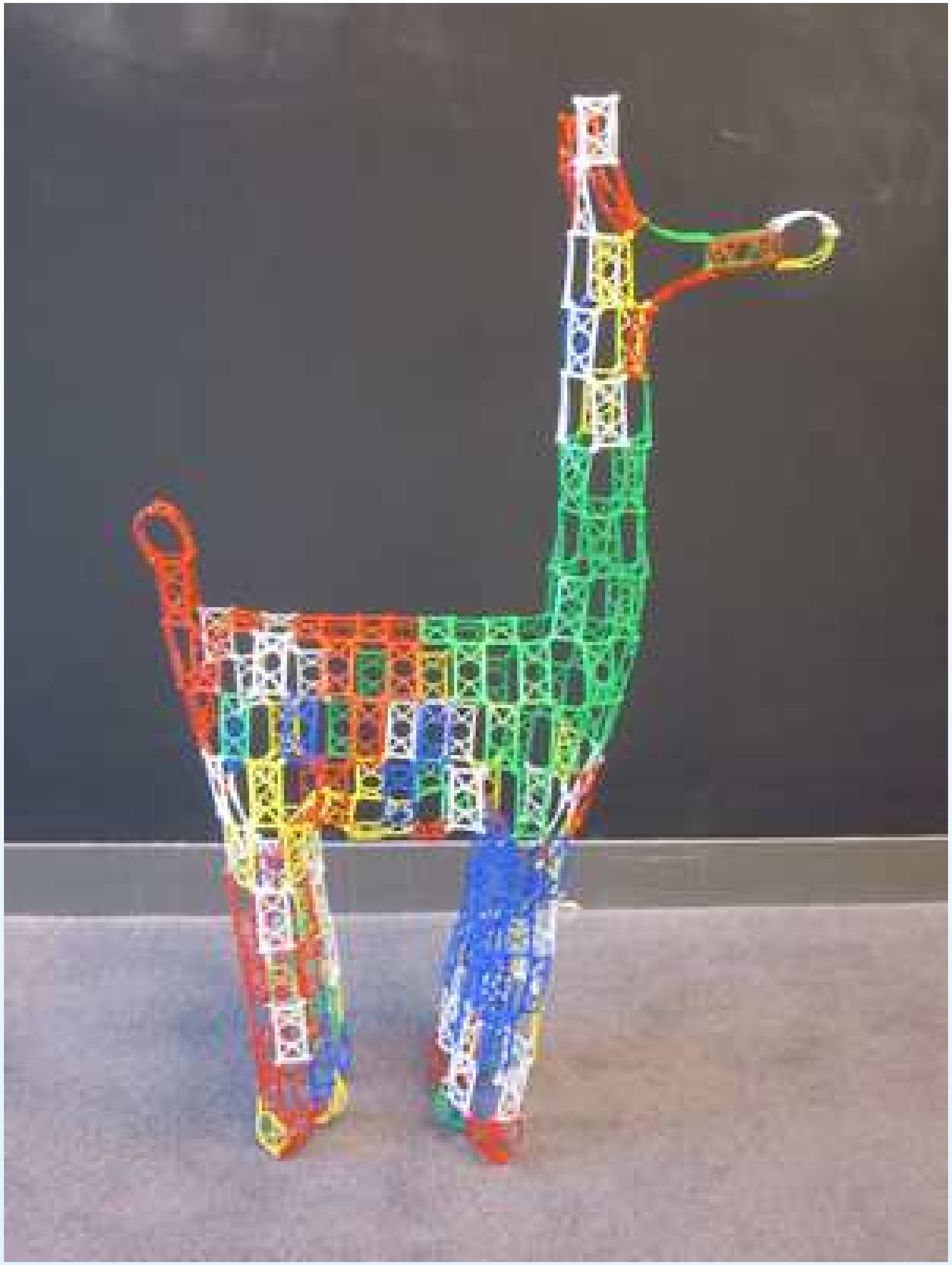


**Theorem.** *The asymptotic probability to get a positive meander after a random gluing of a system of arcs as above is* $\frac{c_1^+(g)}{c_g^+}$*. We have*

$$\frac{c_1^+(g)}{c_g^+} = \frac{1}{4g} + o\left(\frac{1}{g}\right) \text{ as } g \to +\infty\,.$$

# Square-tiled surfaces

## An example of a square-tiled surface



## Square-tiled surfaces: formal definition

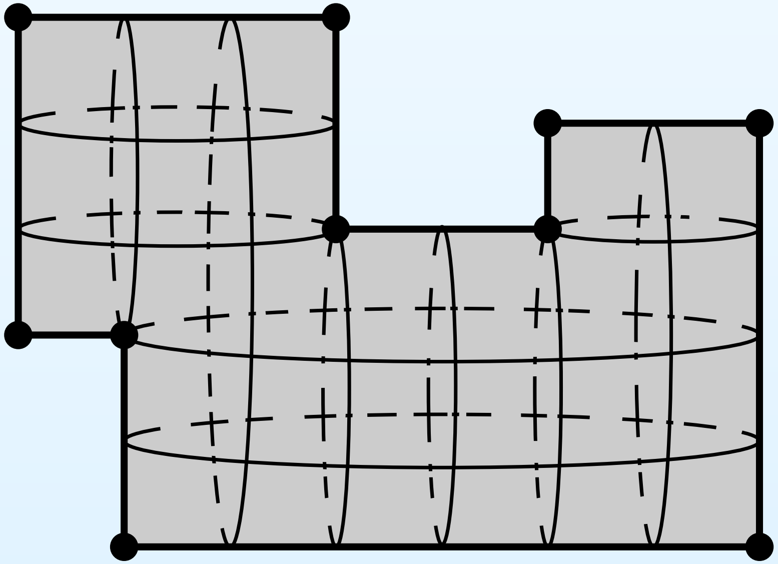
Take a finite set of copies of identical oriented squares for which two opposite sides are chosen to be horizontal and the remaining two sides are declared to be vertical. Identify pairs of sides of the squares by isometries in such way that horizontal sides are glued to horizontal sides and vertical sides to vertical. We get a topological surface $S$ without boundary. We consider only those surfaces obtained in this way which are connected and oriented. The form $dz^2$ on each square is compatible with the gluing and endows $S$ with a complex structure and with a non-zero quadratic differential $q = dz^2$ with at most simple poles. We call such a surface a *square-tiled surface*.

Fix the orientation of the horizontal and of the vertical sides of the initial square compatible with the orientation of the coordinate rays $Ox$ and $Oy$, where $z = x + iy$. The quadratic differential $q = dz^2$ is a square of a globally defined Abelian differential $\omega = dz$ if and only if all identifications of the sides respect the orientation.

## Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

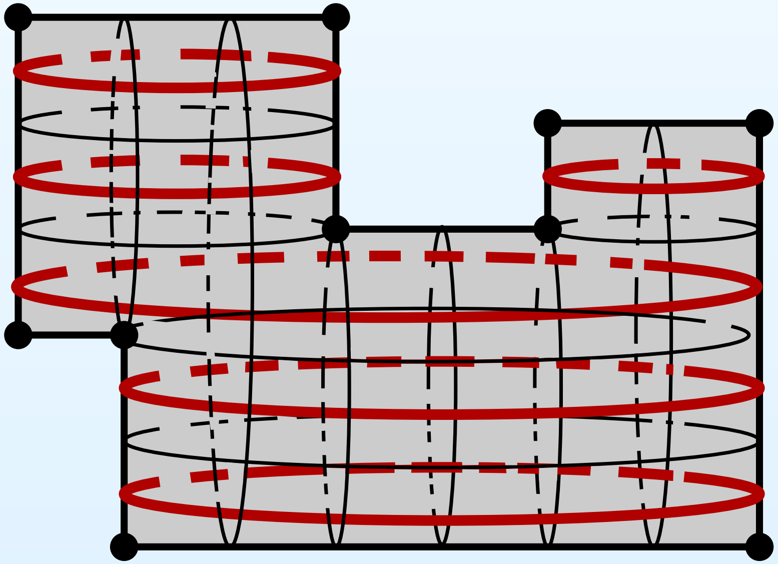
Consider a square-tiled sphere.



## Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

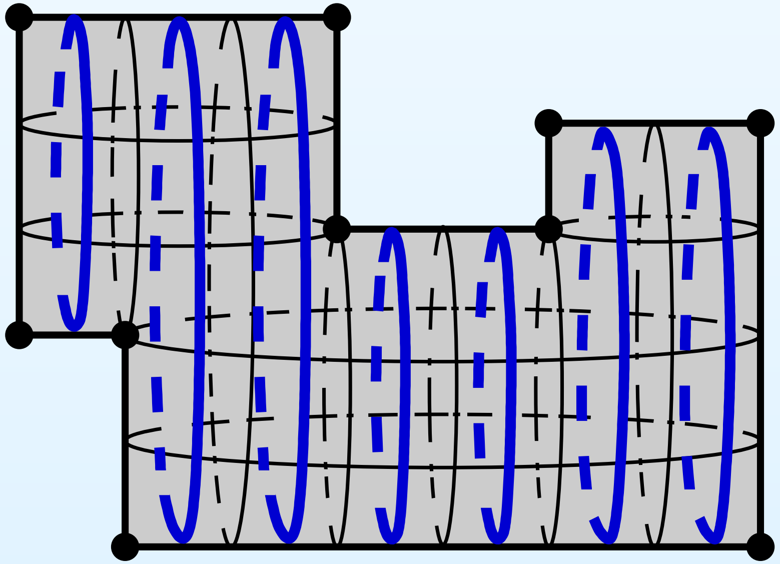
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve.



## Pairs of transverse multicurves as square-tiled surfaces

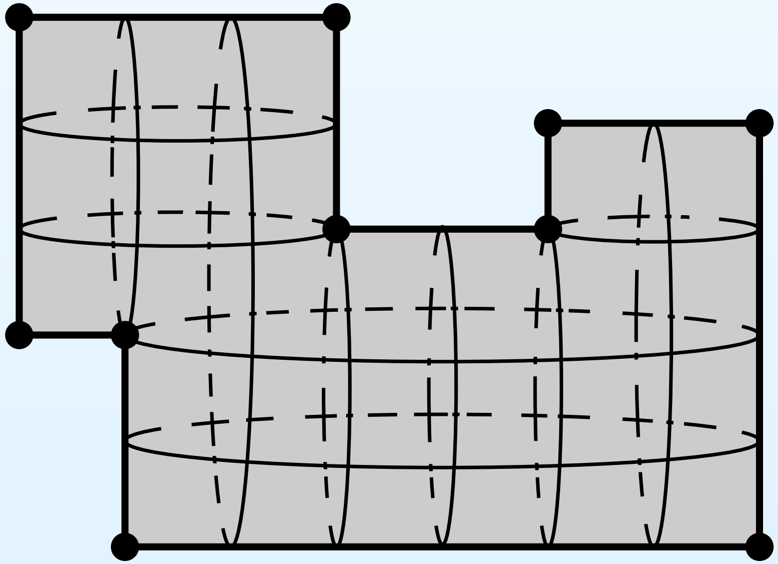
There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

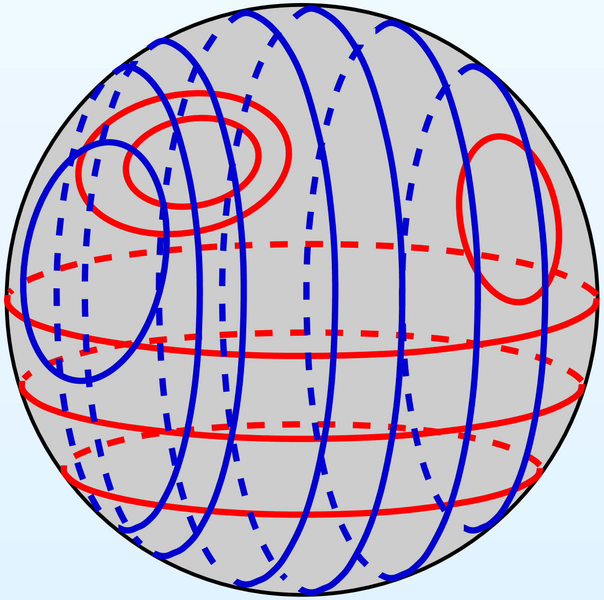
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve.



## Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres. Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve. Reciprocally, any transverse connected pair of multicurves on a sphere defines a square-tiling given by the graph dual to the graph formed by the pair of multicurves.





# Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

## Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs $(C,\omega)$, where $C$ is a complex curve and $\omega$ is a holomorphic 1-form on $C$ having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area $1$ form a real hypersurface $\mathcal{H}_1=\mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1=\text{area}(S)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

Any flat surface $S$ can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu=r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

**Theorem (H. Masur; W. Veech, 1982).** *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

## Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs $(C,\omega)$, where $C$ is a complex curve and $\omega$ is a holomorphic 1-form on $C$ having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area $1$ form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega\wedge\bar\omega = \sum_{i=1}^{g}(A_i\bar B_i - \bar A_i B_i)\,.$$

Any flat surface $S$ can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

**Theorem (H. Masur; W. Veech, 1982).** *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

## Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs $(C,\omega)$, where $C$ is a complex curve and $\omega$ is a holomorphic 1-form on $C$ having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area $1$ form a real hypersurface $\mathcal{H}_1=\mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1=\text{area}(S)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

Any flat surface $S$ can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu=r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

**Theorem (H. Masur; W. Veech, 1982).** *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

## Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface $S$ is defined by a holomorphic 1-form $\omega$ such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover $p$ over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left( \int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}\,,$$

where $P_1$ is a zero of $\omega$. The ramification points of the cover are exactly the zeroes of $\omega$.

## Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface $S$ is defined by a holomorphic 1-form $\omega$ such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover $p$ over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z}\right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}\,,$$

where $P_1$ is a zero of $\omega$. The ramification points of the cover are exactly the zeroes of $\omega$.

Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous "pillowcase covers" over $\mathbb{C}\mathrm{P}^1$ branched at four points. Thus, counting Masur–Veech volumes of strata $\operatorname{Vol}\mathcal{H}$ or $\operatorname{Vol}\mathcal{Q}$ is equivalent to counting the coefficient $c$ in the polynomial asymptotics $c \cdot N^d$ for the number of square-tiled surfaces in the stratum $\mathcal{H}$ or $\mathcal{Q}$ respectively, where $d$ is the complex dimension of the stratum.

## Brief history of evaluation of Masur–Veech volumes

● ’98, Kontsevich–Zorich: first several low-dimensional strata (combinatorics).
● ’00, ’05, Eskin–Okounkov–Pandharipande: algorithm for any stratum (representation theory); values for low-dimensional strata of Abelian differentials.
● ’16, Goujard: same algorithm; values in low dimensional strata of quadratic differentials.
● ’16, Athreya–Eskin–Zorich (conjectured by Kontsevich): close formula for volumes in genus $0$ (dynamics+analytic Riemann Roch theorem).
● ’18, Chen–Möeller–Zagier (conjectured by Eskin–Zorich): large genus asymptotics for the principal stratum of Abelian differentials (developing Eskin–Okounkov).
● ’20, Aggarwal (conjectured by Eskin–Zorich): large genus asymptotics for any stratum of Abelian differentials (developing Eskin–Okounkov+combinatorics).
● ’20, Chen–Möeller–Sauvaget–Zagier: efficient recursive formula for the strata of Abelian differentials (intersection theory).

## Recent developments in evaluation of Masur–Veech volumes

● ’20, Delecroix–Goujard–Zograf–Zorich; Zograf: volume contribution of one-cylinder square-tiled surfaces for any stratum of Abelian differentials; for the principal stratum of quadratic differentials; for all low-dimensional strata of quadratic differentials.

● ’21, Delecroix–Goujard–Zograf–Zorich: volume of the principal stratum of quadratic differentials (through Kontsevich–Witten correlators).

● ’21, Andersen–Borot–Charbonnier–Delecroix–Giacchetto–Lewanski–Wheeler: (inspired by [DGZZ]) same volumes through topological recursion.

● ’21, Chen–Möeller–Sauvaget: same volumes through Hodge integrals. Conjectural analogous formula for general strata.

● ’21, Kazarian; independently Yang-Zagier-Zhang: efficient recursion for these Hodge integrals.

● ’21, Aggarwal (conjectured in [DGZZ]): large genus asymptotics for the principal stratum of quadratic differentials (involved combinatorics+probability theory).

# Meanders count: summary

## Translation to the language of square-tiled surfaces

Every square-tiled surface defines a pair of transverse simple closed multicurves. The number of squares is the number of intersections of the two multicurves.

Reciprocal is not always true since in genera higher than $0$ a pair of transverse multicurves might chop the surface into components more complicated than topological discs. However, it happens rarely in terms of the asymptotic count, so for the purposes of the count we can pretend that we have a bijection.

Bigons arising from intersection of transverse multicurves correspond to simple poles of the associated quadratic differentials. Thus, the count of pairs of transverse multicurves on a surface of genus $g$ with at most $N$ intersections and with $p$ bigons corresponds to the count of square-tiled surfaces of genus $g$ with $p$ poles tiled by at most $N$ squares, i.e. to evaluation of the Masur–Veech volume of the moduli space $\mathcal{Q}_{g,p}$. In this way we get the asymptotics $c(g,p) \cdot N^{6g-6+2p}$ for the number of multicurves and the constant $c(g,p)$.

## How we count meanders

A pair of transverse multicurves associated to a square-tiled surface is orientable if and only if the square-tiled surface is Abelian. Thus, the count of positively intersecting pairs of transverse multicurves in genus $g$ corresponds to the count of Abelian square-tiled surfaces in genus $g$, i.e. to the evaluation of the Masur–Veech volumes of the corresponding moduli space of Abelian differentials. In this way we get the asymptotics $c^+(g) \cdot N^{4g-3}$ and the constant $c_1^+(g)$ for the count of positively intersecting multicurves.

Pairs (simple closed curve, transverse multicurve) correspond to square-tiled surfaces having single horizontal band of squares. We found a way to count such square-tiled surfaces both in the Abelian and in the quadratic case and to evaluate the constants $c_1(g,p)$ and $c_1^+(g)$ in the corresponding asymptotics $c_1(g,p) \cdot N^{6g-6+2p}$ and $c_1^+(g) \cdot N^{4g-3}$ respectively.

Meanders correspond to square-tiled surfaces having single horizontal and single vertical band of squares. We apply our non-correlation theorem to get

$$c_{1,1}(g,p) = \frac{c_1^2(g,p)}{c(g,p)} \quad \text{and} \quad c_{1,1}^+(g) = \frac{\left(c_1^+(g)\right)^2}{c^+(g)} .$$

## Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus $0$.

$$\text{Let} \qquad v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$ , so $v(-1) = 1$ and $v(0) = 2$.

## Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus $0$.

$$\text{Let} \qquad v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$ , so $v(-1) = 1$ and $v(0) = 2$.

## Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus $0$.

$$\text{Let} \qquad v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$ , so $v(-1) = 1$ and $v(0) = 2$.

**Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003)** *The volume of any stratum $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. when $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

## Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus $0$.

$$\text{Let} \qquad v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$ , so $v(-1) = 1$ and $v(0) = 2$.

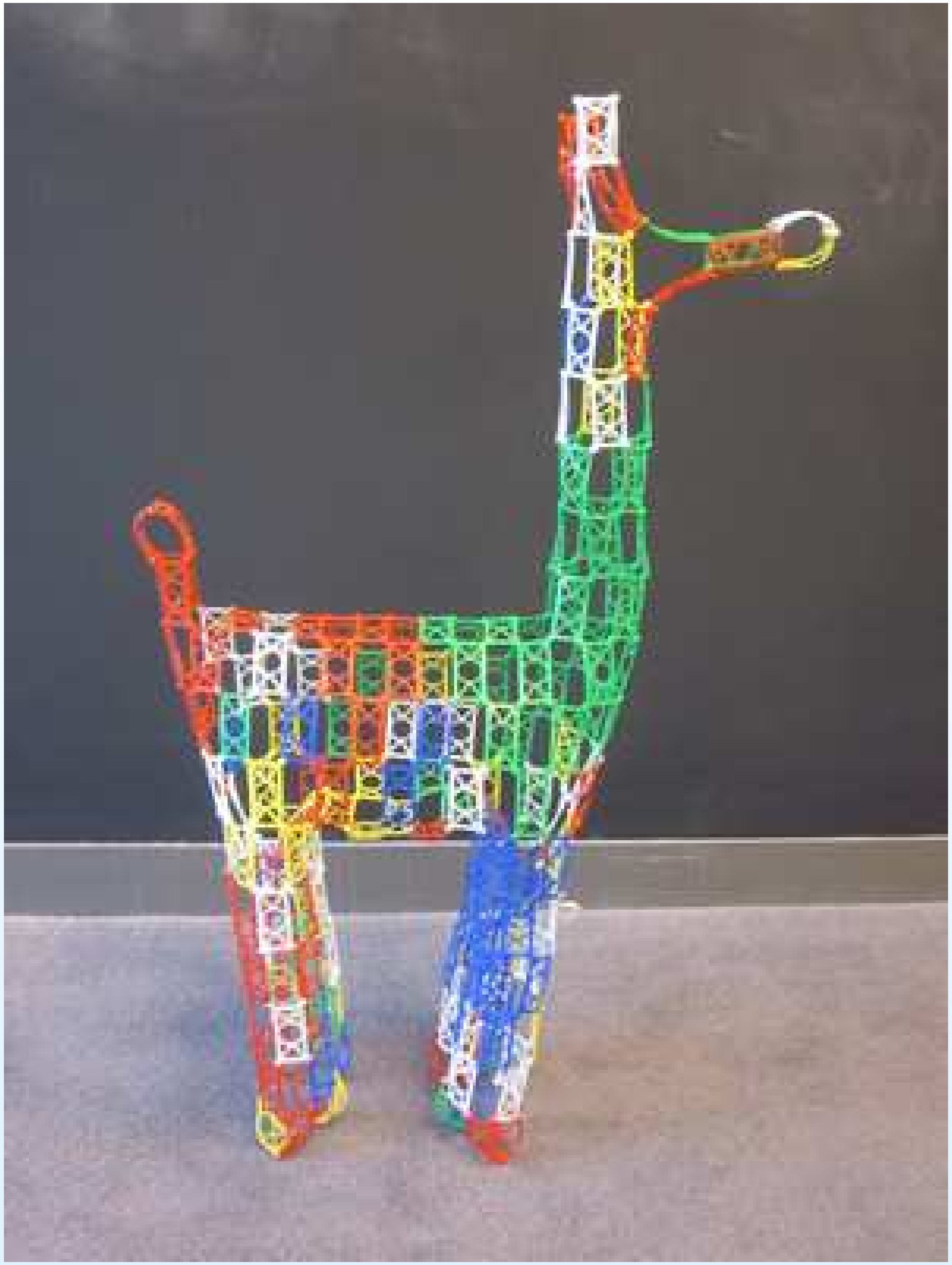
**Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003)**

$$\operatorname{Vol} \mathcal{Q}_{0,n} = 2\pi \cdot \left(\frac{\pi^2}{2}\right)^{n-4} .$$

Applying formula based on Kontsevich polynomials one gets ENORMOUS sum over labeled trees, so this approach does not work. But this formula was reproved by Chen–Möller–Sauvaget through intersection theory.

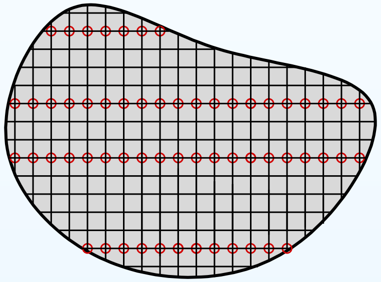
# Non-correlation of vertical and horizontal foliations on square-tiled surfaces

## Cylinder decomposition of a square-tiled surface



## Equidistribution and Non-correlation Theorems

**Theorem.** *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly $k$ maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on $B$.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally $k$-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally $k$-cylinder and vertically $j$-cylinder ones.

**Theorem.** *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

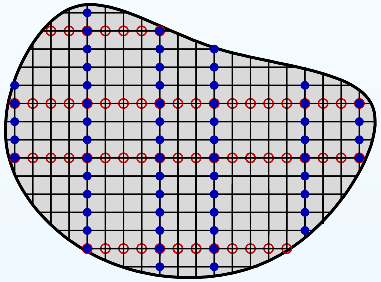
$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula!

Proof: Maryam's counting of volumes $+$ Moore ergodicity theorem

## Equidistribution and Non-correlation Theorems

**Theorem.** *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly $k$ maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on $B$.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally $k$-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L}) / \operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally $k$-cylinder and vertically $j$-cylinder ones.

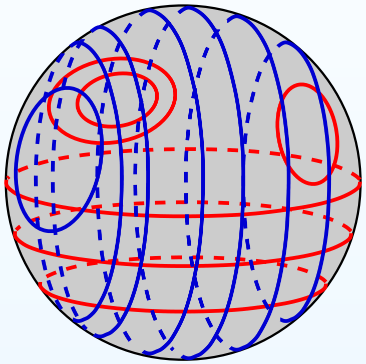
**Theorem.** *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

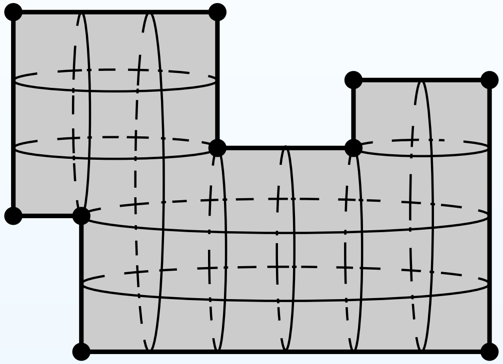
$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Maryam's counting of volumes $+$ Moore ergodicity theorem.

## How to count meanders

**Step 1.** There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





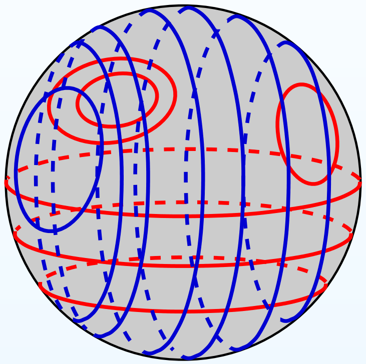
**Step 2.** Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height $1$. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

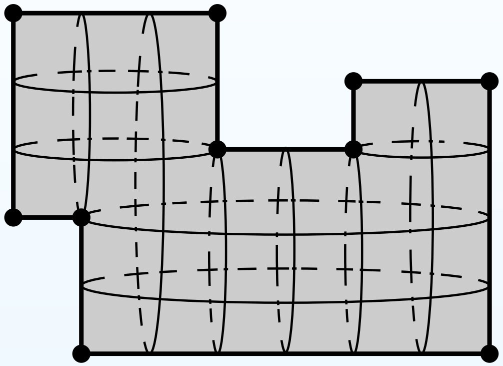
**Step 3.** Fixing the number of minimal arcs ("pimples") we fix the number of simple poles $p$ of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

Hyperbolic metric endowes a multicurve with canonical shape. A pair of multicurves canonically defines a hyperbolic metric. Discrete analog of Hubbard–Masur Theorem.

## How to count meanders

**Step 1.** There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





**Step 2.** Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height $1$. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

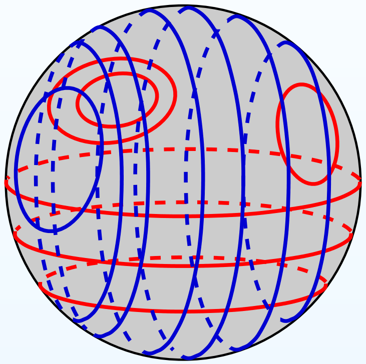
**Step 3.** Fixing the number of minimal arcs ("pimples") we fix the number of simple poles $p$ of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

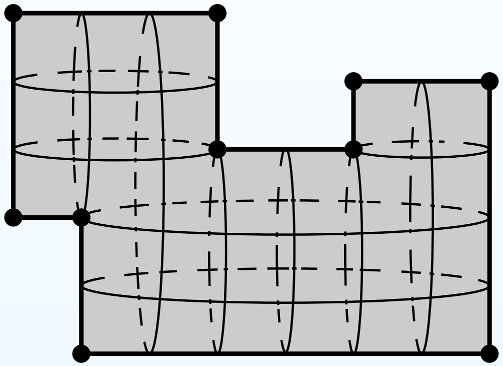


Hyperbolic metric endowes a multicurve with canonical shape. A pair of multicurves canonically defines a hyperbolic metric. Discrete analog of Hubbard–Masur Theorem.

## How to count meanders

**Step 1.** There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





**Step 2.** Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height $1$. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

**Step 3.** Fixing the number of minimal arcs ("pimples") we fix the number of simple poles $p$ of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.



Hyperbolic metric endowes a multicurve with canonical shape. A pair of multicurves canonically defines a hyperbolic metric. Discrete analog of Hubbard–Masur Theorem.

## General philosophy

- Pairs of transverse multicurves correspond to square-tiled surfaces. Thus, count of all pairs of transverse multicurves is equivalent to count of Masur–Veech volumes.

- Count of arc systems, braids, ribbon graphs, pairs: simple closed curve plus transverse multicurve, one-cylinder square-tiled surfaces is another group of (somehow equivalent) problems, which usually admits a more efficient solution.

- Consider the following three counting problems:

1. count of all square-tiled surfaces (i.e. Masur–Veech volume $\mathrm{Vol}$);
2. count of horizontally one-cylinder square-tiled surfaces (i.e. $c_1$);
3. count of horizontally and vertically square-tiled surfaces (i.e. $c_{1,1}$).

By non-correlation, $c_{1,1} = \dfrac{c_1^2}{\mathrm{Vol}}$. Count of $c_1$ usually admits a relatively efficient solution. Hence, as soon as we know the appropriate Masur–Veech volume, we know $c_{1,1}$, and hence we can count meanders, pairs of transverse simple closed curves etc.

# 1-cylinder square-tiled surfaces and permutations

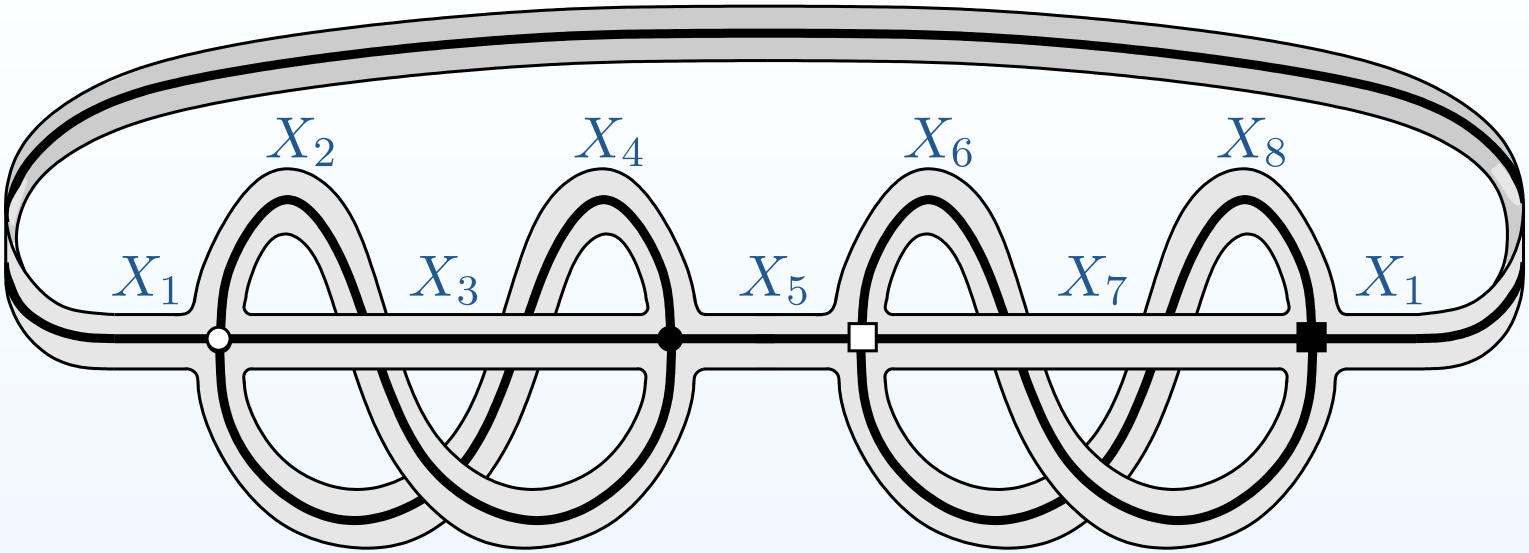
## Count of 1-cylinder square-tiled surfaces

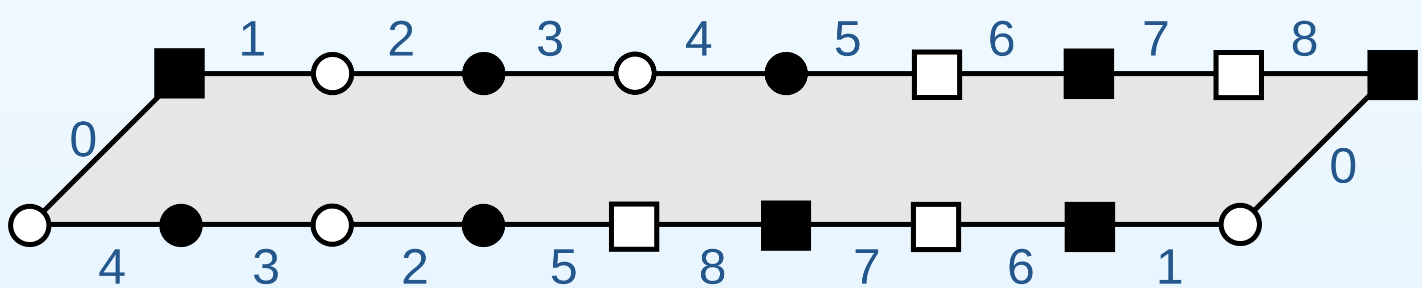
We have rather comprehensive information about Masur–Veech volumes of strata of Abelian differentials. Namely, in all low genera we know them explicitly. In higher genera the volume of the principal stratum $\mathcal{H}(1,\dots,1)$ can be computed exactly up to very high genus. When $g \to +\infty$ it can be computed approximately by the works of Chen–Möller–Zagier and of Chen–Möller–Sauvaget–Zagier. These recent papers as well as the recent work of Aggarwal independently proved our old conjecture with Eskin:

$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n) \sim \frac{4}{(m_1+1)\dots(m_n+1)}\,.$$

By the general philosophy, to compute braids, frequencies of simple closed curves, numbers of pairs of positively intersecting simple closed curves, etc in this context, one has to compute contribution of $1$-cylinder square-tiled surfaces. As in the previous cases, this problem admits a solution.

# $1$-cylinder surface as a pair of permutations




$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8$$

$$4 \to 3 \to 2 \to 5 \to 8 \to 7 \to 6 \to 1$$

$$c_{bot} \cdot c_{top}^{-1} = (1,3)(2,4)(5,7)(6,8)$$

## Frobenius formula

The count of $1$-cylinder $N$-square-tiled surfaces in the stratum $\mathcal{H}(m_1, \dots, m_n)$ is reduced to the count of solutions of the following equation for permutations:

$$(N\text{- cycle}) \cdot (N\text{- cycle}) = \text{product of cycles of lengths } m_1 + 1, \dots, m_n + 1\,.$$

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group $S_n$:

$$\chi_j(g) := \operatorname{tr}(g, \pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0 \le j \le n-1)\,.$$

**Theorem.** *The absolute contribution $c_1(\mathcal{H}(m_1, \dots, m_n))$ of $1$-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1, \dots, m_n)$ equals*

$$c_1 = \frac{2}{(d-1)!} \cdot \prod_k \frac{1}{(k+1)^{\mu_k}} \cdot \sum_{j=0}^{d-2} j!\,(n-1-j)!\,\chi_j(\nu)\,.$$

*Here $d = \dim \mathcal{H}(m_1, \dots, m_n)$; $\nu \in S_n$ is any permutation with decomposition into cycles of lengths $(m_1+1), \dots, (m_n+1)$; $\mu_i$ is the number of zeroes of order $i$, i.e. the multiplicity of the entry $i$ in the multiset $\{m_1, \dots, m_n\}$.*

## Frobenius formula

The count of $1$-cylinder $N$-square-tiled surfaces in the stratum $\mathcal{H}(m_1,\dots,m_n)$ is reduced to the count of solutions of the following equation for permutations:

$$(N\text{-cycle})\cdot(N\text{-cycle}) = \text{product of cycles of lengths } m_1+1,\dots,m_n+1\,.$$

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group $S_n$:

$$\chi_j(g) := \operatorname{tr}(g,\pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0\le j\le n-1)\,.$$

**Theorem.** *The absolute contribution $c_1(\mathcal{H}(m_1,\dots,m_n))$ of $1$-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ satisfies the following bounds:*

$$\frac{1}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)} \le c_1(\mathcal{H}) \le \frac{1}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}$$

We were able to replace the formula in characters by this much more efficient estimate using the results of Zagier.

## Count of one-cylinder square-tiled surfaces: answers

For permutations $\nu$ representing the principal and the minimal strata the characters $\chi_j(\nu)$ admit easier computation which leads to the following formulae:

$$c_1(\mathcal{H}(1^{2g-2})) = \frac{1}{4g-2} \cdot \frac{4}{2^{2g-2}}$$

$$c_1(\mathcal{H}(2g-2)) = \frac{1}{2g} \cdot \frac{4}{2g-1}$$

## Count of one-cylinder square-tiled surfaces: answers

For permutations $\nu$ representing the principal and the minimal strata the characters $\chi_j(\nu)$ admit easier computation which leads to the following formulae:

$$c_1(\mathcal{H}(1^{2g-2})) = \frac{1}{4g-2} \cdot \frac{4}{2^{2g-2}} \approx \frac{1}{d} \cdot \operatorname{Vol}\mathcal{H}(1^{2g-2})$$

$$c_1(\mathcal{H}(2g-2)) = \frac{1}{2g} \cdot \frac{4}{2g-1} \approx \frac{1}{d} \cdot \operatorname{Vol}\mathcal{H}(2g-2)$$