Lecture 1. Count of square-tiled surfaces

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HSE, February 10, 2022

Count of metric ribbon graphs and of square-tiled surfaces

Epigraph: “We are counting cards”

If you are annoyed while waiting for the beginning of the lecture, please type “Very sparkly” in YouTube search and watch the three-minutes extract from the movie “Rainman” (pay attention starting from the conversation with Iris):

https://www.youtube.com/watch?v=Wjc58nT4hUA

This is the best epigraph which I can imagine for my lectures. I fully identify myself with Rainman, except that he and his brother were counting cards for a day and, together with my collaborators, we were obsessively counting square-tiled surfaces for many years... We have not finished yet.

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.

Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$. The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} .$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

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Recursive relations

Initial data: $\langle \tau_0^3 \rangle = 1, \qquad \langle \tau_1 \rangle = \frac{1}{24}\,.$

String equation:

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n+1} = \langle \tau_{d_1 - 1} \ldots \tau_{d_n} \rangle_{g,n} + \cdots + \langle \tau_{d_1} \ldots \tau_{d_n - 1} \rangle_{g,n}\,.$$

Dilaton equation:

$$\langle \tau_1 \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n+1} = (2g - 2 + n) \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k \geq 1$):

$$\langle \tau_{k+1} \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \frac{1}{(2k+3)!!} \Bigg[\sum_{j=1}^{n} \frac{(2k + 2d_j + 1)!!}{(2d_j - 1)!!} \langle \tau_{d_1} \cdots \tau_{d_j + k} \cdots \tau_{d_n} \rangle_g$$

$$+ \frac{1}{2} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} (2r+1)!!(2s+1)!! \sum_{\{1,\ldots,n\} = I \coprod J} \langle \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \Bigg]\,.$$

Uniform large genus asymptotics

We stated in August 2019 a conjecture which was proved by Amol Aggarwal already in April 2020.

Theorem (Aggarwal). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} = \\ = \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$, $d_1 + \dots + d_n = 3g-3+n$, *as* $g \to +\infty$.

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

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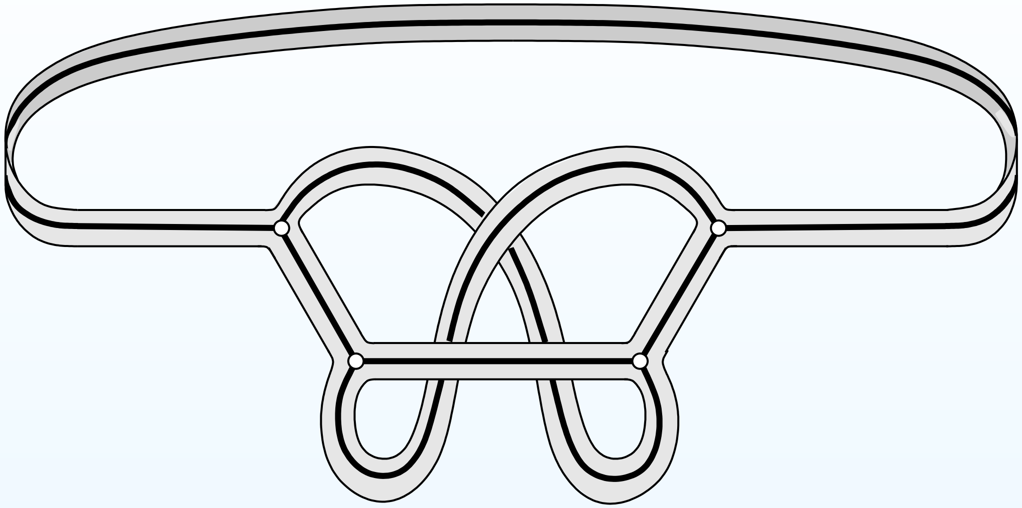
$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\quad \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Kontsevich's count of metric ribbon graphs

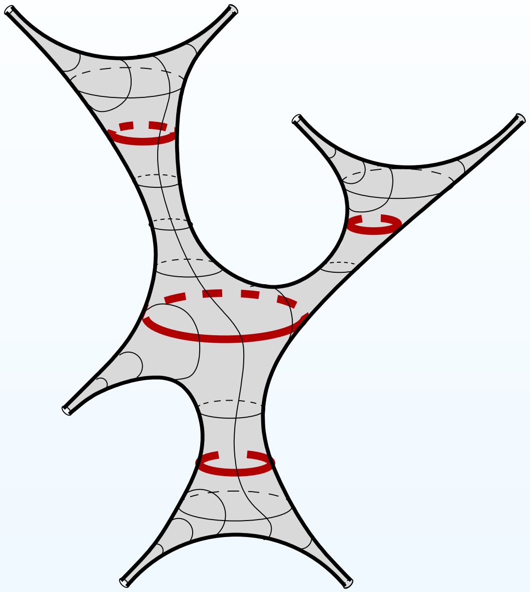
Theorem (M. Kontsevich; in this form — P. Norbury). *Consider a collection of positive integers* $b_1, \ldots, b_n$ *such that* $\sum_{i=1}^n b_i$ *is even. The weighted count of genus* g *connected trivalent metric ribbon graphs* Γ *with integer edges and with* n *labeled boundary components of lengths* $b_1, \ldots, b_n$ *is equal to* $N_{g,n}(b_1, \ldots, b_n)$ *up to the lower order terms:*

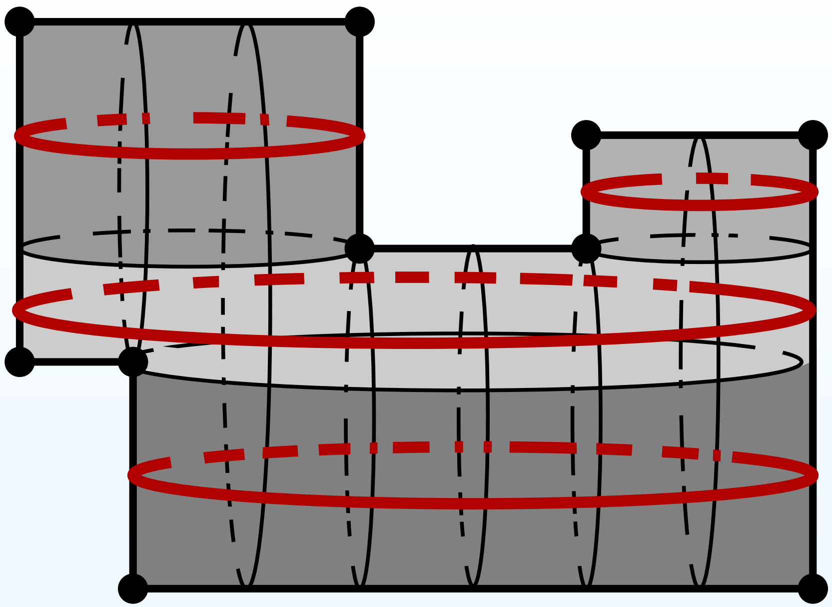
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ *denote the set of (nonisomorphic) trivalent ribbon graphs* Γ *of genus* g *and with* n *boundary components.*

This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

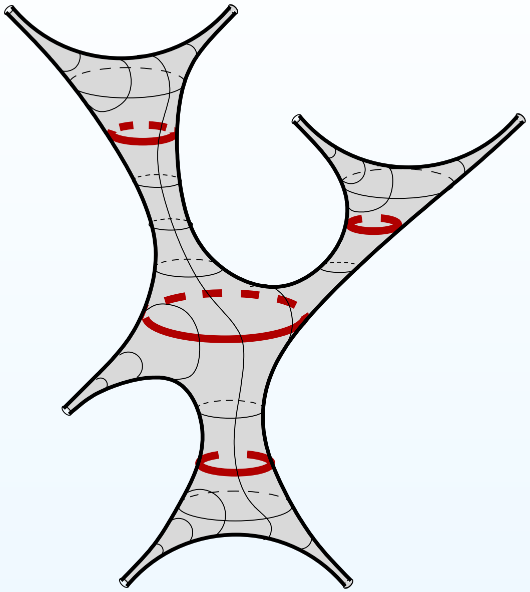
Stable graph associated to a square-tiled surface

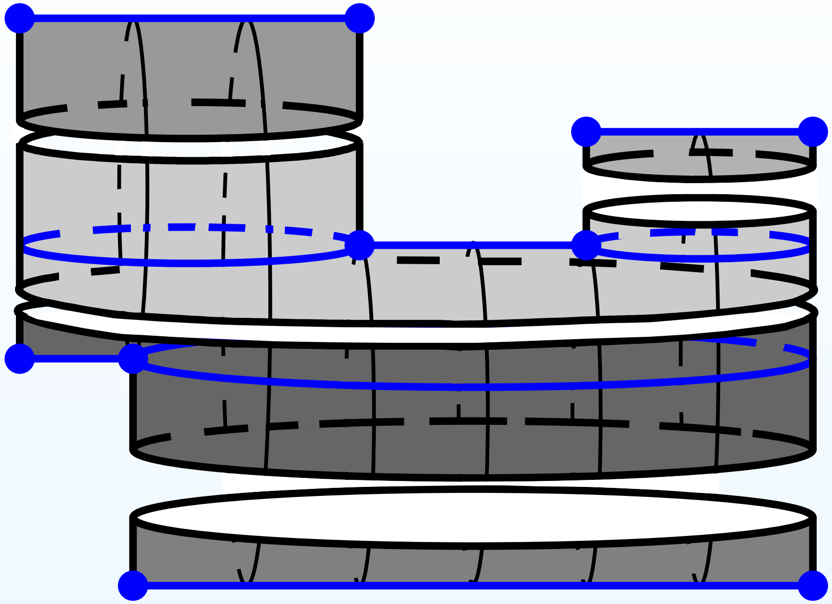


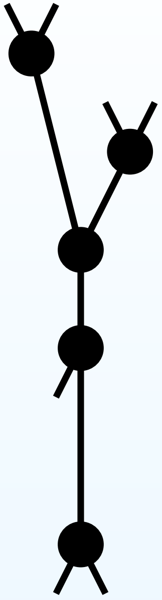


Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders.

Stable graph associated to a square-tiled surface

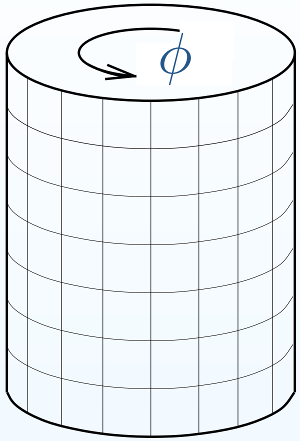


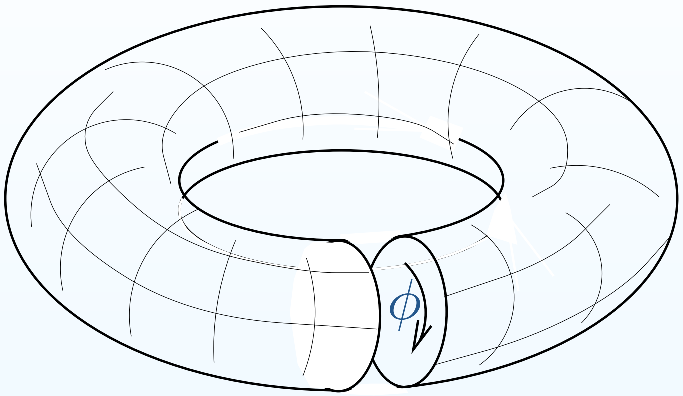




Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. The associated *stable graph* Γ is the dual graph to the multicurve. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by “legs” — half-edges of the dual graph.

Number of square-tiled tori

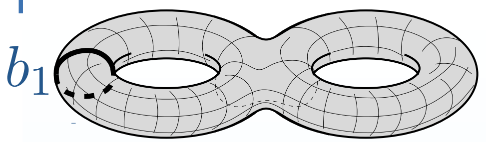




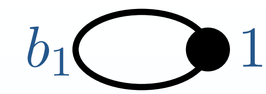
The number of square-tiled tori tiled with at most N squares has asymptotics

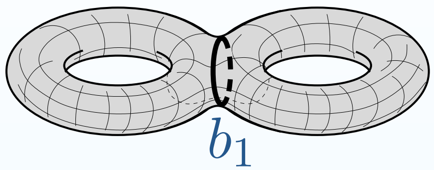
$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\le N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\le \frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\frac{\pi^2}{6} = \frac{N^2}{2}\zeta(2) =$$

$$= \frac{N^2}{2}\mathcal{Z}(b)\,, \quad \text{where} \quad \mathcal{Z}\ :\ \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n}\big(m_i!\cdot\zeta(m_i+1)\big)\,.$$

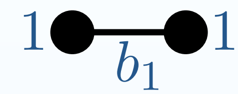


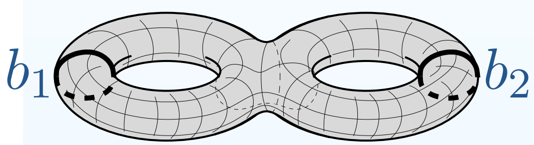
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



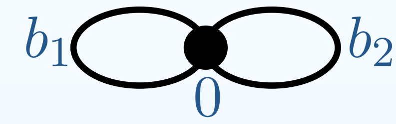


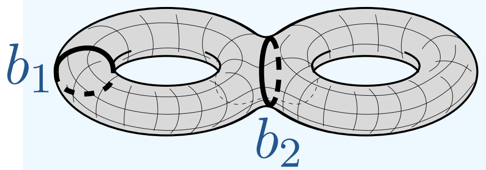
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



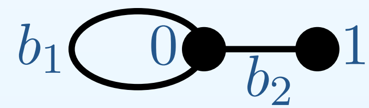


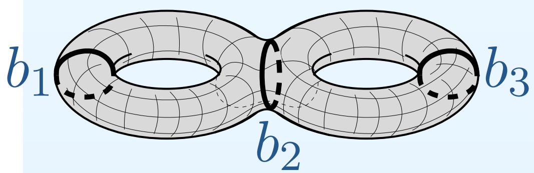
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



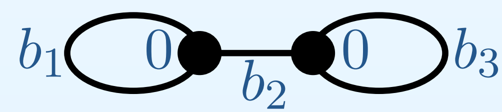


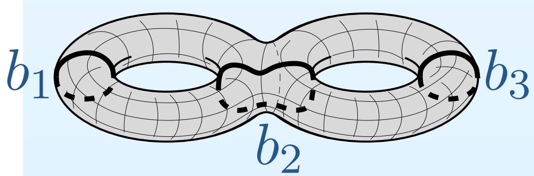
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



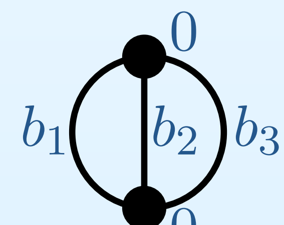


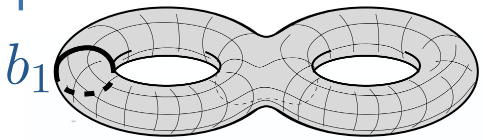
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



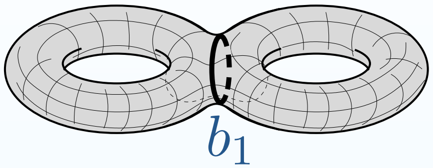


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$

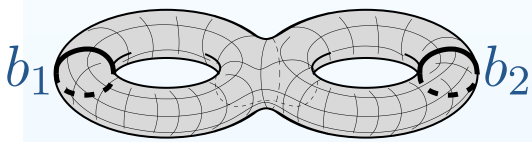




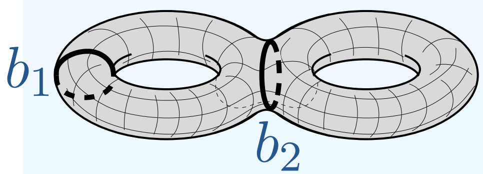
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2)\right)$$



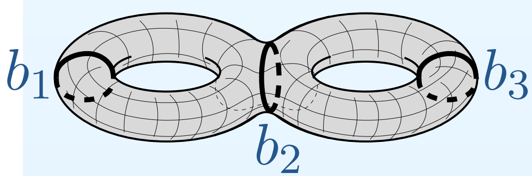
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2\right) \left(\frac{1}{48} b_1^2\right)$$



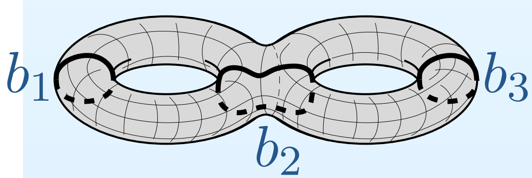
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2)\right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2\right)$$

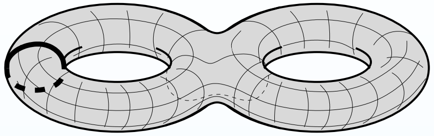


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

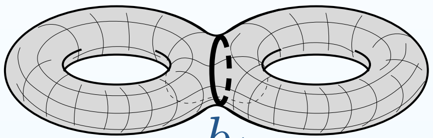


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

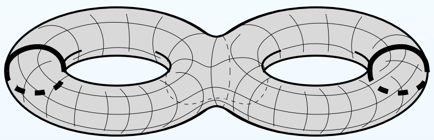
Volume of $\mathcal{Q}_2$



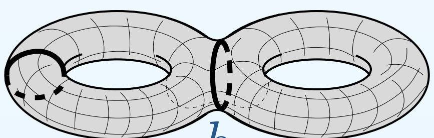
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



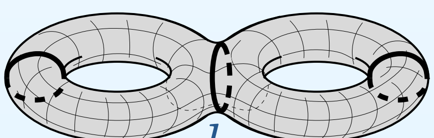
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



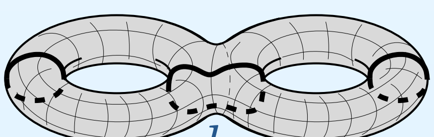
$$\frac{1}{16}(b_1^3 b_2 + {} + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16}b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



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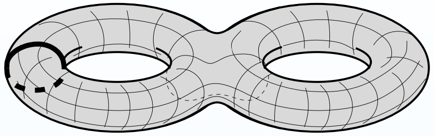
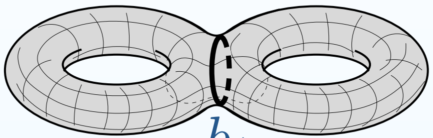
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_2$



$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



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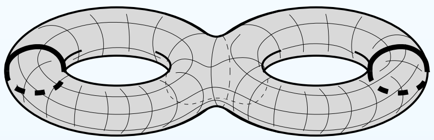
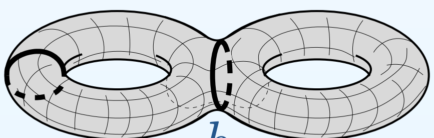


$$\frac{1}{16}(b_1^3 b_2 + +b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



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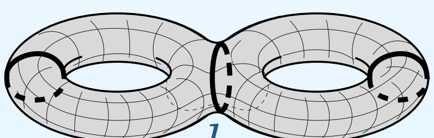

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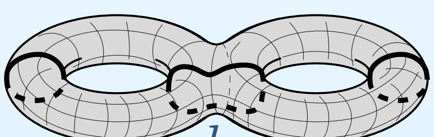

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Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs } \Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2, \underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

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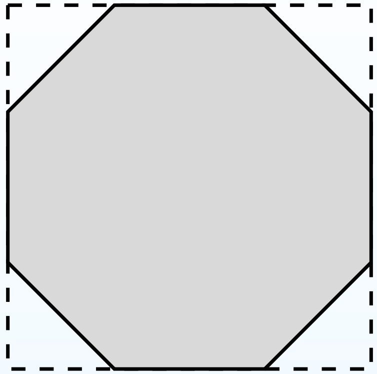
Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials. Square-tiled surfaces

Very flat surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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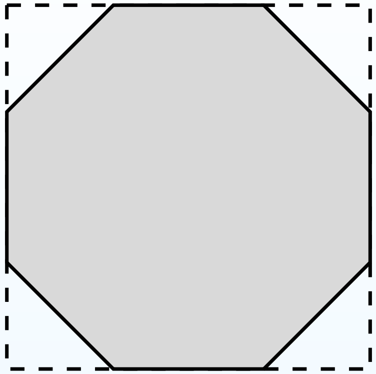

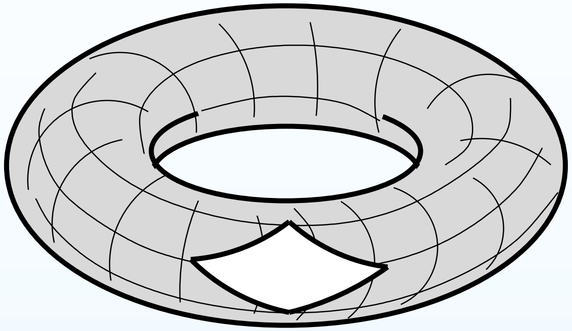

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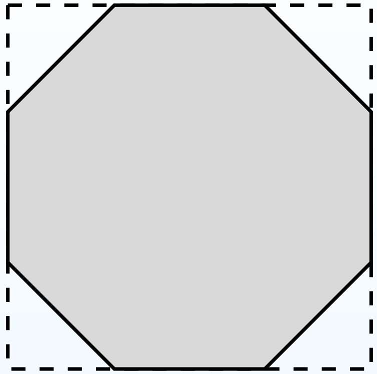

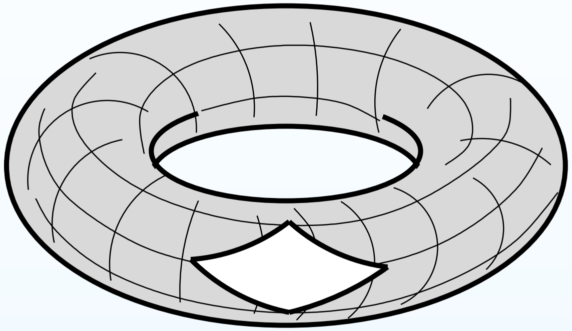

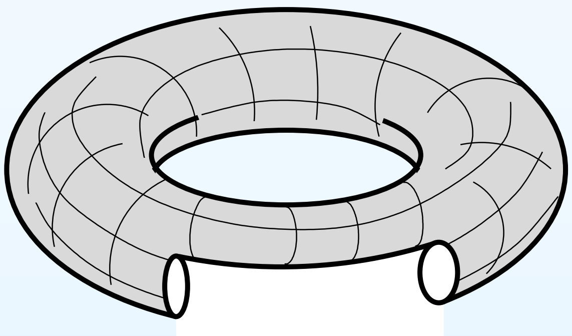

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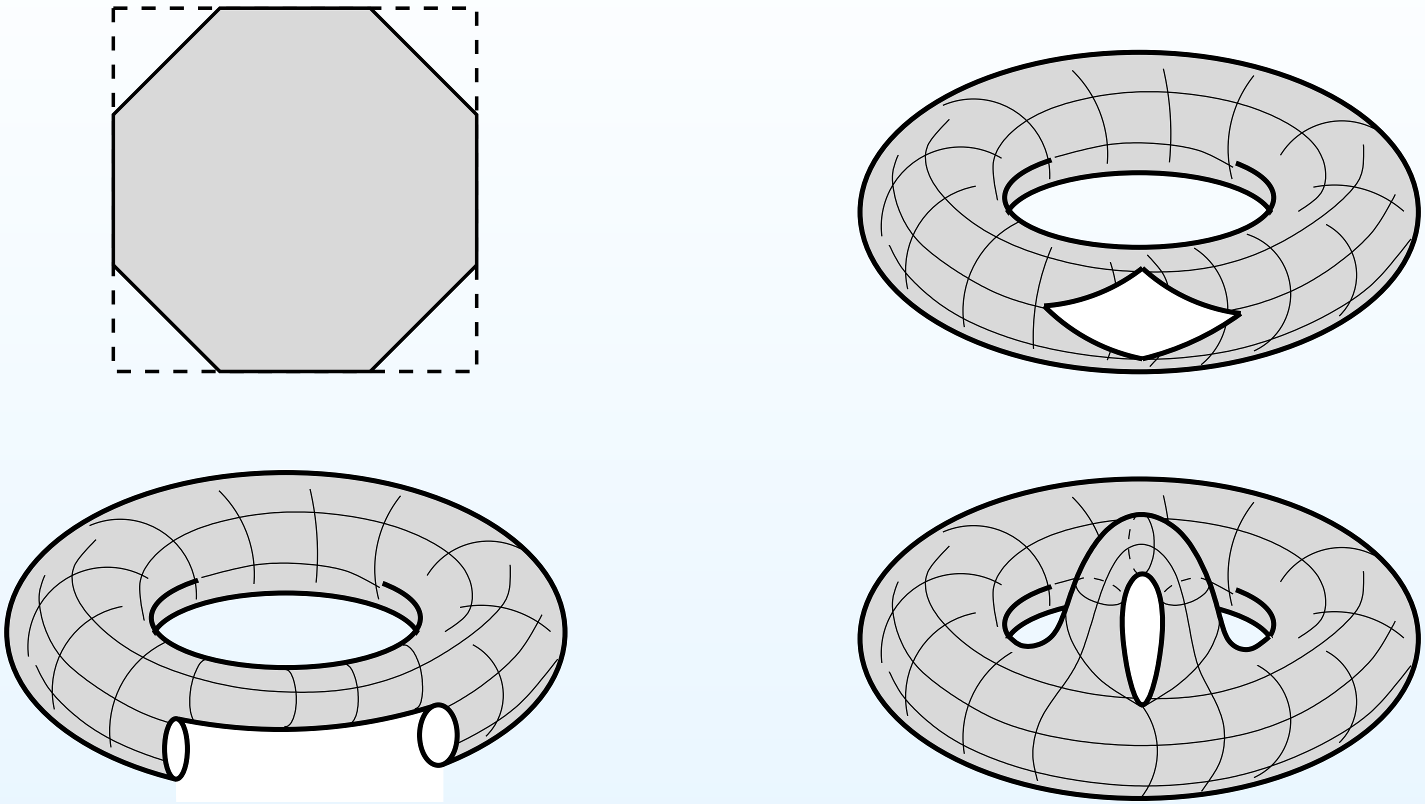

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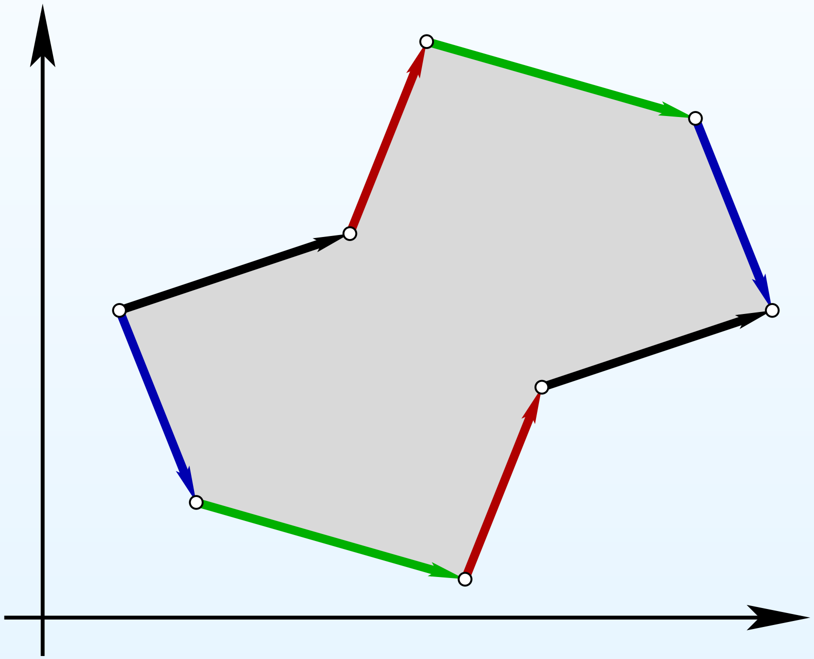
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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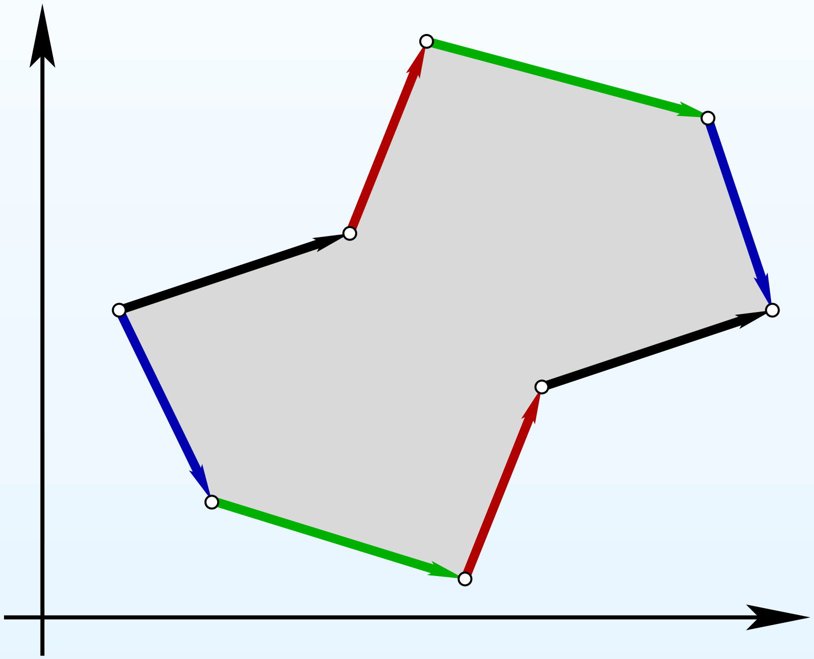

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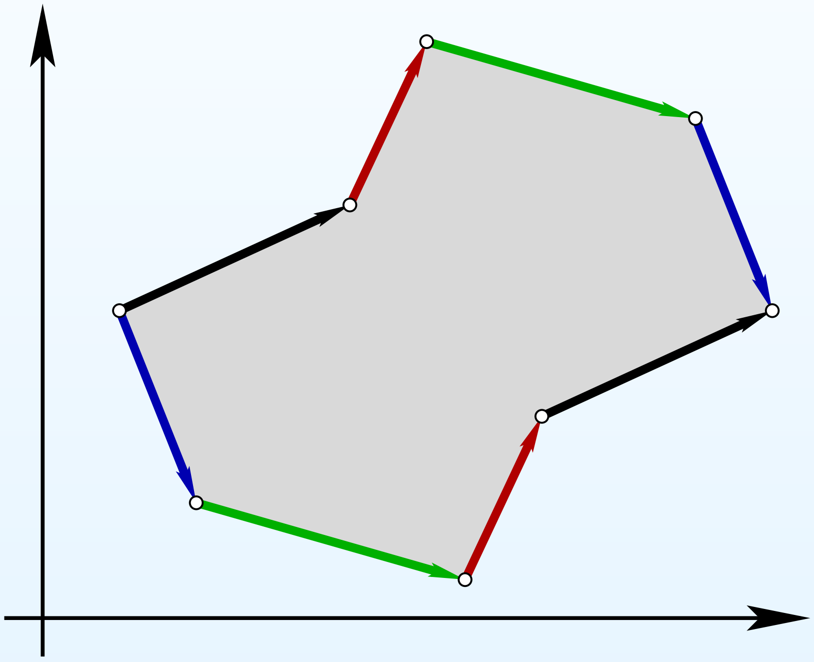

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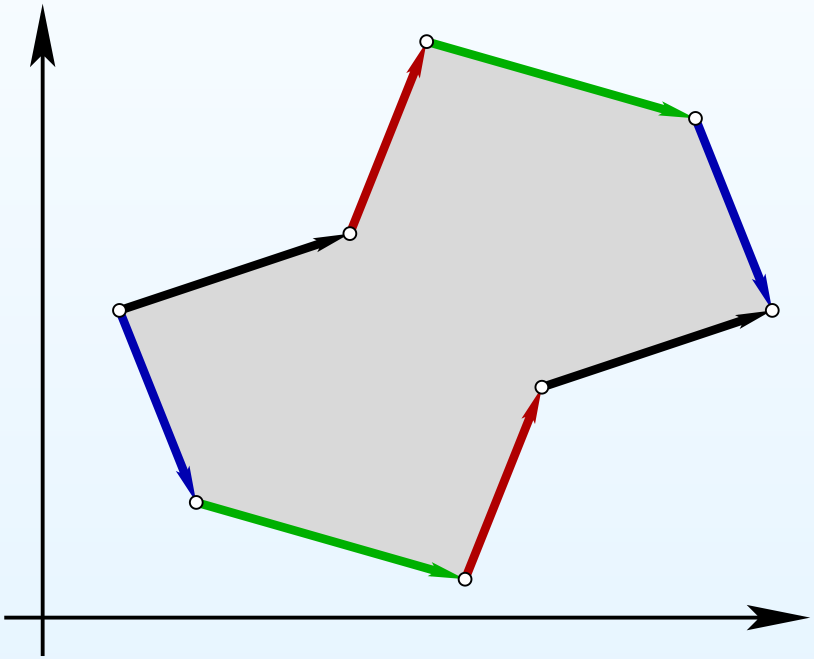
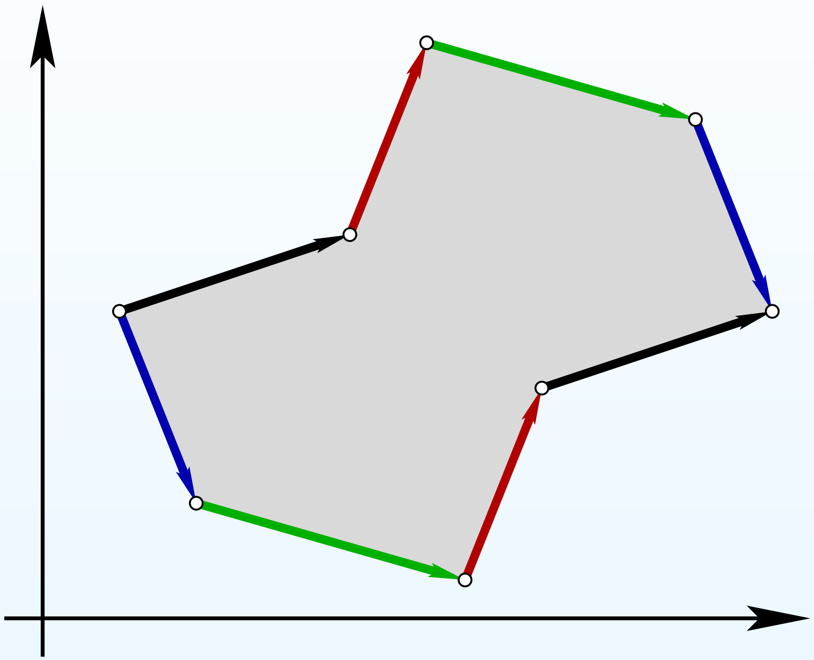


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

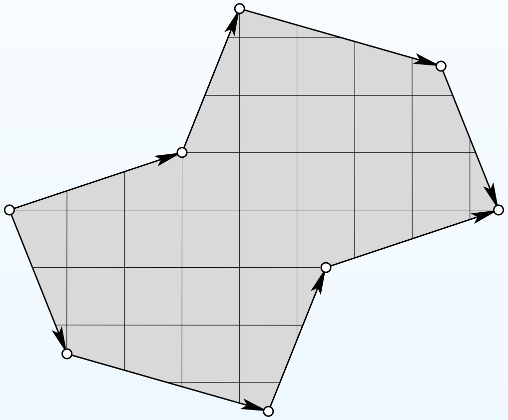
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modeled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any moduli space $\mathcal{H}(m_1, \ldots, m_n)$: given a positive integer $r > 0$ we can rescale a flat surface by factor r. The flat area of the surface gets rescaled by the factor r^2.



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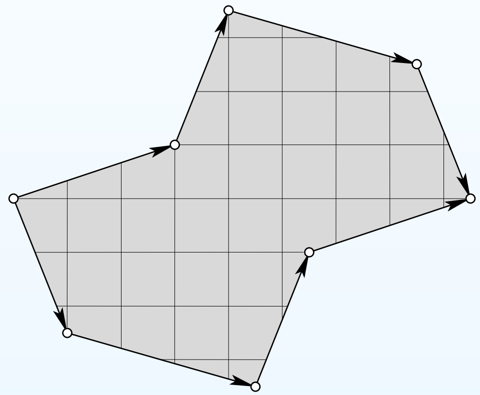

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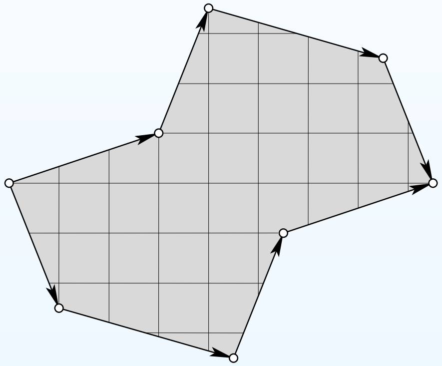

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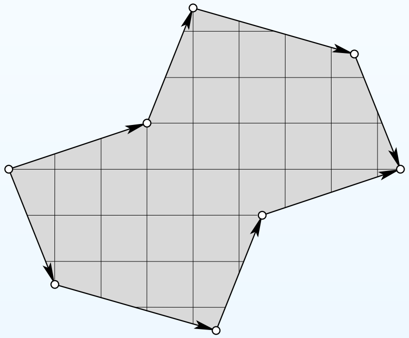

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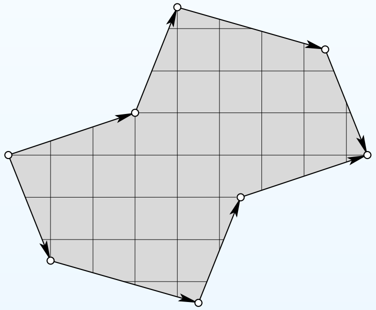

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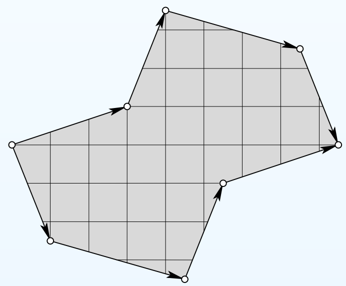


Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \ldots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \ldots, m_n)$. In these "polar coordinates" the volume element disintegrates as $d\nu = r^{2d-1} dr\, d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \ldots, m_n)$.

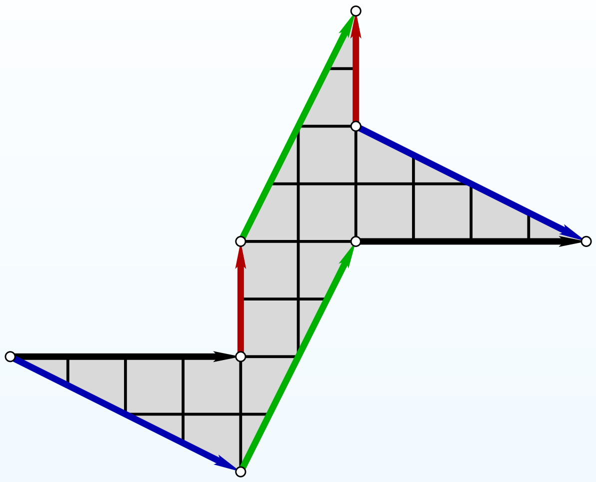
Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ or $\mathcal{Q}_1(m_1, \dots, m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Integer points as square-tiled surfaces



Integer points in period coordinates are represented by *square-tiled surfaces*.

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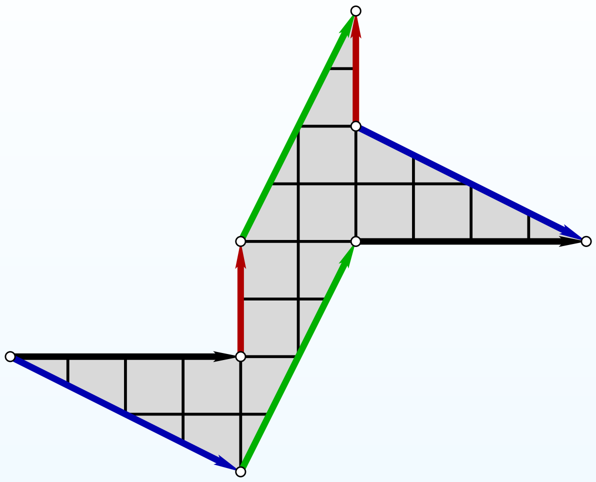

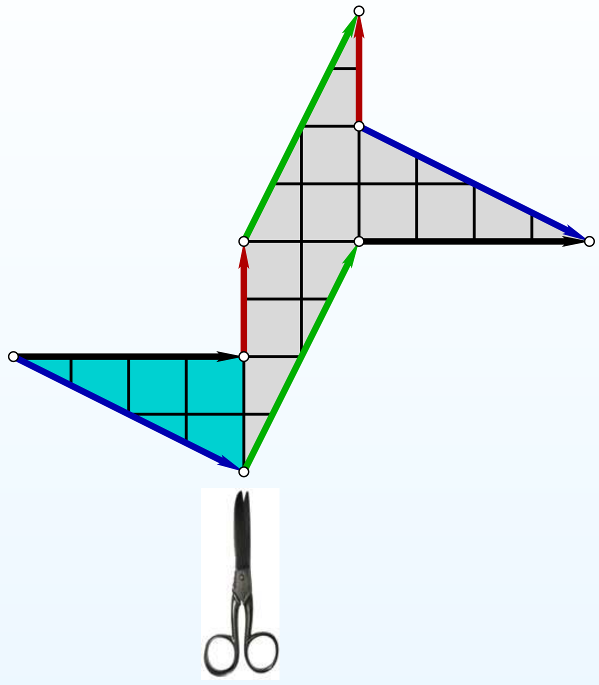

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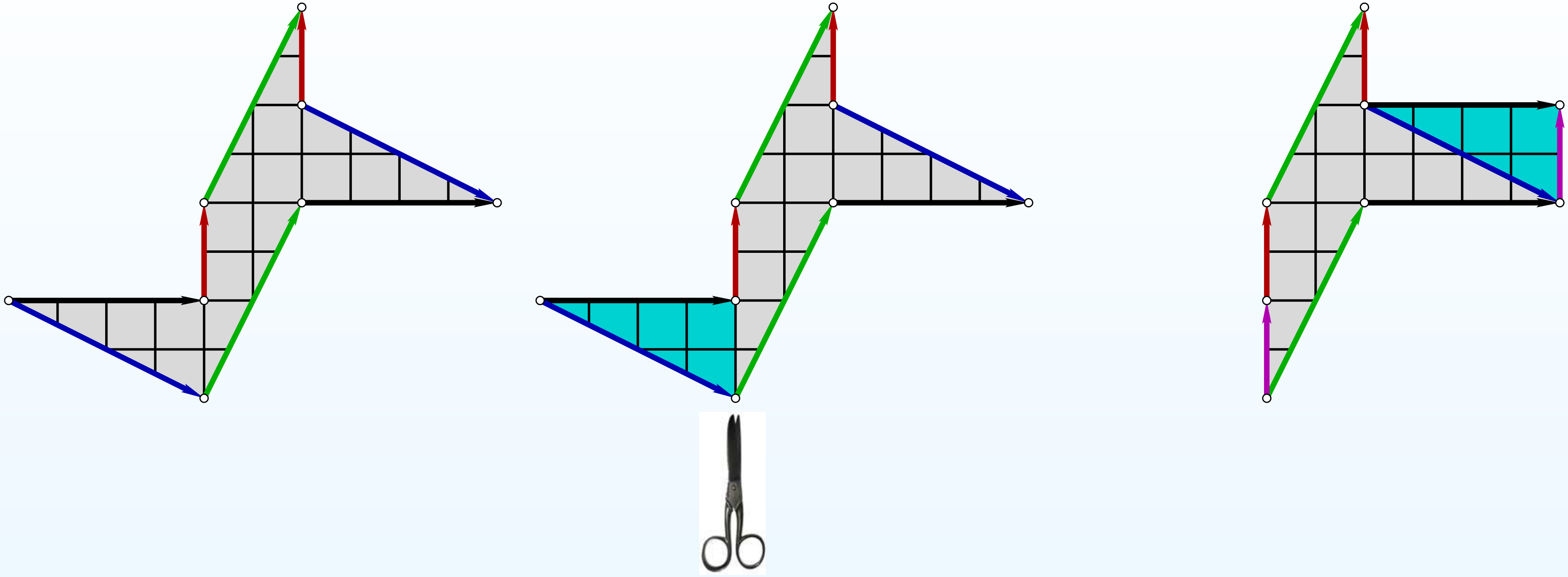

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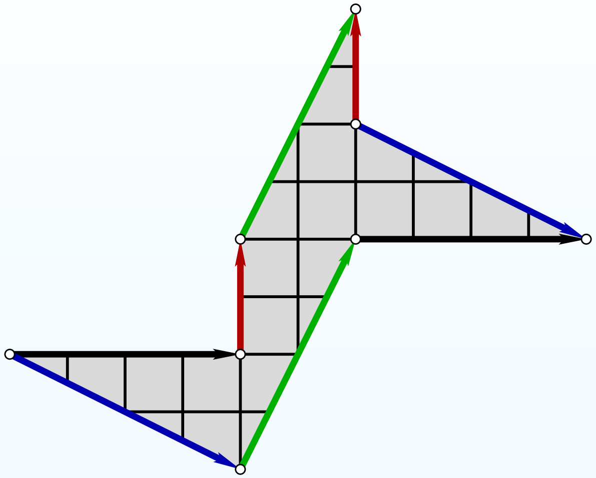

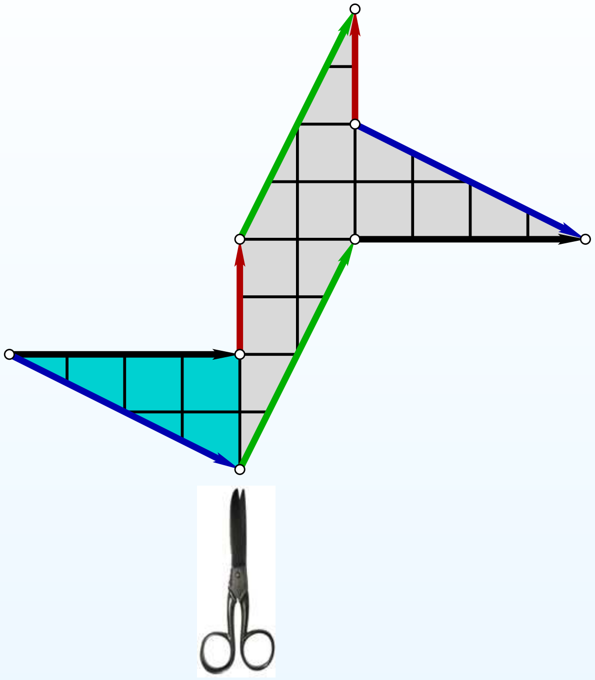

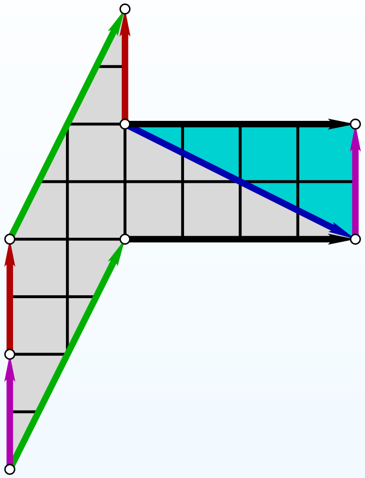

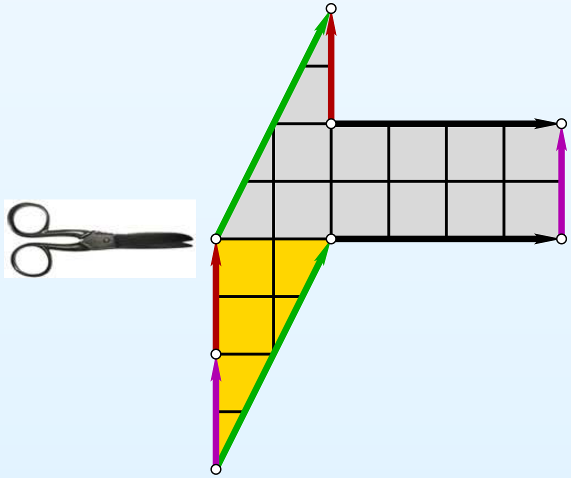

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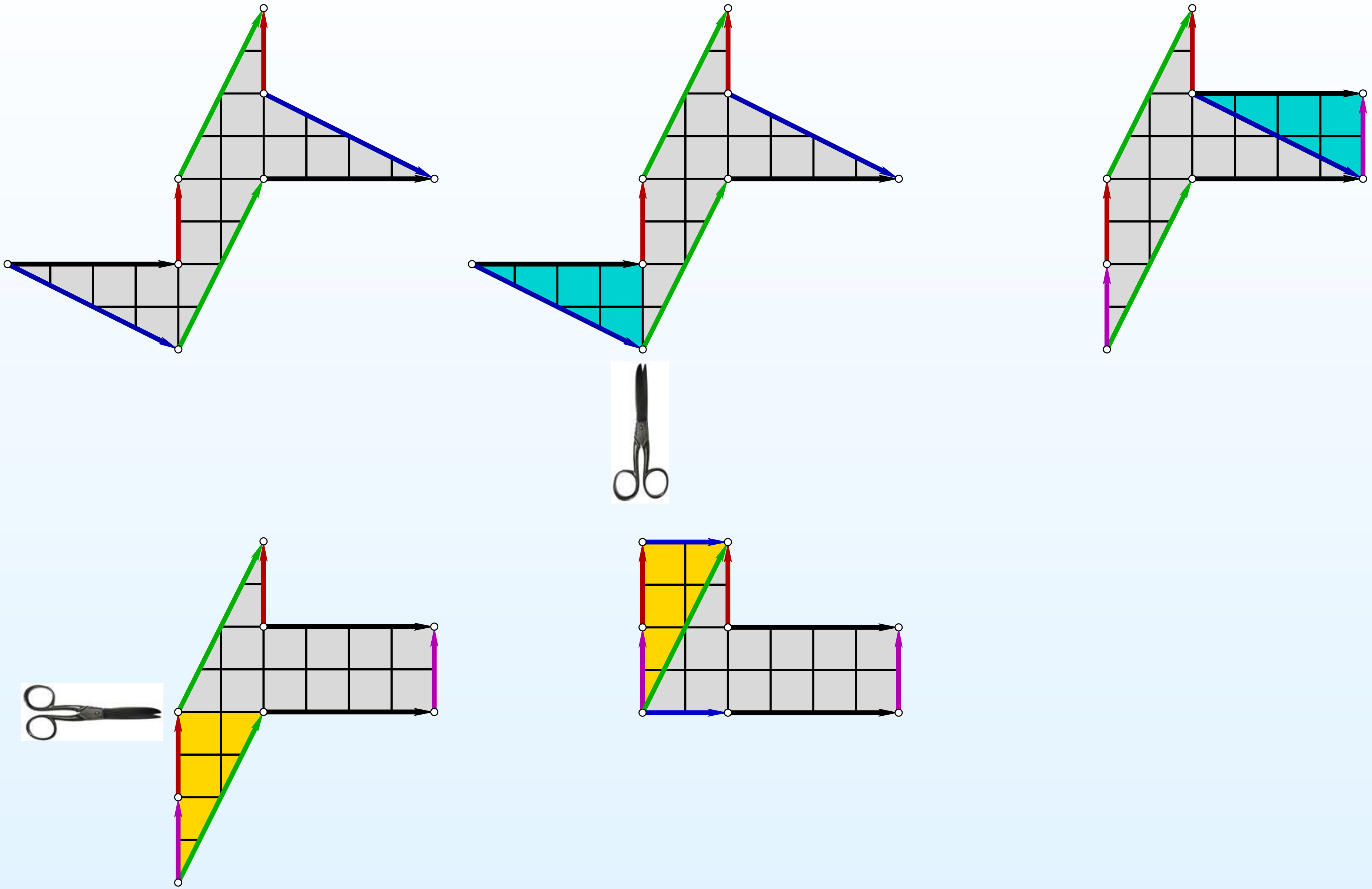

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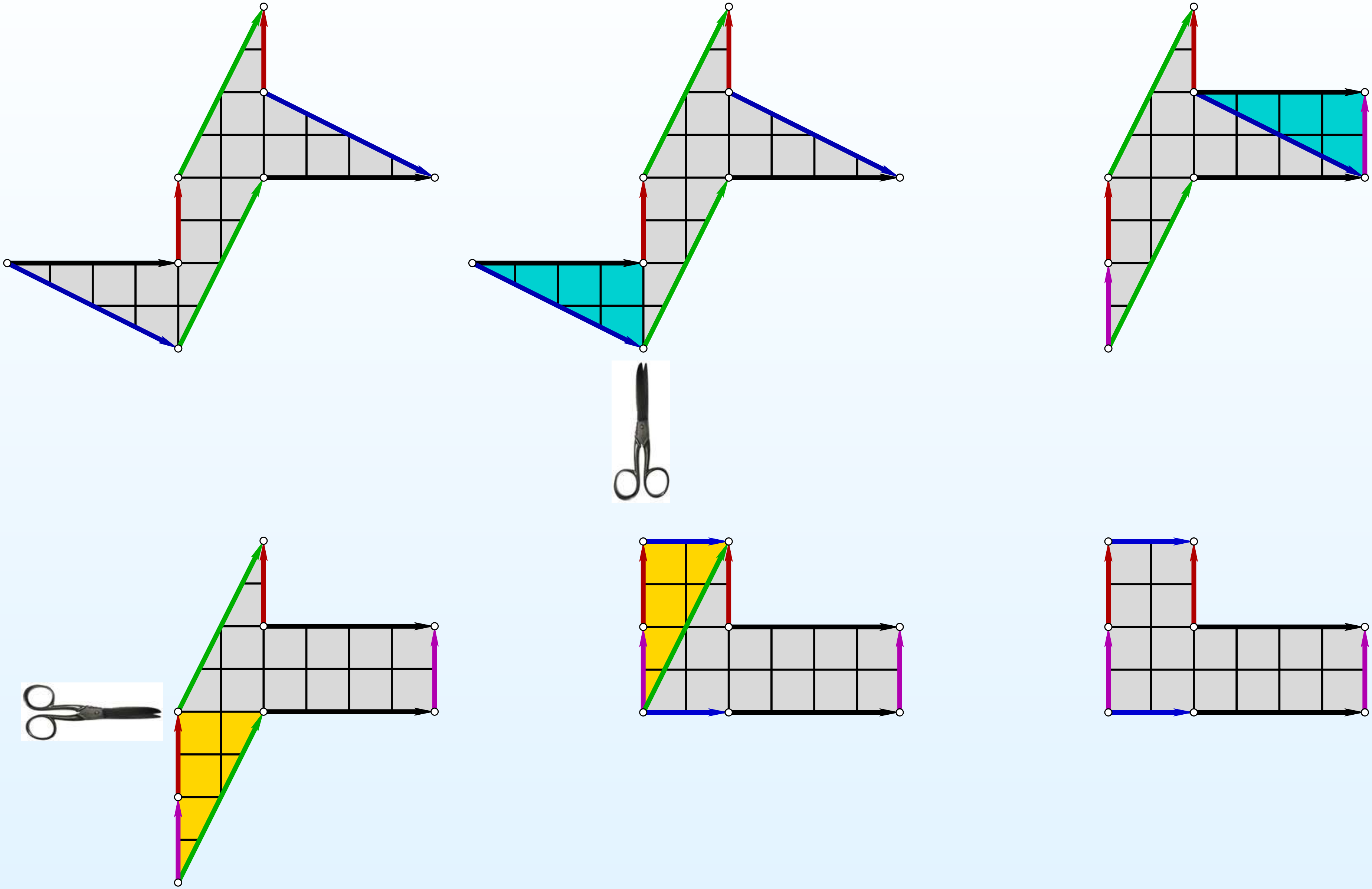

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Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

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The ramification points of the cover are exactly the zeroes of ω.

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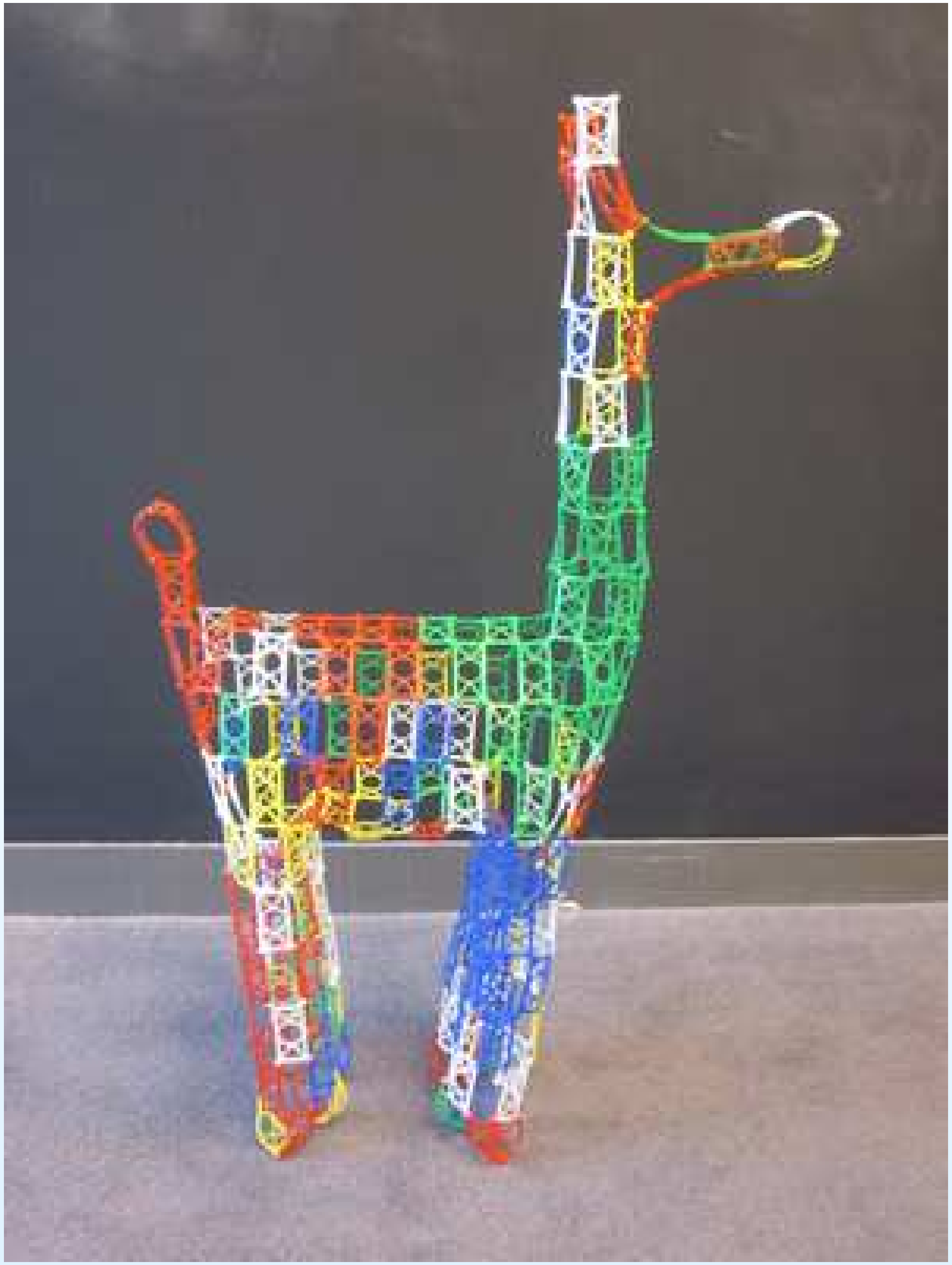
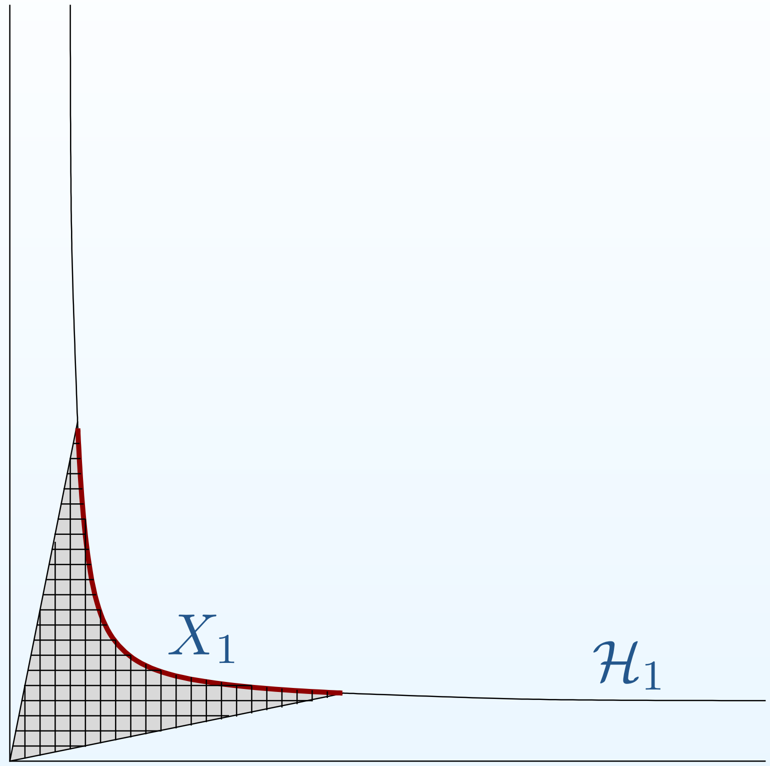
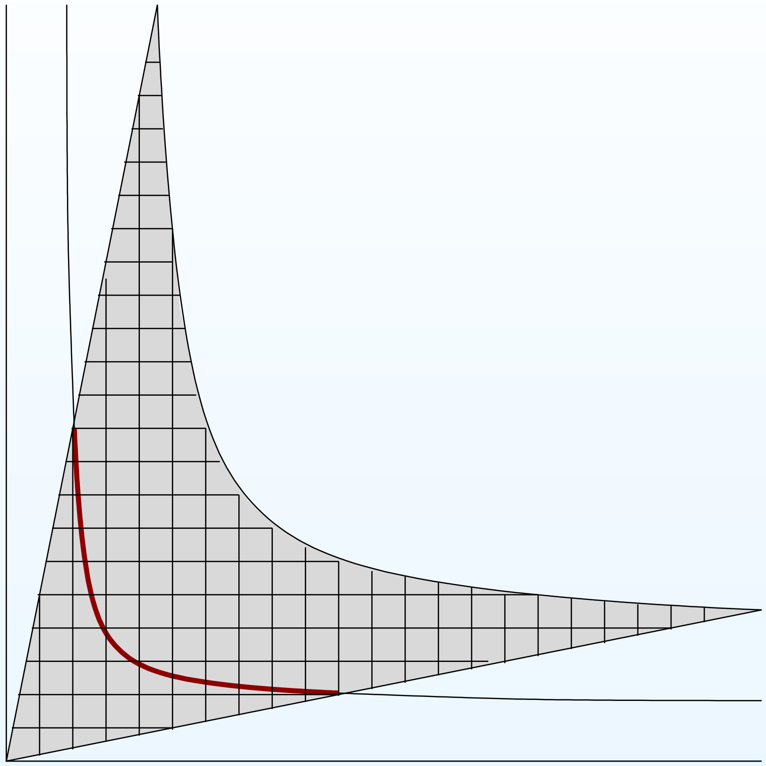
Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous "pillowcase covers" over $\mathbb{CP}^1$ branched at four points. Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

An example of a square-tiled surface



Counting volume by counting integer points in a large cone





To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

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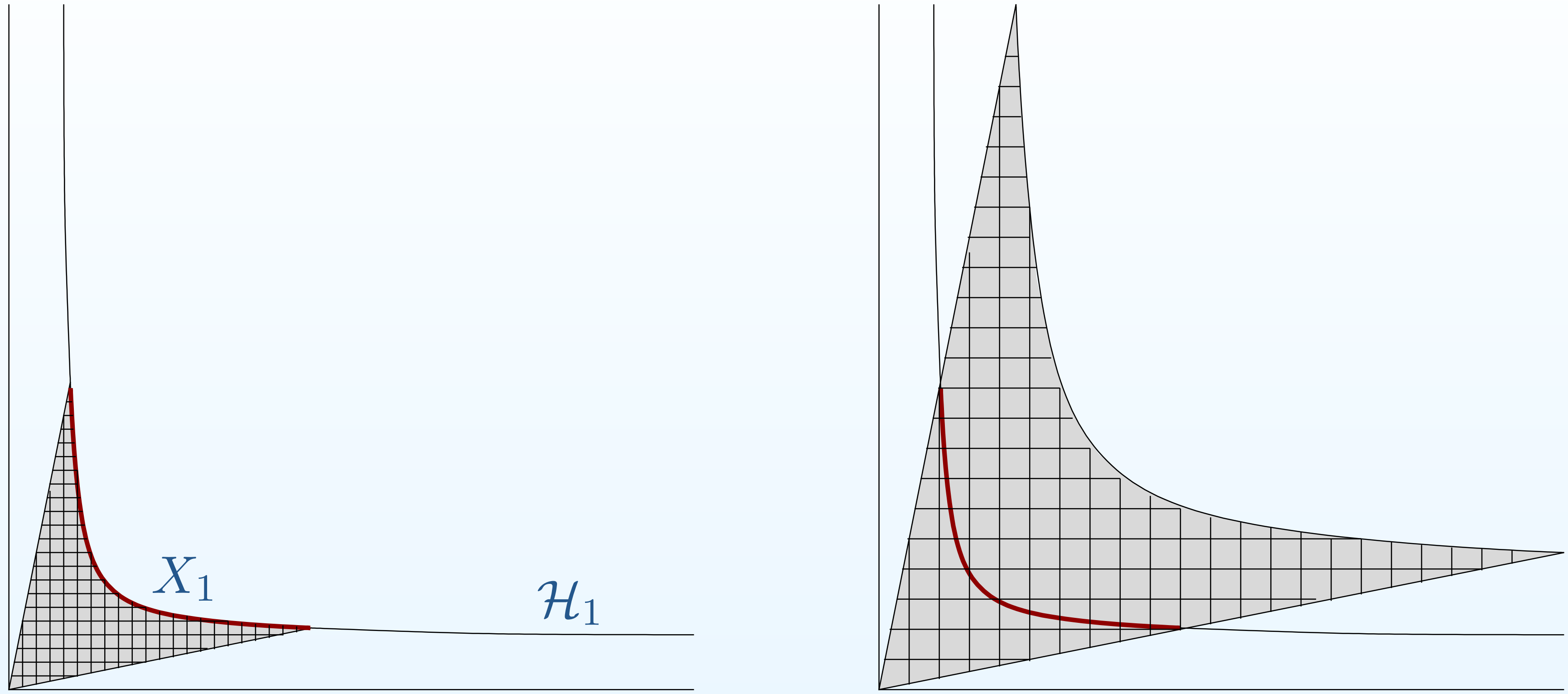

Let $\mathcal{H} = \mathcal{H}(m_1, \ldots, m_n)$; let $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \ldots, m_n) = 2g + n - 1$. We get:

$$\operatorname{Vol} \mathcal{H}_1 = 2d \cdot \lim_{N \to +\infty} \frac{\begin{pmatrix}\text{number of square-tiled surfaces in } \mathcal{H} \\ \text{tiled with at most } N \text{ identical squares}\end{pmatrix}}{N^d}\,.$$

Methods of evaluation of Masur–Veech volumes

● M. Kontsevich–A. Zorich (1998). Straightforward calculation of square-tiled surfaces.

● (A. Eskin–A. Okounkov–R. Pandharipande; D. Chen–M. Möller–D. Zagier; E. Goujard) A. Eskin and A. Okounkov observed in 2000 that the generating function for the count of square-tiled surfaces is a quasimodular form.

● D. Chen–M. Möller–A. Sauvaget; M. Kazarian; Di Yang–D. Zagier–Y. Zhang (2018–) Using recent BCGGM smooth compactification of the moduli space, one can work with the volume element as with the cohomology class. Intersection theory.

● V. Delecroix–E. Goujard–P. Zograf–A. Zorich (2018) (F. Arana–Herrera): volume of the principal stratum of quadratic differentials through Kontsevich's count of metric ribbon graphs in terms of Witten–Kontsevich correlators.

● D. Chen–M. Möller–A. Sauvaget–D. Zagier; A. Aggarwal (2018–) Large genus asymptotics for any stratum of Abelian differentials (proving conjectures of Eskin–Zorich and of Delecroix–Goujard–Zograf–Zorich).

● Andersen–Borot–Charbonnier–Delecroix–Giacchetto–Lewanski–Wheeler, 2020 (inspired by the formula of Delecroix-Goujard–Zograf–Zorich): topological recursion.


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