On the Canonical Ramsey Theorem and Ramsey ultrafilters

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Ramsey theorem (basic infinite version)

Notation and definitions

- For any set X and $n \in \omega$ the set of all n-element subsets of X is denoted by $[X]^n$:
$$[X]^n = \{\boldsymbol{x} \subseteq X : |\boldsymbol{x}| = n\}.$$
- A set $\mathcal{P} \subseteq \mathscr{P}(Z)$ is called a *partition* of a set Z if
  1. $\bigcup \mathcal{P} = Z$, and
  2. $(\forall X, Y \in \mathcal{P})\, X \cap Y = \varnothing \vee X = Y$.

  For convenience, we assume that one of the elements of the partition can be empty.
- For any partition $\mathcal{P}$ of $[X]^n$, a set $Y \subseteq X$ is called *homogeneous* for $\mathcal{P}$ if there is a set $P \in \mathcal{P}$ such that $[Y]^n \subseteq P$.

Theorem (Ramsey, 1930)

For any n, $1 \leqslant n < \omega$, and finite partition $\mathcal{P}$ of $[\omega]^n$ there is an infinite set $Y \subseteq \omega$ that is homogeneous for $\mathcal{P}$.

Canonical Ramsey Theorem of Erdős and Rado

Notation and definitions

- For any partition $\mathcal{P}$ of a set Z, the corresponding equivalence relation is denoted by $\approx_{\mathcal{P}}$:
  $$x \approx_{\mathcal{P}} y \Leftrightarrow (\exists P \in \mathcal{P})\, x, y \in P$$
  for all $x, y \in Z$.
- For any $X \subseteq \omega$ and $i < |X|$, the i-th (in the natural ordering) element $x \in X$ is denoted by $X_{[i]}$:
  $$x = X_{[i]} \Leftrightarrow (x \in X \wedge |x \cap X| = i).$$
- Let $\mathcal{P}$ be a partition of $[\omega]^n$, $1 \leqslant n < \omega$, and $I \subseteq n$. A set $X \subseteq \omega$ is called *I-canonical* for $\mathcal{P}$ if
  $$\boldsymbol{p} \approx_{\mathcal{P}} \boldsymbol{q} \Leftrightarrow \bigwedge_{i \in I} (\boldsymbol{p}_{[i]} = \boldsymbol{q}_{[i]})$$
  for all $\boldsymbol{p}, \boldsymbol{q} \in [X]^n$. A set $X \subseteq \omega$ is called *canonical* for $\mathcal{P}$ if there is a set $I \subseteq n$ such that X is *I-canonical* for $\mathcal{P}$.

Theorem (Erdős and Rado, 1950)

For any n, $1 \leqslant n < \omega$, and any partition $\mathcal{P}$ of $[\omega]^n$ there is an infinite set $Y \subseteq \omega$ that is canonical for $\mathcal{P}$.

Remarks

- As usual, the empty conjunction is true, so any $\varnothing$-canonical set X is homogeneous (for any partition $\mathcal{P}$ of $[\omega]^n$).
- Any infinite canonical set $X \subseteq \omega$ for a finite partition $\mathcal{P}$ of $[\omega]^n$ is $\varnothing$-canonical for $\mathcal{P}$. Therefore, Ramsey Theorem (RT) immediately follows from Canonical Ramsey Theorem (CRT).

P. Erdős, R. Rado, *A combinatorial theorem.* J. London Math. Soc. 25 (1950).

Reformulation in terms of functions $f : \omega^n \to \omega$

A function $f : \omega^n \to \omega$ is called

i. *selectively injective upward on a set $X \subseteq \omega$ w.r.t. a set (of indices) $I \subseteq n$* if

$$f(x_0, x_1, \ldots, x_{n-1}) = f(y_0, y_1, \ldots, y_{n-1}) \Leftrightarrow \bigwedge_{i \in I} (x_i = y_i)$$

for all $x_0 < x_1 < \ldots < x_{n-1}$ and $y_0 < y_1 < \ldots < y_{n-1}$ from X.

ii. *selectively injective upward* on a set $X \subseteq \omega$ if it is selectively injective on a set $X \subseteq \omega$ w.r.t. some non-empty set of indices $J \subseteq n$,

iii. *constant upward* on a set $X \subseteq \omega$ if it is selectively injective on a set $X \subseteq \omega$ w.r.t. $\varnothing$, i.e.,

$$f(x_0, x_1, \ldots, x_{n-1}) = f(y_0, y_1, \ldots, y_{n-1})$$

for all $x_0 < x_1 < \ldots < x_{n-1}$ and $y_0 < y_1 < \ldots < y_{n-1}$ from X.

- Ramsey Theorem staits that any function $f : \omega^n \to k$ (where $1 \leqslant n, k < \omega$) is constant upward on some infinite set $X \subseteq \omega$.
- Canonical Ramsey Theorem staits that any function $f : \omega^n \to \omega$ (where $1 \leqslant n < \omega$) is either selectively injective upward or constant upward on some infinite set $X \subseteq \omega$.

In monographs, Canonical Ramsey Theorem is often cited cithout proof, or with just the (not so representative) proof for $n = 2$.

Proof (for $n = 2$).
For $n = 1$ the proof is trivial. We will use this case.
Let $n = 2$. We will prove the following fact.

Fact

Let X be an infinite subset of ω and $f : X^2 \to Y$. Then there is an infinite set $X' \subseteq X$ for which one of the four following cases holds:

i. $f(x_0, x_1) = f(y_0, y_1)$ *for all* $x_0 < x_1 \in X'$ *and* $y_0 < y_1 \in X'$,
ii. $f(x_0, x_1) = f(y_0, y_1) \Leftrightarrow x_0 = y_0$ *for all* $x_0 < x_1 \in X'$ *and* $y_0 < y_1 \in X'$,
iii. $f(x_0, x_1) = f(y_0, y_1) \Leftrightarrow x_1 = y_1$ *for all* $x_0 < x_1 \in X'$ *and* $y_0 < y_1 \in X'$,
iv. $f(x_0, x_1) = f(y_0, y_1) \Leftrightarrow (x_0 = y_0 \wedge x_1 = y_1)$ *for all* $x_0 < x_1 \in X'$ *and* $y_0 < y_1 \in X'$.

If there is a set $X'' \subseteq X$ and a function $g : \omega \to \omega$ such that

$$(\forall x < y \in X'')f(x, y) = g(x) \text{ or } (\forall x < y \in X'')f(x, y) = g(y),$$

use the case $n = 1$. We will assume that this fact does not hold, and show that there is an infinite set $X' \subseteq X$ such that

$$f(x_0, x_1) = f(y_0, y_1) \Leftrightarrow (x_0 = y_0 \wedge x_1 = y_1)$$

for all $x_0 < x_1 \in X'$ and $y_0 < y_1 \in X'$.

We will construct the X' set in three steps.

1. First, we construct an infinite set $X_0 \subseteq X$, for which

$$f(x, x_1) = g(x, y_1) \Rightarrow x_1 = y_1 \qquad (*)$$

for all $x < x_1 \in X_0$ and $x < y_1 \in X_0$.

To do this, we construct a sequence $A_0 \supset A_1 \supset \dots$ of infinite subsets of X and two sequences $B_0 \subseteq B_1 \subseteq \dots$ and $C_0 \subseteq C_1 \subseteq \dots$ of finite subsets of X as follows. Put $A_0 = X$, $B_0 = C_0 = \varnothing$. If $A_0, A_1, \dots, A_{i-1}$, $B_0, B_1, \dots, B_{i-1}$, and $C_0, C_1, \dots, C_{i-1}$ are constructed, choose $c \in A_{i-1}$ such that $c > \max(B_{i-1} \cup C_{i-1})$. Denote $Z = \{x \in A_{i-1} : x > c\}$. Consider the function $g : Z \to Y$ defined by

$$g(x) = f(c, x).$$

- If there is an infinite set $Z' \subseteq Z$ such that $f\restriction_{Z'}$ is constant, put $A_i = Z'$, $B_i = B_{i-1} \cup \{c\}$, $C_i = C_{i-1}$.
- If there is an infinite set $Z' \subseteq Z$ such that $f\restriction_{Z'}$ is injective, put $A_i = Z'$, $B_i = B_{i-1}$, $C_i = C_{i-1} \cup \{c\}$.

At least one of the sets $B = \bigcup\limits_{i<\omega} B_i$ and $C = \bigcup\limits_{i<\omega} C_i$ is infinite. The case $|B| < \omega$ contradicts the assumption. Put $X_0 = C$. Condition $(*)$ holds.

2. Now, we construct an infinite set $X_1 \subseteq X_0$, for which

$$f(x_0, x) = g(y_0, x) \Rightarrow x_0 = y_0 \qquad (**)$$

for all $x < x_1 \in X_0$ and $x < y_1 \in X_0$.

To do this, we construct a sequence $A_0 \supset A_1 \supset \dots$ of infinite subsets of X_0, two sequences $B_0 \subseteq B_1 \subseteq \dots$ and $C_0 \subseteq C_1 \subseteq \dots$ of finite subsets of X_0 and sequence $P_0 \subseteq P_1 \subseteq \dots$ of equivalence relations on $B_0, B_1, \dots$ as follows. Put $A_0 = X_0$, $B_0 = C_0 = P_0 = \varnothing$. If $A_0, A_1, \dots, A_{i-1}$, $B_0, B_1, \dots, B_{i-1}$, $C_0, C_1, \dots, C_{i-1}$, and $P_0, P_1, \dots, P_{i-1}$ are constructed, choose $c \in A_{i-1}$ such that $c > \max B_{i-1}$. Put $B_i = B_{i-1} \cup \{c\}$. Denote $Z = \{x \in A_{i-1} : x > c\}$. For any $b \in B_{i-1}$ denote $Z_b = \{x \in Z : f(c, x) = f(b, x)\}$.

- If there is an infinite set Z_b, $b \in B_{i-1}$, put $A_i = Z_b$, $C_i = C_{i-1}$, $P_i = (P_{i-1} \cup \{(b, c)\})^*$ where P^* is reflexive, transitive and symmetric closure of P.
- Otherwize, put $A_i = Z \setminus \bigcup_{b \in B_{i-1}} Z_b$, $C_i = C_{i-1} \cup \{c\}$, $P_i = P_{i-1} \cup \{(c, c)\}$.

Let $B = \bigcup_{i<\omega} B_i$, $C = \bigcup_{i<\omega} C_i$, and $P = \bigcup_{i<\omega} P_i$. Either C or one of the P-equivalence classes is infinite. The case $|C| < \omega$ contradicts the assumption. Put $X_1 = C$. Condition $(**)$ holds.

3. Now let us construct the required set $X' \subseteq X_1$. Note that X_1 satisfies both conditions $(*)$ and $(**)$. Construct a sequences $B_0 \subseteq B_1 \subseteq \ldots$ of finite subsets of X_1 as follows. Let $B_0 \subseteq X_1$ be an arbitrary two-element set. Let B_{i-1} de constructed. Let $U = \{f(x,y) : x, y \in B_{i-1}, x < y\}$. Denote

$$V = \{y \in X_1 : y \geqslant \max(B_{i-1}) \wedge (\exists x \in B_{i-1})\, f(x,y) \in U\}.$$

From $(*)$ we have that any equation

$$f(a,y) = b$$

has at most one solution with $a < y$. So, $|V| < \omega$. Choose $c \in X_1$ such that $c \geqslant \max(B_{i-1})$ and $c \notin V$, and put $B_i = B_{i-1} \cup \{c\}$. Denote $X' = \bigcup\limits_{i<\omega} B_i$. From $(*)$ and $(**)$ we have that X' is a required set.

Remark

Difficulties start with $n \geqslant 3$.

Background

- Original proof: P. Erdős, R. Rado. *A combinatorial theorem.* J. London Math. Soc. 25 (1950).
- P. Erdős, R. Rado. *Combinatorial Theorems on Classifications of Subsets of a Given Set.* Proc. London Math. Soc. s3–2:1 (1952).
  The modified proof also covers the finite version.
- R. Rado. *Note on Canonical Partitions.* Bul. of the London Math. Soc. 18:2 (1986).
  Simplified version of the proof.
- H. Lefmann, V. Rödl. *On Erdős-Rado numbers.* Combinatorica 15 (1995).
  Estimates are obtained for the Erdős-Rado numbers (analogous to the Ramsey numbers).
- J. R. Mileti. *The canonical Ramsey theorem and computability theory.*, Trans. Amer. Math. Soc. 360 (2008)
  The author derives CRT from Koenig's lemma. CRT is studied in the context of reverse mathematics.
- P. Matet. *An easier proof of the Canonical Ramsey Theorem.* Colloquium Mathematicum 145 (2016).
  The author gives an elegant proof using the antilexicographical order on $[\omega]^n$.

Theorem (P., 2022)

For any natural number $n \geqslant 1$ and partition $\mathcal{P}$ of $[\omega]^n$ there is a finite partition $\mathcal{Q}$ of $[\omega]^{2n}$ such that any set $X \subseteq \omega$ that is homogeneous for $\mathcal{Q}$ is a finite union of sets that are canonical for $\mathcal{P}$.

Remarks

- CRT immediately follows from RT and this Theorem.
- The proof of this Theorem is quite elementary and does not use RT. Therefore, informally speaking, we divide CRT into a Ramseyan and a non-Ramseyan parts.
- This approach is particularly useful in the theory of ultrafilters.
- In fact, a theorem has been proven that gives a little more information.

Theorem (P., 2022)

For any natural number $n \geqslant 1$ there are natural numbers k and m (we can put $k = 2^{\frac{1}{2}\binom{2n}{n}\left(\binom{2n}{n}-1\right)}$ and $m = n^{n\binom{2n}{n}\left(\binom{2n}{n}-1\right)}$) such that for any partition $\mathcal{P}$ of $[\omega]^n$ there is a finite partition $\mathcal{Q}$ of $[\omega]^{2n}$ of cardinality no more than k for which for any set $Q \in \mathcal{Q}$ there is a set $I \subseteq n$ such that for any infinite set $X \subseteq \omega$ with $[X]^{2n} \subseteq Q$ there is a partition $\mathcal{R} = \{R_0, R_1, \ldots, R_m\}$ of X such that

1. *the set R_0 is finite of cardinality at most m,*
2. *for any i, $1 \leqslant i \leqslant m$, the set R_i is an infinite I-canonical set for $\mathcal{P}$.*

Application in theory of ultrafilters

Notation and definitions

- An ultrafilter on $\mathscr{P}(X)$ (or over X) is a set $\mathfrak{u} \subseteq \mathscr{P}(X)$ such that
  1. $\varnothing \notin \mathfrak{u}$,
  2. if $A \in \mathfrak{u}$ and $B \in \mathfrak{u}$, then $A \cap B \in \mathfrak{u}$,
  3. if $A \in \mathfrak{u}$ and $A \subseteq B$, then $B \in \mathfrak{u}$,
  4. $A \in \mathfrak{u}$ or $X \setminus A \in \mathfrak{u}$

  for all $A, B \subseteq X$.
- An ultrafilter $\mathfrak{u}$ on $\mathscr{P}(X)$ is principal if $\mathfrak{u} = \{A \subseteq X : a \in A\}$ for some $a \in X$.
- An ultrafilter $\mathfrak{u}$ on $\mathscr{P}(\omega)$ is called a *Ramsey ultrafilter* if it is nonprincipal and for any n, $1 \leqslant n < \omega$, and finite partition $\mathcal{P}$ of $[\omega]^n$, $\mathfrak{u}$ contains some set $X \subseteq \omega$ that is homogenous for $\mathcal{P}$.

There are many equivalent characterizations of Ramsey ultrafilters. In particular, an ultrafilter $\mathfrak{u}$ is Ramsey if and only if it is selective, and if and only if it is minimal (characterizations in terms of unary functions and their ultrafilter extensions).

W. W. Comfort, S. Negrepontis. *The theory of ultrafilters.* Springer, Berlin, 1974.

Definition

An ultrafilter $\mathfrak{u}$ on $\mathscr{P}(\omega)$ is *selective* if for every function $f : \omega \to \omega$ there is $X \in \mathfrak{u}$ such that the restriction $f \upharpoonright X$ of f to X is either one-to-one or constant.

The concept of a minimal ultrafilter is based on the notion of ultrafilter extension of unary functions and the Rudin-Keisler (pre)order. For any set A, the set of all ultrafilters on $\mathscr{P}(A)$ is denoted by $\boldsymbol{\beta} A$.

Definition

For any function $f : A \to B$ the ultrafilter extension $\widetilde{f}$ of f is the function from $\boldsymbol{\beta} A$ to $\boldsymbol{\beta} B$ defined by

$$\widetilde{f}(\mathfrak{u}) = \{S \subseteq B : (\forall X \in \mathfrak{u})(\exists x \in X)\, f(x) \in S\}$$

for all $\mathfrak{u} \in \boldsymbol{\beta} A$.

Definition

The *Rudin-Keisler preorder* on $\boldsymbol{\beta} A$ is the binary relation $\leqslant_{\mathrm{RK}}$ defined by

$$\mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v} \Leftrightarrow \widetilde{f}(\mathfrak{v}) = \mathfrak{u} \text{ for some } f : A \to A$$

for all $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta} A$.

Definition

An ultrafilter $\mathfrak{u} \in \boldsymbol{\beta} A$ is called *minimal* if

$$\mathfrak{v} \leqslant_{\mathrm{RK}} \mathfrak{u} \;\Rightarrow\; \mathfrak{v} \text{ is principal or } \mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v}$$

for any $\mathfrak{v} \in \boldsymbol{\beta} A$.

In other words, $\mathfrak{u}$ is minimal if for any function $f : A \to A$ either $\widetilde{f}(\mathfrak{u})$ is principal or there is a function $g : A \to A$ such that $\widetilde{g}\left(\widetilde{f}(\mathfrak{u})\right) = \mathfrak{u}$.

The equivalence relation $\leqslant_{\mathrm{RK}} \cap \leqslant_{\mathrm{RK}}^{-1}$ is denoted by $\approx_{\mathrm{RK}}$. The Rudin-Keisler preorder naturally extends to the quotient set $\boldsymbol{\beta} A / \approx_{\mathrm{RK}}$:

$$\tau(\mathfrak{u}) \leqslant_{\mathrm{RK}} \tau(\mathfrak{v}) \;\Leftrightarrow\; \mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v}$$

for all equivalence class $\tau(\mathfrak{u})$ and $\tau(\mathfrak{v})$ of ultrafilters $\mathfrak{u}$ and $\mathfrak{v}$, respectively.

The relation $\leqslant_{\mathrm{RK}}$ is a (partial) order on $\boldsymbol{\beta} A / \approx_{\mathrm{RK}}$, and ultrafilter $\mathfrak{u}$ is *minimal* iff the equivalence class $\tau(\mathfrak{u})$ is a minimal element of the poset $(\boldsymbol{\beta} A / \approx_{\mathrm{RK}}) \setminus \{\tau(a)\}$ where a is any principal ultrafilter on A (all principal ultrafilters on A are equivalent w.r.t. $\approx_{\mathrm{RK}}$).

Ultrafilter extention of n-ary functions

Ultrafilter extensions of binary maps, especially of group and semigroup operations, have been considered since the 60s of the 20th century. The results obtained in this field have found numerous Ramsey-theoretic applications in number theory, algebra, topological dynamics, and ergodic theory.
Ultrafilter extensions of arbitrary n-ary maps (and, more broadly, of first-order models) have been introduced independently in recent works by V. Goranko and D. I. Saveliev.

- N. Hindman, D. Strauss. *Algebra in the Stone–Čech Compactification.* 2nd ed., revised and expanded, W. de Gruyter, Berlin–N.Y., 2012. Springer, Berlin, 1974.
- V. Goranko. *Filter and ultrafilter extensions of structures: universal-algebraic aspects.* Preprint, 2007.
- D. I. Saveliev. *On ultrafilter extensions of models.* In: S.-D. Friedman et al. (eds.). The Infinity Project Proc. CRM Documents 11, Barcelona, 2012, 599–616.
- D. I. Saveliev, S. Shelah. *Ultrafilter extensions do not preserve elementary equivalence*. Math. Log. Quart., 65 (2019): 511–516.
- Poliakov, N.L., Saveliev, D.I. On ultrafilter extensions of first-order models and ultrafilter interpretations. Arch. Math. Logic 60 (2021), 625–681.

For a map $f : A^n \to B$, the extended map $\widetilde{f} : (\boldsymbol{\beta} A)^n \to \boldsymbol{\beta} B$ can be defined by recursion.

Definition

- A nullary function f is identified with a constant $c_f \in B$. For $n = 0$, we define $\widetilde{f}$ as the principal ultrafilter generated by c_f, i.e.

$$\widetilde{f} = \{S \subseteq B : c_f \in S\}.$$

- For $n > 0$ we define

$$\widetilde{f}(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n) = \{S \subseteq B : (\forall X \in \mathfrak{u}_1)(\exists x \in X)\, S \in \widetilde{f_x}(\mathfrak{u}_2, \dots, \mathfrak{u}_n)\},$$

  where $f_x(x_2, \dots, x_n) = f(x, x_2, \dots, x_n)$ for all $x, x_2, \dots, x_n \in A$.

It is easy to verify that for $n = 1$ we have the definition equivalent to that given above.

The combinatorial theorem allows one to obtain a characterization of Ramsey ultrafilters in terms of arbitrary n-ary functions and their ultrafilter extensions.

Fact

For all ultrafilters $\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{n-1} \in \boldsymbol{\beta} A$ *and one-to-one functions* $f, g : A^n \to A$ *ultrafilters* $\widetilde{f}(\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{n-1})$ *and* $\widetilde{g}(\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{n-1})$ *are* RK*-equivalent.*

Considering ultrafilters up to equivalence relation $\approx_{\mathrm{RK}}$, we denote by

$$\mathfrak{u}_0 \times \mathfrak{u}_1 \times \ldots \times \mathfrak{u}_{n-1}$$

the ultrafiter $\widetilde{f}(\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_{n-1})$ for some one-to-one map $f : A^n \to A$.

Theorem (P., 2022)

Let $\mathfrak{u}$ *be a non-principal ultrafilter on* $\mathscr{P}(\omega)$*. Then the following conditions are equivalent:*

1. $\mathfrak{u}$ *is Ramsey ultrafilter;*
2. *for every* n, $1 \leqslant n < \omega$, *and partition* $\mathcal{P}$ *of* $[\omega]^n$, $\mathfrak{u}$ *contains some set* X *that is canonical for* $\mathcal{P}$*;*
3. *for every* n, $1 \leqslant n < \omega$, *and function* $f : \omega^n \to \omega$, $\mathfrak{u}$ *contains some set* X *such that* f *is either selectively injective upward or constant upward on* X*;*
4. *for every* n, $1 \leqslant n < \omega$, *and function* $f : \omega^n \to \omega$, *either* $\widetilde{f}(\mathfrak{u}, \mathfrak{u}, \ldots, \mathfrak{u})$ *is principal or* $\widetilde{f}(\mathfrak{u}, \mathfrak{u}, \ldots, \mathfrak{u}) \approx_{\mathrm{RK}} \underbrace{\mathfrak{u} \times \mathfrak{u} \times \ldots \times \mathfrak{u}}_{m \text{ times}}$ *for some* m, $1 \leqslant m \leqslant n$.

In combinatorial applications of the theory of ultrafilters, non-principal *idempotents* are of great importance. It is well known that among Ramsey ultrafilters $\mathfrak{u}$ there are no one such that $\mathfrak{u} \mathbin{\widetilde{+}} \mathfrak{u} = \mathfrak{u}$ or $\mathfrak{u} \mathbin{\widetilde{\cdot}} \mathfrak{u} = \mathfrak{u}$. It can be shown that this property of Ramsey ultrafilters extends to any function $f : \omega^n \to \omega$, except for trivial cases.

Proposition (P., 2022)

Let $\mathfrak{u}$ be a Ramsey ultrafilter on $\mathscr{P}(\omega)$, and let $f : \omega^n \to \omega$, $1 \leqslant n < \omega$. Then $\widetilde{f}(\mathfrak{u}, \mathfrak{u}, \ldots, \mathfrak{u}) = \mathfrak{u}$ if and only if there are $X \in \mathfrak{u}$ and $i < n$ such that

$$f(x_0, x_1, \ldots, x_{n-1}) = x_i$$

for all $x_0 < x_1 < \ldots < x_{n-1} \in X$.

Discussion

The shortest and most elegant proof of the Ramsey Theorem uses the ultrafilter technique. Can combinatorial theorem (or the Canonical Ramsey Theorem) be proved in a similar way?

D. I. Saveliev. *On idempotents in compact left topological universal algebras*. Topology Proc. 43 (2014), 37–46.

Appendix: infinitary functions

Can these results be more or less extended to functions $f:\omega^\omega\to\omega$?

Trivial note. Yes, if the function only depends on a finite number of arguments, i.e.

$$(\exists n<\omega)(\forall \boldsymbol{x},\boldsymbol{y}\in\omega^\omega)\,\boldsymbol{x}\restriction_n=\boldsymbol{y}\restriction_n\Rightarrow f(\boldsymbol{x})=f(\boldsymbol{y}).$$

Consider the set $\mathcal{B}$ of all functions $f:\omega^\omega\to\omega$ satisfying

$$(\forall \boldsymbol{x}\in\omega^\omega)(\exists n<\omega)(\forall \boldsymbol{y}\in\omega^\omega)\,\boldsymbol{x}\restriction_n=\boldsymbol{y}\restriction_n\Rightarrow f(\boldsymbol{x})=f(\boldsymbol{y}).$$

Example.

$$f(x_0,x_1,x_2,\ldots)=x_1+x_2+\ldots+x_{x_0}.$$

Facts.

- $\mathcal{B}$ is the set of all continuous functions from the topological space (ω^ω,τ_0) with the base $\{\{\boldsymbol{x}\in\omega^\omega:\boldsymbol{x}\restriction_n=\boldsymbol{a}\}:n\in\omega,\boldsymbol{a}\in\omega^n\}$ to the topological space (ω,τ_1) with discrete topology.

D.I. Saveliev (joint work with P.). Between the Rudin–Keisler and Comfort preorders. Report at the conference Ultramat 2022 (Pisa). https://www.ultramath.it/

- The set $\mathcal{B}$ allows an ordinal hierarchy: let
    1. $\mathcal{B}_0$ be the set of constant functions $f : \omega^\omega \to \omega$, and
    2. for any ordinal $\alpha > 0$, $\mathcal{B}_\alpha$ is the set of functions $f : \omega^\omega \to \omega$ such that for any $a \in \omega$ the function $f_a(x_0, x_1, \ldots) = f(a, x_0, x_1, \ldots)$ belongs to $\bigcup\limits_{\beta<\alpha} \mathcal{B}_\beta$.

  So, $\mathcal{B} = \bigcup\limits_{\alpha<\omega_1} \mathcal{B}_\alpha$.

- The ultrafilter extension of functions $f \in \mathcal{B}$ is correctly defined as follows:
    i. if $f \in \mathcal{B}_0$, and $f(x_0, x_1, \ldots) \equiv c$, then

    $$\widetilde{f}(\mathfrak{u}_0, \mathfrak{u}_1, \ldots) \equiv \langle c \rangle,$$

    where $\langle c \rangle = \{S \subseteq \omega : c \in S\}$,

    ii. for any ordinal α, $0 < \alpha < \omega_1$,

    $$\widetilde{f}(\mathfrak{u}_0, \mathfrak{u}_1, \mathfrak{u}_2, \ldots) = \{S \subseteq \omega : (\forall X \in \mathfrak{u}_0)(\exists x \in X)\, S \in \widetilde{f}_x(\mathfrak{u}_1, \mathfrak{u}_2, \ldots,)\}.$$

The ultrafilter extension of functions can be used to prove (or re-prove) some combinatorial results.

Definition

Let $f[X]$ be the image of X under f, and let $I = \{\boldsymbol{x} \in \omega^\omega : \boldsymbol{x} \text{ is increasing}\}$.

If $X \subseteq \omega$ and $f : \omega^\omega \to \omega$, we say that f

- is *constant upward on* X iff $|f[X \cap I]| = 1$,
- is *quasi-invertible upward on* X iff there exists $g : Y \to \omega$ such that for any infinite $A \subseteq X$ we have $g(f[A \cap I]) \subseteq A$ and $|A \setminus g(f[A \cap I])| < \omega$.

Theorem

For any function $f \in \mathcal{B}$ with $|\mathrm{ran}\, f| < \omega$ there is an infinite set $X \subseteq \omega$ such that f is constant upward on X.

Theorem

For any function $f \in \mathcal{B}$ there is an infinite set $X \subseteq \omega$ such that f is either constant upward or quasi-invertible upward on X.

Theorem

Let $\mathfrak{u} \in \beta\omega$ be Ramsey ultrafilter. Then for any function $f \in \mathcal{B}$ there is a set $X \in \mathfrak{u}$ such that $f \in \mathcal{B}$ is either constant upward or quasi-invertible upward on X.

THANK YOU!